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SCHRÖDINGER EQUATION WITH MULTIPARTICLE POTENTIAL AND CRITICAL NONLINEARITY

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Abstract. We study the existence and non-existence of ground states for the Schrödinger equations $-\Delta u - \lambda \sum_{i<j} u/|x_i - x_j|^2 = |u|^{2^*-2}u$, $x = (x_1, \ldots, x_m) \in \mathbb{R}^{mN}$, and $-\Delta u - \lambda u/|y|^2 = |u|^{2^*-2}u$, $x = (y, z) \in \mathbb{R}^N$. In both cases we assume $\lambda \neq 0$ and $\lambda < \overline{\lambda}$, where $\overline{\lambda}$ is the Hardy constant corresponding to the problem.

1. Introduction and statement of main results

Let $x_1, \ldots, x_m$ represent m particles in $\mathbb{R}^N$, denote $x = (x_1, \ldots, x_m) \in \mathbb{R}^{mN}$ and let

$$V(x) := \sum_{i<j} \frac{1}{|x_i - x_j|^2}. \tag{1.1}$$

It has been shown in a recent paper by M. Hoffmann-Ostenhof et al. [7] that the following Hardy inequality holds if $m \geq 2$ and $N \geq 3$:

$$\overline{\lambda} := \inf_{u \in H^1(\mathbb{R}^{mN}) \setminus \{0\}} \frac{\int_{\mathbb{R}^{mN}} |\nabla u|^2 \, dx}{\int_{\mathbb{R}^{mN}} V(x) u^2 \, dx} > 0. \tag{1.2}$$

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For $N = 1$ (1.2) remains valid if $H^1(\mathbb{R}^m)$ is replaced by $H_0^1(\mathbb{R}^m \setminus N_m)$, where

$$N_m := \{x = (x_1, \dots, x_m) \in \mathbb{R}^m : x_i = x_j \text{ for some } i \neq j\}, \tag{1.3}$$

and in this latter case $\overline{\lambda} = 1/2$, see [7].

In the present paper we study the Schrödinger equation

$$-\Delta u - \lambda V(x)u = |u|^{2^*-2}u \quad \text{in } \mathbb{R}^{mN}, \tag{1.4}$$

where $\lambda < \overline{\lambda}$, $\lambda \neq 0$ and $2^* := 2mN/(mN-2)$ is the critical Sobolev exponent.

Let $\|\cdot\|_p$ denote the usual $L^p(\mathbb{R}^l)$-norm and $\mathcal{D}^{1,2}(\mathbb{R}^l)$ the closure of $C_0^\infty(\mathbb{R}^l)$ in the norm $\|\nabla u\|_2$ ($l = mN$ or N depending on whether we consider (1.5) or (1.7) below). Let $m \geq 2$, $N \geq 3$ and

$$S_\lambda := \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^{mN}) \setminus \{0\}} \frac{\int_{\mathbb{R}^{mN}} |\nabla u|^2 \, dx - \lambda \int_{\mathbb{R}^{mN}} V(x)u^2 \, dx}{\|u\|_{2^*}^2}. \tag{1.5}$$

Assuming $\lambda < \overline{\lambda}$, it follows from (1.2) and the Sobolev inequality that $S_\lambda > 0$. Moreover, if there exists a minimizer $\overline{u}$, then $\overline{u}$, normalized by $\|\overline{u}\|_{2^*}^{2^*-2} = S_\lambda$, is a solution of (1.4). It will be called a ground state. Obviously, $S_0 = S$, where S denotes the best Sobolev constant for the embedding $\mathcal{D}^{1,2}(\mathbb{R}^{mN}) \hookrightarrow L^{2^*}(\mathbb{R}^{mN})$.

Our main result is the following

Theorem 1.1. *Suppose $m \geq 2$ and $N \geq 3$. If $0 < \lambda < \overline{\lambda}$, then $S_\lambda < S$ and there exists a ground state $u \in \mathcal{D}^{1,2}(\mathbb{R}^{mN})$ for* (1.4)*. If $\lambda < 0$, then $S_\lambda = S$ and there is no ground state.*

In Remark 3.3 we make comments on the cases $N = 1$ and 2. For the moment we only note that if $N = 1$, $m \geq 3$ and $0 < \lambda < \overline{\lambda} \equiv 1/2$, then S_λ is still well defined and positive; however, $\mathcal{D}^{1,2}(\mathbb{R}^{mN})$ in (1.5) must be replaced by $\mathcal{D}_0^{1,2}(\mathbb{R}^{mN} \setminus N_m)$.

In the two-particle case we can change the variables to $x = (y, z)$, where $y = (x_1 - x_2)/\sqrt{2}$ and $z = (x_1 + x_2)/\sqrt{2}$ (cf. Lemma 4.6 in [7]). Then $\Delta u(x_1, x_2) = \Delta u(y, z)$ and

$$V(x_1, x_2) = V(y, z) = \frac{2}{|y|^2}.$$

Motivated by this, we let $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, $1 \leq k < N$, $2^* := 2N/(N-2)$ and consider the equation

$$-\Delta u - \lambda \frac{u}{|y|^2} = |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N. \tag{1.6}$$

The corresponding minimization problem is

$$\widehat{S}_\lambda := \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \lambda \int_{\mathbb{R}^N} (u^2/|y|^2) \, dx}{\|u\|_{2^*}^2}. \tag{1.7}$$

It is well known from the Hardy–Sobolev–Maz'ja inequality [8, Corollary 3, Section 2.1.6] that if

$$\overline{\lambda} := \inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx}{\int_{\mathbb{R}^N} (u^2/|y|^2)\,dx},$$

then $\overline{\lambda} = ((k-2)/2)^2$ and $\widehat{S}_\lambda > 0$ for $k \geq 3$, $\lambda \leq \overline{\lambda}$. The same is true for $k = 1$, but with $\mathcal{D}^{1,2}(\mathbb{R}^N)$ replaced by $\mathcal{D}^{1,2}_0(\mathbb{R}^N \setminus (\{0\}\times\mathbb{R}^{N-k}))$. In [13] it has been shown that $\widehat{S}_\lambda$ is attained (in a larger space) if $\lambda = \overline{\lambda}$; here we assume $\lambda < \overline{\lambda}$.

THEOREM 1.2. *Suppose $3 \leq k < N$. If $0 < \lambda < \overline{\lambda}$, then $\widehat{S}_\lambda < S$ and there exists a ground state $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ for* (1.6)*. If $\lambda < 0$, then $\widehat{S}_\lambda = S$ and there is no ground state.*

Let $\mathcal{D}^{1,2}_{\mathrm{sym}}(\mathbb{R}^N) := \{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) : u = u(|y|,|z|)\}$, i.e. u is radially symmetric in each of the variables y and z but not necessarily in x. Denote the infimum in (1.7), taken with respect to $u \in \mathcal{D}^{1,2}_{\mathrm{sym}}(\mathbb{R}^N)$, by $\widehat{S}_{\lambda,\mathrm{sym}}$.

THEOREM 1.3. *Suppose $3 \leq k < N$. If $0 < \lambda < \overline{\lambda}$, then $\widehat{S}_{\lambda,\mathrm{sym}}$ is attained and $\widehat{S}_{\lambda,\mathrm{sym}} = \widehat{S}_\lambda$. If $\lambda < 0$, then $\widehat{S}_\lambda < \widehat{S}_{\lambda,\mathrm{sym}}$ and $\widehat{S}_{\lambda,\mathrm{sym}}$ is attained* (*while $\widehat{S}_\lambda$ is not as follows from the preceding theorem*).

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Theorems 1.2 and 1.3 also hold for $k = N$. However, since this case has already been considered in [10], [12], we do not discuss it here. We would also like to mention some problems which are somewhat related to our work: to minimize

$$\frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx}{(\int_{\mathbb{R}^N} (|u|^q/|y|^\beta)\,dx)^{2/q}},$$

where $q = 2(N-\beta)/(N-2)$, see e.g. [3], [4], [11], to minimize

$$\frac{1}{\|u\|^2_{2^*}}\biggl(\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \sum_{i=1}^m \lambda_i \int_{\mathbb{R}^N} \frac{u^2}{|x-a_i|^2}\,dx\biggr),$$

where $(a_1,\ldots,a_m)$ is fixed in $\mathbb{R}^{mN}$ [6], and to find nonnegative solutions $u \in H^1(\mathbb{R}^N)$ for the equation

$$-\Delta u + \frac{u}{|y|^2} = f(u),$$

where f is of subcritical growth [1].

Finally we note that if u is a minimizer for (1.5) or (1.7), then so is $|u|$. Therefore we may assume without loss of generality the ground states we have found are non-negative.

When this paper was already written, the authors have learned about recent work [2] and [9]. Our Theorem 1.3 is similar to Theorem 1 in [2] and Theorem 1.2 is similar to Theorem 2 in [9]. However, since our arguments are different and simpler than those contained in [2], [9], we include them in this paper.

2. Proofs of Theorems 1.2 and 1.3

Let $\mathcal{M}(\mathbb{R}^N)$ denote the space of finite measures on $\mathbb{R}^N$ and recall that $\mu_n \rightharpoonup \mu$ in $\mathcal{M}(\mathbb{R}^N)$ if $\langle \mu_n, \varphi\rangle \to \langle \mu, \varphi\rangle$ for all $\varphi \in C_0(\mathbb{R}^N)$, where $C_0(\mathbb{R}^N)$ is the closure, in the $L^\infty(\mathbb{R}^N)$-norm, of the set of continuous and compactly supported functions. For each $R > 0$, let $\psi_R \in C^\infty(\mathbb{R}^N, [0,1])$ be a radially symmetric function such that $\psi_R(x) = 0$ as $|x| \le R$ and $\psi_R(x) = 1$ as $|x| \ge R + 1$. Given $\lambda < \overline{\lambda}$ and a sequence $u_n \rightharpoonup u$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, we introduce the measures at infinity

$$\mu_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 - \lambda \frac{u_n^2}{|y|^2}\right) \psi_R^2 \, dx \tag{2.1}$$

and

$$\nu_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2^*} \psi_R^{2^*} \, dx. \tag{2.2}$$

Originally the definition of ν_∞ has been given by the expression

$$\nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\ge R} |u_n|^{2^*} \, dx,$$

and these two definitions are known to be equivalent, see [5] or the proof of Lemma 1.40 in [15]. The corresponding two definitions of μ_∞ are equivalent when $\lambda \le 0$, and obviously, $\mu_\infty \ge 0$ in this case. However, if $0 < \lambda < \overline{\lambda}$, this is no longer clear, the reason being that the inequality $|\nabla u_n|^2 - \lambda u_n^2/|y|^2 \ge 0$ may not hold almost everywhere. By the same reason it is not clear that the limit as $R \to \infty$ exists in the definition of μ_∞, see Remark 2.2 below.

Lemma 2.1. *Let $(u_n) \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ be a sequence such that $u_n \rightharpoonup u$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, $u_n \to u$ almost everywhere in $\mathbb{R}^N$,*

$$|\nabla(u_n - u)|^2 - \lambda \frac{(u_n - u)^2}{|y|^2} \rightharpoonup \mu \quad \text{and} \quad |u_n - u|^{2^*} \rightharpoonup \nu \quad \text{in } \mathcal{M}(\mathbb{R}^N). \tag{2.3}$$

Then

$$\|\nu\|^{2/2^*} \le \widehat{S}_\lambda^{-1} \|\mu\|, \quad \nu_\infty^{2/2^*} \le \widehat{S}_\lambda^{-1} \mu_\infty,$$

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 - \lambda \frac{u_n^2}{|y|^2}\right) dx = \int_{\mathbb{R}^N} \left(|\nabla u|^2 - \lambda \frac{u^2}{|y|^2}\right) dx + \|\mu\| + \mu_\infty \tag{2.4}$$

and

$$\limsup_{n\to\infty} \|u_n\|_{2^*}^{2^*} = \|u\|_{2^*}^{2^*} + \|\nu\| + \nu_\infty.$$

Moreover, if $u = 0$ *and* $\|\nu\|^{2/2^*} = \widehat{S}_\lambda^{-1}\|\mu\|$, *then* μ *and* ν *are concentrated at a single point.*

This is a variant of the concentration-compactness lemma [15]. Below we shall show that μ and μ_∞ are positive measures. Assuming this, the proof of Lemma 2.1 is exactly the same as that of Lemma 1.40 in [15]. We note in particular that the expressions for μ_∞ and ν_∞ employed in the proof are those given by (2.1) and (2.2).

REMARK 2.2. It follows from (2.4) that μ_∞ is independent of the particular choice of the functions ψ_R satisfying the required properties. As we have mentioned above, it is not clear whether the limit in (2.1) exists as $R \to \infty$. Therefore when adapting the proof of Lemma 1.40 in [15] to our case, we need to replace this limit with either $\limsup_{R\to\infty}$ or $\liminf_{R\to\infty}$. Since we obtain the same equality (2.4) in both cases, these limits must be equal and μ_∞ is well defined.

LEMMA 2.3. *The measures* μ *and* μ_∞ *are positive.*

PROOF. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$, $\varphi \ge 0$, and put $\varphi_\varepsilon := \sqrt{\varphi + \varepsilon^2} - \varepsilon$, $\varepsilon > 0$. Since $u_n - u \rightharpoonup 0$ in $L^2_{loc}(\mathbb{R}^N)$ and $\varphi_\varepsilon \in C_0^1(\mathbb{R}^N)$, we have

$$\begin{aligned} 0 &\le \lim_{n\to\infty}\int_{\mathbb{R}^N}\left(|\nabla(\varphi_\varepsilon(u_n-u))|^2 - \lambda\,\frac{(\varphi_\varepsilon(u_n-u))^2}{|y|^2}\right)dx \\ &= \lim_{n\to\infty}\int_{\mathbb{R}^N}\left(|\nabla(u_n-u)|^2 - \lambda\,\frac{(u_n-u)^2}{|y|^2}\right)\varphi_\varepsilon^2\,dx \to \langle \mu, \varphi_\varepsilon^2\rangle. \end{aligned}$$

Since $\varphi_\varepsilon^2 \to \varphi$ in $L^\infty(\mathbb{R}^N)$ as $\varepsilon \to 0$, $\langle\mu,\varphi\rangle \ge 0$ and therefore $\mu \ge 0$.

Let ψ_R be as in the definition of μ_∞. Then

$$\begin{aligned} 0 \le \int_{\mathbb{R}^N}\left(|\nabla(\psi_R u_n)|^2 - \lambda\,\frac{(\psi_R u_n)^2}{|y|^2}\right)dx &= \int_{\mathbb{R}^N}\left(|\nabla u_n|^2 - \lambda\,\frac{u_n^2}{|y|^2}\right)\psi_R^2\,dx \\ &+ 2\int_{\mathbb{R}^N} u_n\psi_R\nabla u_n\cdot\nabla\psi_R\,dx + \int_{\mathbb{R}^N} u_n^2|\nabla\psi_R|^2\,dx. \end{aligned}$$

By Hölder's inequality and since $\|\nabla u_n\|_2 \le c$ for some $c > 0$,

$$\int_{\mathbb{R}^N}|u_n\psi_R\nabla u_n\cdot\nabla\psi_R|\,dx \le c\|u_n\nabla\psi_R\|_2 \to c\|u\nabla\psi_R\|_2 \quad \text{as } n\to\infty.$$

Letting $R \to \infty$ we see that the right-hand side above tends to 0. Similarly,

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N} u_n^2|\nabla\psi_R|^2\,dx = 0,$$

and it follows that $\mu_\infty \ge 0$. □

PROOF OF THEOREM 1.2. If $\lambda < 0$, then it is clear that $S \le \widehat{S}_\lambda$. Let

$$U_\varepsilon(x) = (N(N-2))^{(N-2)/4}\left(\frac{\varepsilon}{\varepsilon^2+|x|^2}\right)^{(N-2)/2},$$

choose $\widetilde{x} = (\widetilde{y}, \widetilde{z})$ with $\widetilde{y} \neq 0$ and let $\varphi \in C_0^\infty(\mathbb{R}^N, [0,1])$ be a function such that $\varphi(x) = 1$ in a neighbourhood of $\widetilde{x}$ and $\operatorname{supp}\varphi \subset B(\widetilde{x}, r)$ for some $r < |\widetilde{y}|$ ($B(\widetilde{x}, r)$ is the open ball centered at $\widetilde{x}$ and having radius r). Then, setting $u_\varepsilon(x) := \varphi(x) U_\varepsilon(x - \widetilde{x})$, we see by an easy calculation that for a suitable $C > 0$,

$$\int_{\mathbb{R}^N} \frac{u_\varepsilon^2}{|y|^2}\, dx \le C \int_{B(\widetilde{x}, r)} U_\varepsilon^2(x - \widetilde{x})\, dx \to 0 \quad \text{as } \varepsilon \to 0.$$

Hence, using the estimates on p. 35 in [15],

$$S \le \frac{\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2\, dx}{\|u_\varepsilon\|_{2^*}^2} \le \frac{\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2\, dx - \lambda \int_{\mathbb{R}^N} (u_\varepsilon^2/|y|^2)\, dx}{\|u_\varepsilon\|_{2^*}^2} = \frac{S^{N/2} + o(1)}{S^{(N-2)/2} + o(1)} \to S$$

as $\varepsilon \to 0$, and it follows that $\widehat{S}_\lambda = S$. If u is a minimizer for (1.7) and $\|u\|_{2^*} = 1$, then

$$S = \widehat{S}_\lambda = \int_{\mathbb{R}^N} |\nabla u|^2\, dx - \lambda \int_{\mathbb{R}^N} \frac{u^2}{|y|^2}\, dx > \int_{\mathbb{R}^N} |\nabla u|^2\, dx \ge S,$$

a contradiction.

Suppose now $0 < \lambda < \overline{\lambda}$. Since

$$\int_{\mathbb{R}^N} \left(|\nabla U_\varepsilon|^2 - \lambda \frac{U_\varepsilon^2}{|y|^2} \right) dx < \|\nabla U_\varepsilon\|_2^2 = S \|U_\varepsilon\|_{2^*}^2,$$

$\widehat{S}_\lambda < S$ and it remains to show that $\widehat{S}_\lambda$ is attained. We modify the argument of Theorem 1.41 in [15].

Let (u_n) be a minimizing sequence for (1.7) such that $\|u_n\|_{2^*} = 1$ and let

$$Q_n(r) := \sup_{\widetilde{x} = (0, \widetilde{z})} \int_{B(\widetilde{x}, r)} |u_n|^{2^*}\, dx$$

(this is a variant of Lévy's concentration function). It is clear that $Q_n(r) \to 0$ as $r \to 0$ and $Q_n(r) \to 1$ as $r \to \infty$ (n fixed), hence $Q_n(r_n) = 1/2$ for some r_n. Moreover, since $\int_{B(\widetilde{x}, r)} |u_n|^{2^*}\, dx \to 0$ as $|\widetilde{x}| = |\widetilde{z}| \to \infty$ (n and r fixed), $Q_n(r_n)$ is attained at some $\widetilde{x}_n = (0, \widetilde{z}_n)$. It follows that setting

$$v_n(x) := r_n^{(N-2)/2} u_n(r_n x + \widetilde{x}_n), \tag{2.5}$$

we obtain

$$\int_{B(0,1)} |v_n|^{2^*} dx = \sup_{\widetilde{x} = (0, \widetilde{z})} \int_{B(\widetilde{x}, 1)} |v_n|^{2^*}\, dx = \frac{1}{2}. \tag{2.6}$$

Since

$$\int_{\mathbb{R}^N} \left(|\nabla v_n|^2 - \lambda \frac{v_n^2}{|y|^2} \right) dx = \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 - \lambda \frac{u_n^2}{|y|^2} \right) dx$$

and

$$\|v_n\|_{2^*} = \|u_n\|_{2^*} = 1,$$

(v_n) is a minimizing sequence for (1.7). In particular, it is bounded, hence $v_n \rightharpoonup v$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, $v_n \to v$ almost everywhere and (2.3) holds for v_n, v and some μ, ν after passing to a subsequence. As

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 - \lambda \frac{v_n^2}{|y|^2}\right) dx = \widehat{S}_\lambda = \widehat{S}_\lambda \lim_{n\to\infty} \|v_n\|_{2^*}^2,$$

it follows using Lemma 2.1 and the definition of $\widehat{S}_\lambda$ that

$$\begin{aligned}(2.7)\quad \int_{\mathbb{R}^N}\left(|\nabla v|^2 - \lambda \frac{v^2}{|y|^2}\right) dx + \|\mu\| + \mu_\infty &= \widehat{S}_\lambda(\|v\|_{2^*}^{2^*} + \|\nu\| + \nu_\infty)^{2/2^*} \\ \le \widehat{S}_\lambda(\|v\|_{2^*}^2 + \|\nu\|^{2/2^*} + \nu_\infty^{2/2^*}) &\le \int_{\mathbb{R}^N}\left(|\nabla v|^2 - \lambda \frac{v^2}{|y|^2}\right) dx + \|\mu\| + \mu_\infty.\end{aligned}$$

Hence

$$1 = (\|v\|_{2^*}^{2^*} + \|\nu\| + \nu_\infty)^{2/2^*} = \|v\|_{2^*}^2 + \|\nu\|^{2/2^*} + \nu_\infty^{2/2^*},$$

so exactly one of $\|v\|_{2^*}$, $\|\nu\|$, ν_∞ is 1 and the other two are 0. Since ν_∞ cannot be 1 according to (2.6), it must be 0. If $v = 0$, then $\|\mu\| = \widehat{S}_\lambda \|\nu\|^{2/2^*}$ as follows from (2.7), and μ, ν are concentrated at a single point $\widetilde{x}$. If $\widetilde{x} = (0, \widetilde{z})$, then, employing (2.6),

$$(2.8)\qquad \frac{1}{2} = \int_{B(0,1)} |v_n|^{2^*} dx \ge \int_{B(\widetilde{x},1)} |v_n|^{2^*} dx \to \|\nu\| = 1,$$

a contradiction. Suppose $\widetilde{x} = (\widetilde{y}, \widetilde{z})$, $\widetilde{y} \ne 0$, and let $\varphi \in C_0^\infty(\mathbb{R}^N, [0,1])$ be such that $\varphi(x) = 1$ in a neighbourhood of $\widetilde{x}$ and $\operatorname{supp}\varphi \subset B(\widetilde{x}, r)$, $r < |\widetilde{y}|$. Since $\mu_\infty = 0$ and μ concentrates at $\widetilde{x}$, we have

$$(2.9)\qquad \lim_{n\to\infty} \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 - \lambda \frac{v_n^2}{|y|^2}\right)(1 - \varphi^2)\, dx = 0.$$

Moreover, $\int_{\mathbb{R}^N} (v_n^2/|y|^2)\varphi^2\, dx \to 0$ because $v_n \to 0$ in $L^2_{\rm loc}(\mathbb{R}^N)$ and y is bounded away from 0 on $\operatorname{supp}\varphi$. Since also ν concentrates at $\widetilde{x}$, it follows using (2.9) that

$$\begin{aligned}(2.10)\qquad \widehat{S}_\lambda &= \lim_{n\to\infty} \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 - \lambda \frac{v_n^2}{|y|^2}\right)\varphi^2\, dx \\ &= \lim_{n\to\infty} \|\nabla(\varphi v_n)\|_2^2 \ge S \lim_{n\to\infty} \|\varphi v_n\|_{2^*}^2 = S,\end{aligned}$$

a contradiction again. Hence $\nu = 0$, $\|v\|_{2^*} = 1$ and

$$\widehat{S}_\lambda = \int_{\mathbb{R}^N} \left(|\nabla v|^2 - \lambda \frac{v^2}{|y|^2}\right) dx. \qquad \square$$

Proof of Theorem 1.3. That $\widehat{S}_{\lambda,\mathrm{sym}} = \widehat{S}_\lambda$ for $0 < \lambda < \overline{\lambda}$ follows immediately by the argument of Theorem 3.1 in [11]. More precisely, in this case $\widehat{S}_\lambda$ is attained at some $u \ge 0$ as follows from Theorem 1.2 and the comment at the end of the introduction. If $u^*(\,\cdot\,, z)$ denotes the Schwarz symmetrization of $u(\,\cdot\,, z)$ and $u^{**}(y,\,\cdot\,)$ the Schwarz symmetrization of $u^*(y,\,\cdot\,)$, then $u^{**} = u^{**}(|y|, |z|) \in \mathcal{D}^{1,2}_{\mathrm{sym}}(\mathbb{R}^N)$ and $\widehat{S}_\lambda$ is attained at u^{**}.

Suppose $\lambda < 0$, let $\widetilde{\mathcal{D}}^{1,2}_{\mathrm{sym}}(\mathbb{R}^N) := \{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) : u = u(|y|, z)\}$ and let $\widetilde{S}_{\lambda,\mathrm{sym}}$ be the corresponding infimum as in (1.7). Lemma 2.1 requires some modification now. Since the measures μ and ν are invariant with respect to the action of $O(k)$ given by $\widetilde{g}(y,z) := (gy, z)$, $g \in O(k)$, one sees by inspecting the proof in [15] that if $u = 0$ and $\|\nu\|^{2/2^*} = \widetilde{S}^{-1}_{\lambda,\mathrm{sym}}\|\mu\|$, then μ and ν are concentrated at a single orbit $\{(gy, z) : g \in O(k)\}$, cf. [14]. Taking this into account, the same argument as in the proof of Theorem 1.2 shows that there exists a minimizing sequence $v_n \rightharpoonup v$ such that $\nu_\infty = 0$ and if $v = 0$, then $\|\nu\| = 1$ and ν is concentrated at an orbit $\{(g\widetilde{y}, \widetilde{z}) : g \in O(k)\}$. Since ν consists of at most countably many atoms, $\widetilde{y}$ must be 0. But this leads to a contradiction as in (2.8). So $\|v\|_{2^*} = 1$ and $\widetilde{S}_{\lambda,\mathrm{sym}}$ is attained. Finally, Schwarz symmetrization shows that $\widehat{S}_{\lambda,\mathrm{sym}} = \widetilde{S}_{\lambda,\mathrm{sym}}$, and since $\widehat{S}_\lambda$ is not attained, $\widehat{S}_\lambda < \widehat{S}_{\lambda,\mathrm{sym}}$. □

3. Proof of Theorem 1.1

Now we have $x = (x_1, \ldots, x_m) \in \mathbb{R}^{mN}$ and $V(x)$ is given by (1.1). It will be convenient to introduce the following notation:

$$\widetilde{J} := \{(i,j) : 1 \le i < j \le m\},$$

$$J_p := \{J \subset \widetilde{J} : J \text{ contains } p \text{ pairs } (i,j)\}$$

and

$$V_J(x) := \sum_{(i,j)\in J} \frac{1}{|x_i - x_j|^2}.$$

We also set $J_0 := \emptyset$ and $V_J := 0$ if $J \in J_0$. Clearly, $J_{m(m-1)/2} = \widetilde{J}$ and $V_J = V$ if $J \in J_{m(m-1)/2}$. Let

$$S_{\lambda,p} := \min_{J\in J_p} \inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^{mN})\setminus\{0\}} \frac{\int_{\mathbb{R}^{mN}} (|\nabla u|^2 - \lambda V_J(x) u^2)\,dx}{\|u\|^2_{2^*}},$$

and for $\lambda < \overline{\lambda}$, a sequence $u_n \rightharpoonup u$ in $\mathcal{D}^{1,2}(\mathbb{R}^{mN})$ and $J \in J_p$, let

$$\mu_{J,\infty} := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^{mN}} (|\nabla u_n|^2 - \lambda V_J(x) u_n^2)\psi_R^2\,dx$$

and

$$\nu_{J,\infty} := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^{mN}} |u_n|^{2^*} \psi_R^2\,dx,$$

where $\psi_R \in C^\infty(\mathbb{R}^{mN}, [0,1])$ is radially symmetric, $\psi_R = 0$ for $|x| \le R$ and $\psi_R = 1$ for $|x| \ge R+1$. Inspecting the proof of Lemma 1.40 in [15] once more we obtain the following

LEMMA 3.1. *Let* $(u_n) \subset \mathcal{D}^{1,2}(\mathbb{R}^{mN})$ *be a sequence such that* $u_n \rightharpoonup u$ *in* $\mathcal{D}^{1,2}(\mathbb{R}^{mN})$, $u_n \to u$ *almost everywhere in* $\mathbb{R}^{mN}$,

$$|\nabla(u_n-u)|^2 - \lambda V_J(x)(u_n-u)^2 \rightharpoonup \mu_J \quad \text{and} \quad |u_n-u|^{2^*} \rightharpoonup \nu_J \quad \text{in } \mathcal{M}(\mathbb{R}^{mN}).$$

Then

$$\|\nu_J\|^{2/2^*} \le S_{\lambda,p}^{-1}\|\mu_J\|, \quad \nu_{J,\infty}^{2/2^*} \le S_{\lambda,p}^{-1}\mu_{J,\infty},$$

$$\limsup_{n\to\infty}\int_{\mathbb{R}^{mN}}(|\nabla u_n|^2 - \lambda V_J(x)u_n^2)\,dx = \int_{\mathbb{R}^{mN}}(|\nabla u|^2 - \lambda V_J(x)u^2)\,dx + \|\mu_J\| + \mu_{J,\infty}$$

and

$$\limsup_{n\to\infty}\|u_n\|_{2^*}^{2^*} = \|u\|_{2^*}^{2^*} + \|\nu_J\| + \nu_{J,\infty}.$$

Moreover, if $u=0$ *and* $\|\nu_J\|^{2/2^*} = S_{\lambda,p}^{-1}\|\mu_J\|$, *then* μ_J *and* ν_J *are concentrated at a single point.*

That $\mu_{J,\infty}$ is well defined and μ_J, $\mu_{J,\infty}$ are positive is seen as in Remark 2.2 and Lemma 2.3.

PROPOSITION 3.2. *Let* $\lambda \in (0,\overline{\lambda})$. *Then* $S_{\lambda,p} < S_{\lambda,p-1}$ *and* $S_{\lambda,p}$ *is attained for each* $p = 1,\ldots,m(m-1)/2$.

We note that $S_{\lambda,0} = S$ (and is attained) while $S_{\lambda,m(m-1)/2} = S_\lambda$. Hence the existence part of Theorem 1.1 is an immediate consequence of Proposition 3.2. The non-existence part is shown as in Theorem 1.2 except that now $\widetilde{x} = (\widetilde{x}_1,\ldots,\widetilde{x}_m)$ and r need to be chosen so that $x_i \ne x_j$ for any $i \ne j$ and $x = (x_1,\ldots,x_m) \in B(\widetilde{x},r)$.

PROOF OF PROPOSITION 3.2. We proceed by (finite) induction. Suppose it has been shown that $S_{\lambda,p-1}$ is attained. If $\overline{u}$ is a minimizer for $S_{\lambda,p-1}$, $\|\overline{u}\|_{2^*} = 1$, then

$$S_{\lambda,p-1} = \int_{\mathbb{R}^{mN}}(|\nabla\overline{u}|^2 - \lambda V_J(x)\overline{u}^2)\,dx$$

for some $J \in J_{p-1}$, hence

$$\int_{\mathbb{R}^{mN}}(|\nabla\overline{u}|^2 - \lambda V_{J^*}(x)\overline{u}^2)\,dx < S_{\lambda,p-1}$$

for all $J^* \in J_p$, $J^* \supset J$. So $S_{\lambda,p} < S_{\lambda,p-1}$ and it remains to show that $S_{\lambda,p}$ is attained. Choose $J \in J_p$ so that

$$S_{\lambda,p} = \inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^{mN})\setminus\{0\}} \frac{\int_{\mathbb{R}^{mN}}(|\nabla u|^2 - \lambda V_J(x)u^2)\,dx}{\|u\|_{2^*}^2} \tag{3.1}$$

and assume for notational convenience that the indices $1,\dots,l$ but not $l+1,\dots,m$ appear in J. Let (u_n) be a minimizing sequence for (3.1), $\|u_n\|_{2^*}=1$,

$$X := \{x=(x_1,\dots,x_m)\in\mathbb{R}^{mN} : x_1=\dots=x_l\}$$

and

$$Q_n(r) := \sup_{\widetilde{x}\in X}\int_{B(\widetilde{x},r)}|u_n|^{2^*}\,dx.$$

Define v_n as in (2.5), with N replaced by mN. Then (2.6) holds except that this time the supremum is taken over all $\widetilde{x}\in X$. Since the right-hand side of (3.1) is invariant with respect to dilations and translations by elements of X, (v_n) is a minimizing sequence for (3.1). As in the proof of Theorem 1.2, we see that $\nu_{J,\infty}=0$ and if the weak limit of (v_n) is 0, then $\|\mu_J\| = S_{\lambda,p}\|\nu_J\|_2^{2/2^*}$ and μ_J, ν_J are concentrated at a single point $\widetilde{x}$. If $\widetilde{x}\in X$, then (2.8) holds and we have a contradiction. If $\widetilde{x}\notin X$, then we may assume (for notational convenience again) that $\widetilde{x}_1\neq\widetilde{x}_2$, and we set $I := J\setminus\{(1,2)\}$. By the same argument as in (2.9) and (2.10) (with φ such that $\operatorname{supp}\varphi\cap X=\emptyset$) we see that

$$\lim_{n\to\infty}\int_{\mathbb{R}^{mN}}(|\nabla v_n|^2-\lambda V_I(x)v_n^2)(1-\varphi^2)\,dx = 0$$

and

$$\begin{aligned}S_{\lambda,p} &= \lim_{n\to\infty}\int_{\mathbb{R}^{mN}}(|\nabla v_n|^2-\lambda V_J(x)v_n^2)\varphi^2\,dx\\ &= \lim_{n\to\infty}\int_{\mathbb{R}^{mN}}(|\nabla(\varphi v_n)|^2-\lambda V_I(x)(\varphi v_n)^2)\,dx \geq S_{\lambda,p-1}\lim_{n\to\infty}\|\varphi v_n\|_{2^*}^2 = S_{\lambda,p-1},\end{aligned}$$

a contradiction. So $\|v\|_{2^*}=1$ and the conclusion follows. □

Remark 3.3. If $m\geq 4$, $N=1$ and $0<\lambda<\overline{\lambda}$, then the Hardy inequality (1.2) still holds (with $\overline{\lambda}=1/2$) for a smaller class of functions as we have already mentioned at the beginning of the introduction. In this case S_λ will be attained if $\mathcal{D}^{1,2}(\mathbb{R}^{mN})$ is replaced by $\mathcal{D}_0^{1,2}(\mathbb{R}^{mN}\setminus N_m)$, where N_m is as in (1.3). This follows by inspection of the argument of Theorem 1.1. If $N=2$, then $\overline{\lambda}=0$, cf. Remark 2.2(i) in [7]. For $\lambda<0$ there are no ground states if $m\geq 3$, $N=1$ or $m\geq 2$, $N=2$. The proof is the same as for $m\geq 2$, $N\geq 3$.

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