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COMPARISON RESULTS AND EXISTENCE OF BOUNDED SOLUTIONS TO STRONGLY NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract. We investigate the existence of bounded solutions on the whole real line of the following strongly non-linear non-autonomous differential equation

$$(a(x(t))x'(t))' = f(t, x(t), x'(t)) \quad \text{a.e. } t \in \mathbb{R} \tag{E}$$

where $a(x)$ is a generic continuous positive function, f is a Carathéodory right-hand side.

We get existence results by combining the upper and lower-solutions method to fixed-point techniques. We also provide operative comparison criteria ensuring the well-ordering of pairs of upper and lower-solutions.

1. Introduction

The study of differential equations governed by general nonlinear differential operators has been a great develop in the last years, due to various applications to physics, engineering and other fields.

Probably the most known differential operator is the p-Laplacian, to which a lot of paper have been devoted. Several extensions of the p-Laplacian have been studied; in general equations of the type $(\Phi(u'))' = f(t, u, u')$ have been

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investigated, where Φ is a monotone operator (see, e.g. [1], [4]–[9], [12], [13], [15], [19], [20]).

Different types of nonlinear differential operators have been introduced, like $(A(u))''$, where A is a C^1 strictly monotone increasing function, vanishing at 0. This kind of differential operator appears in various relevant autonomous equations, such as reaction-diffusion equations with non-constant diffusion or the porous media equation. Moreover, non-autonomous differential equations governed by such a type of operator model various phenomena, such as the semiconductor fabrication ([10]), infiltration of water from reservoirs ([16]) and the diffusion of a dopant through a semiconductor (see [17]).

Motivated by these several applications, the study of the differential equations

$$(A(u(t))'' = f(t, u(t), u'(t)) \tag{E}$$

with various boundary conditions had a great impulse in recent years. Equations with right-hand side independent on u' have been investigated by Cabada and others ([2], [3]), moreover Papageorgiou and others carried out investigations for equation (E) in compact intervals (see [11], [18]).

Recently, we proved results on the existence or non-existence of heteroclinic solutions for equation (E) on the whole real line (see [14]) using fixed-point techniques combined with the method of upper and lower-solutions.

The aim of this paper is to investigate the existence of bounded solutions (not necessarily heteroclinic) for equation (E), which can be re-written as

$$(a(u)u')' = f(t, u, u') \tag{E}$$

where $f\colon \mathbb{R}^3 \to \mathbb{R}$ is a Carathéodory function and $a\colon \mathbb{R} \to \mathbb{R}$ is a positive, continuous function.

Our first goal is to prove, by using a fixed-point theorem, existence results under the assumption of the existence of a well-ordered pair of upper and lower-solutions (see Theorems 3.2 and 3.3 in Section 3).

When dealing with autonomous problem it is rather usual to take constant functions as upper and lower solutions, but this is not possible in general for non-autonomous equations. Moreover, when it is required the attainment of boundary conditions then one needs to force the asymptotic behavior of the solution by using non-constant upper and lower-solutions. In these situations, it is not simple to find upper and lower-solutions and even if one can asserts their existence, in general their analytic expression is unknown. Hence, even if one gets the existence of a pair of upper and lower-solutions, it is not trivial to establish if they are *well-ordered*.

Actually, in literature various comparison results are available in compact intervals, for differential equations with usual linear differential operator, but no result seems known for equations governed by nonlinear differential operator like that in (E), and very few results have been proved for usual differential equations in non-compact intervals.

To this aim, we devote Section 4 to establish comparison results stating sufficient conditions which guarantee that a pair of upper and lower-solutions is well-ordered (see Theorems 4.1, 4.4–4.6 and Corollary 4.3).

Finally, in Section 5 we combine the above ingredients and present some concrete solvability results for general classes of nonlinear equations (see Corollaries 5.1 and 5.3).

2. Auxiliary results

In this section, for the convenience of the reader, we present in details some results that we will use in the next sections. We start by considering a two points problem for a functional differential equation in a compact interval.

Let $I = [a, b] \subset \mathbb{R}$ be a compact interval and let $A: C^1(I) \to C(I)$, $x \mapsto A_x$, and $F: C^1(I) \to L^1(I)$, $x \mapsto F_x$, be two continuous functionals. Let us consider the following functional boundary value problem on $[a, b]$

(P) $$\begin{cases} (A_u(t)u'(t))' = F_u(t) & \text{a.e. on } I, \\ u(a) = \nu_1, \ u(b) = \nu_2, \end{cases}$$

where $\nu_1, \nu_2 \in \mathbb{R}$ are given.

Assume the following hypotheses on the functionals A and F:

(F1) there exists $m, M > 0$ such that $m \le A_x(t) \le M$ for every $x \in C^1(I)$, $t \in I$;

(F2) A maps bounded sets of $C^1(I)$ into uniformly continuous sets in $C(I)$, i.e. for every bounded set $D \subset C^1(I)$ and every $\varepsilon > 0$ there exists a real $\rho = \rho(\varepsilon) > 0$ such that $|A_x(t_1) - A_x(t_2)| < \varepsilon$ for every $x \in D$ and $t_1, t_2 \in I$ with $|t_1 - t_2| < \rho$;

(F3) there exists $\eta \in L^1_+(I)$ such that $|F_x(t)| \le \eta(t)$, almost everywhere on I, for every $x \in C^1(I)$.

In [14] we proved the following existence result for problem (P).

Theorem 2.1. *Under the assumptions* (F1)–(F3), *for every* $\nu_1, \nu_2 \in \mathbb{R}$ *there exists a function* $u \in C^1(I)$ *such that* $A_u \cdot u' \in W^{1,1}(I)$ *and*

(P) $$\begin{cases} (A_u(t)u'(t))' = F_u(t) & \text{a.e. on } I, \\ u(a) = \nu_1, \ u(b) = \nu_2, \end{cases}$$

i.e. u is a solution of problem (P).

Let us now consider the equation

$$(a(x(t))x'(t))' = f(t, x(t), x'(t)) \quad \text{a.e. } t \in \mathbb{R}, \tag{E}$$

where $f\colon \mathbb{R}^3 \to \mathbb{R}$ is a given Carathéodory function and $a\colon \mathbb{R} \to \mathbb{R}$ is a positive continuous function.

The following result concerns the convergence of sequences of functions related, in a certain sense, to solutions of the previous equation.

For all $n \in \mathbb{N}$ let $I_n := [-n, n]$ and $u_n \in C^1(I_n)$ be such that $a(u_n)u_n' \in W^{1,1}(I_n)$ and

$$(a(u_n(t))u_n'(t))' = f(t, u_n(t), u_n'(t)) \quad \text{a.e. } t \in I_n.$$

Consider the following sequences of functions $(y_n)_n$, $(z_n)_n$, $(x_n)_n$ defined by

$$\begin{aligned}
y_n(t) &:= \begin{cases} u_n'(t) & \text{for } t \in I_n, \\ 0 & \text{for } t \notin I_n, \end{cases} \\
z_n(t) &:= \begin{cases} (a(u_n(t))u_n'(t))' & \text{for a.e. } t \in I_n, \\ 0 & \text{elsewhere in } \mathbb{R}, \end{cases} \\
x_n(t) &:= u_n(0) + \int_0^t y_n(s)\,ds.
\end{aligned}$$

Lemma 2.2. *Assume that:*

(i) *the sequences $(u_n(0))_n$ and $(u_n'(0))_n$ are bounded;*
(ii) *there exist two functions $H, \gamma \in L^1_{\text{loc}}(\mathbb{R})$ such that $|y_n(t)| \le H(t)$ and $|z_n(t)| \le \gamma(t)$ almost everywhere on $\mathbb{R}$ and for all $n \in \mathbb{N}$.*

Then there exist three subsequences $(y_{n_k})_k$, $(z_{n_k})_k$, $(x_{n_k})_k$ and a function $x \in C^1(\mathbb{R})$, with $a(x)x'$ differentiable almost everywhere in $\mathbb{R}$ and $(a(x)x')' \in L^1_{\text{loc}}(\mathbb{R})$, such that

(a) *$x_{n_k} \to x$ uniformly on compact sets of $\mathbb{R}$;*
(b) *$y_{n_k} \to x'$ pointwise on $\mathbb{R}$ and in the norm of L^1 on the compact subsets of $\mathbb{R}$;*
(c) *$z_{n_k} \rightharpoonup (a(x)x')'$ weakly in L^1 on the compact subsets of $\mathbb{R}$;*
(d) *$(a(x(t))x'(t))' = f(t, x(t), x'(t))$ almost everywhere on $\mathbb{R}$.*

Proof. By applying Lemma 1 of [14] on the intervals I_n, $n = 1, 2, \ldots$, and using the diagonal process we obtain the assertion. $\square$

Remark 2.3. If there exists $L > 0$ such that $u_n'(t) \ge 0$ for every $|t| > L$ and $n \in \mathbb{N}$, then x is definitively increasing, since $x'(t) \ge 0$ for every $|t| > L$.

Remark 2.4. If there exist $\alpha, \beta \in C(\mathbb{R})$ such that $\alpha(t) \le u_n(t) \le \beta(t)$ for every $t \in I_n, n \in \mathbb{N}$, then $\alpha(t) \le x(t) \le \beta(t)$ for every $t \in \mathbb{R}$.

3. Existence theorems

In this section we investigate the existence of bounded solutions for the equation (E), under very mild conditions on the right-hand side. Our approach is based on fixed-point techniques suitably combined to the method of upper and lower solutions, according to the following definition.

Definition 3.1. A lower (upper) solution for equation (E) is a bounded function $\alpha \in C^1(\mathbb{R})$ such that $a(\alpha)\alpha' \in W^{1,1}(\mathbb{R})$ and

$$(a(\alpha(t))\alpha'(t))' \geq (\leq) f(t,\alpha(t),\alpha'(t)), \quad \text{for a.e. } t \in \mathbb{R}.$$

Throughout this section we will assume the existence of an ordered pair of lower and upper solutions α, β, i.e. satisfying $\alpha(t) \leq \beta(t)$ for every $t \in \mathbb{R}$, and we will adopt the following notations:

$$\mathcal{I} := [\inf_{t\in\mathbb{R}} \alpha(t), \sup_{t\in\mathbb{R}} \beta(t)], \qquad \nu := |\mathcal{I}| = \sup_{t\in\mathbb{R}} \beta(t) - \inf_{t\in\mathbb{R}} \alpha(t)$$

$$m := \min_{x\in\mathcal{I}} a(x) > 0, \qquad M := \max_{x\in\mathcal{I}} a(x),$$

$$d := \max\{|\alpha'(t)| + |\beta'(t)| : t \in \mathbb{R}\}.$$

Note that the value d is well-defined, in fact $\lim_{|t|\to\infty} \alpha'(t) = \lim_{|t|\to\infty} \beta'(t) = 0$, since $a(\alpha)\alpha' \in W^{1,1}(\mathbb{R})$ and $m > 0$ (the same argument holds for β').

Moreover, in what follows, x^+ and x^- will denote the positive and negative part of the real number x, respectively, and we will put $x \wedge y := \min\{x, y\}$, $x \vee y := \max\{x, y\}$.

The first theorem furnishes a sufficient condition for the existence of bounded solutions to equation (E).

Theorem 3.2. *Let the following assumptions hold:*

(H1) *there exist a pair of lower and upper solutions $\alpha, \beta \in C^1(\mathbb{R})$ of the equation* (E)*, satisfying $\alpha(t) \leq \beta(t)$, for every $t \in \mathbb{R}$;*

(H2) *for every $s > 0$ there exists a function $\eta_s \in L^1_{\text{loc}}(\mathbb{R})$ such that*

$$|f(t,x,y)| \leq \eta_s(t) \quad \textit{for a.e. } t \in \mathbb{R}, \ \textit{every } x \in \mathcal{I} \textit{ and } |y| \leq s; \tag{3.1}$$

(H3) *there exist a constant $H > \nu/2$, a continuous function $\theta: \mathbb{R}^+ \to \mathbb{R}^+$ and a function $\lambda \in L^p(\mathbb{R})$ with $1 \leq p \leq \infty$, such that*

$$\int^{\infty} \frac{r^{1-1/p}}{\theta(r)}\, dr = +\infty \tag{3.2}$$

with the position $1/\infty = 0$ and

$$|f(t,x,y)| \leq \lambda(t)\theta(a(x)|y|) \quad \textit{for a.e. } t \in \mathbb{R}, \ \textit{every } x \in \mathcal{I}, \ \textit{every } |y| \geq H. \tag{3.3}$$

Then, there exists a function $x \in C^1(\mathbb{R})$, with $a(x)x' \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$, such that

$$\begin{cases} (a(x(t))x'(t))'f(t,x(t),x'(t)) & \text{a.e. } t \in \mathbb{R}, \\ \alpha(t) \le x(t) \le \beta(t) & \text{for every } t \in \mathbb{R}. \end{cases}$$

Finally, if there exists $\Lambda \in \mathbb{R}$ such that α is increasing in $(-\infty,-\Lambda)$ (β is increasing in (Λ,∞)) and $f(-t,x,y) \ge 0$ ($f(t,x,y) \le 0$) for almost every $t \ge \Lambda$, every $x \in \mathcal{I}$ and $y \le 0$, then the solution x is increasing in $(-\infty,-\Lambda)$ (increasing in (Λ,∞)).

Proof. By (3.2), there exists a constant $C > (M/m)H \ge H$ such that

$$\int_{MH}^{mC} \frac{r^{1-1/p}}{\theta(r)}\,dr > (M\nu)^{1-1/p}\|\lambda\|_p. \tag{3.4}$$

Let us fix an integer $n \in \mathbb{N}$. Let $T\colon W^{1,1}(I_n) \to W^{1,1}(I_n)$ denote the truncating operator defined by

$$T(x) := T_x \quad \text{where } T_x(t) := [\beta(t) \wedge x(t)] \vee \alpha(t).$$

Of course, T is well-defined and $T_x'(t) = x'(t)$ for almost all $t \in I_n$ such that $\alpha(t) < x(t) < \beta(t)$, whereas $T_x'(t) = \alpha'(t)$ for almost every t such that $x(t) \le \alpha(t)$, $T_x'(t) = \beta'(t)$ for almost every t such that $x(t) \ge \beta(t)$.

Set $Q_x(t) := -\overline{C} \vee [T_x'(t) \wedge \overline{C}]$, where $\overline{C} = (M/m)C + d$. Finally, let $u\colon \mathbb{R}^2 \to \mathbb{R}$ denote the penalty function defined by

$$u(t,x) := [x - \beta(t)]^+ - [x - \alpha(t)]^-.$$

Let us consider the following auxiliary boundary value problem on the compact interval I_n:

$$(\mathrm{P}_n^*) \quad \begin{cases} (a(T_x(t))x'(t))' = f(t,T_x(t),Q_x(t)) + \arctan(u(t,x(t)) & \text{a.e. } t \in I_n, \\ x(-n) = \alpha(-n), \ x(n) = \beta(n). \end{cases}$$

Step 1. Let us now prove that if $x \in C^1(I_n)$ is a solution of problem (P_n^*), then $\alpha(t) \le x(t) \le \beta(t)$ for all $t \in I_n$, hence $T_x(t) \equiv x(t)$ and $u(t,x(t)) \equiv 0$.

First we show that $\alpha(t) \le x(t)$ for every $t \in I_n$. If t_0 is such that $x(t_0) - \alpha(t_0) := \min(x(t) - \alpha(t)) < 0$, then t_0 belongs to a compact interval $[t_1,t_2] \subset I_n$ satisfying $x(t_1) - \alpha(t_1) = x(t_2) - \alpha(t_2) = 0$ and $x(t) - \alpha(t) < 0$ for every $t \in (t_1,t_2)$. Hence, $T_x(t) \equiv \alpha(t)$ and $Q_x(t) \equiv \alpha'(t)$ in $[t_1,t_2]$, then for almost every $t \in (t_1,t_2)$ we have

$$(a(\alpha(t))x'(t))' = f(t,\alpha(t),\alpha'(t)) + \arctan(x(t) - \alpha(t)) < (a(\alpha(t))\alpha'(t))'.$$

Thus, the function $a(\alpha(t))(x'(t) - \alpha'(t))$ is strictly decreasing in (t_1,t_2), so we have $a(\alpha(t))(x'(t) - \alpha'(t)) < a(\alpha(t_0))(x'(t_0) - \alpha'(t_0)) = 0$ for $t \in (t_0,t_2)$, then also $x'(t) - \alpha'(t) < 0$ in (t_0,t_2), a contradiction. Similarly one can show that $x(t) \le \beta(t)$ for every $t \in I_n$.

Step 2. Now we prove that if $x \in C^1(I_n)$ is a solution of problem (P_n^*), then $|x'(t)| \le (M/m)C \le \overline{C}$ for every $t \in I_n$.

Since $x \in C^1(I_n)$ and $x(I_n) \subset \mathcal{I}$, we can apply Lagrange Theorem to deduce that for some $\tau_0 \in I_n$ we have

$$|x'(\tau_0)| = \frac{1}{2n}|x(n) - x(-n)| \le \frac{\sup\beta - \inf\alpha}{2n} < H < C.$$

Assume now, by contradiction, the existence of an interval $J = (\tau_1, \tau_2) \subset I_n$, such that $H < |x'(t)| < C$ in J and $|x'(\tau_1)| = H, |x'(\tau_2)| = C$ or viceversa. Of course, $x'(t)$ keeps constant sign in J; assume now $x'(t) > 0$ in J (the proof will proceed similarly if $x'(t) < 0$).

Since $x'(t) < C$ for every $t \in J$, by the definition of (P_n^*) and assumption (3.3), for almost every $t \in J$ it results

$$|(a(x(t))x'(t))'| = |(a(T_x(t))x'(t))'| = |f(t, x(t), x'(t))| \le \lambda(t)\theta(a(x(t))x'(t)).$$

Therefore, by Hölder inequality, if q is the conjugate exponent of p, we deduce

$$\begin{aligned}\int_{MH}^{mC} \frac{r^{1/q}}{\theta(r)}\,dr &\le \int_{\tau_1}^{\tau_2} \frac{(a(x(t))x'(t))^{1/q}}{\theta(a(x(t))x'(t))}|(a(x(t))x'(t))'|\,dt \\ &\le \int_{\tau_1}^{\tau_2} (a(x(t))x'(t))^{1/q}\lambda(t)\,dt \le \left(\int_{\tau_1}^{\tau_2} (a(x(t))x'(t))\,dt\right)^{1/q} \|\lambda\|_p \\ &\le M^{1/q}\|\lambda\|_p \left(\int_{\tau_1}^{\tau_2} x'(t)\,dt\right)^{1/q} \le (M\nu)^{1/q}\|\lambda\|_p\end{aligned}$$

in contradiction with (3.4). Thus, we get $|x'(t)| < C \le (M/m)C$ for every $t \in I_n$.

Step 3. Let us now prove that problem (P_n^*) admits solutions for every $n \in \mathbb{N}$. To this aim, let $A\colon C^1(I_n) \to C(I_n)$, $x \mapsto A_x$, and $F\colon C^1(I_n) \to L^1(I_n)$, $x \mapsto F_x$, be the functionals defined by

$$A_x(t) := a(T_x(t)), \quad F_x(t) := f(t, T_x(t), Q_x(t)) + \arctan(u(t, x(t))).$$

As it is easy to check, by (3.1) the functionals are well-defined. Moreover, if D is a bounded subset of $C^1(I_n)$, i.e. there exists $S > 0$ such that $\|x\|_{C^1(I)} \le S$, then fixed $\varepsilon > 0$, by the uniform continuity of $a(\,\cdot\,)$ in $\mathcal{I}$, there exists $\delta = \delta(\varepsilon) > 0$ such that $|a(\xi_1) - a(\xi_2)| < \varepsilon$ whenever $|\xi_1 - \xi_2| < \delta$. Therefore, put $\rho = \delta/S$, if $|t_1 - t_2| < \rho$ we have

$$|T_x(t_1) - T_x(t_2)| \le |x(t_2) - x(t_1)| \le \left|\int_{t_1}^{t_2} |x'(\tau)|\,d\tau\right| \le S|t_1 - t_2| < \delta$$

for every $x \in D$ and consequently $|A_x(t_1) - A_x(t_2)| < \varepsilon$ for every $x \in D$, whenever $|t_1 - t_2| < \rho$.

Therefore, the functionals A and F satisfy the hypotheses (F1)–(F3) of Theorem 2.1. So, by applying such a result with $\nu_1 = \alpha(-n)$ and $\nu_2 = \beta(n)$, we obtain the existence of a function $u_n \in C^1(I_n)$ such that $a(u_n)u_n' \in W^{1,1}(I_n)$ which is a solution of the problem (P_n^*). Moreover, taking account of the properties proved in Steps 1 and 2, we infer that, for every $n \in \mathbb{N}$,

$$(a(u_n(t))u_n'(t))' = f(t, u_n(t), u_n'(t)) \quad \text{a.e. } t \in I_n.$$

Observe now that the sequence of solutions $(u_n)_n$ satisfies all the assumptions of Lemma 2.2, with $H(t) = \overline{C}$ and $\gamma(t) = \eta_{\overline{C}}(t)$, for $t \in \mathbb{R}$. So, by assertion (d) of such a lemma, we deduce the existence of a solution x of equation (E).

Step 4. Finally, if we assume that α is increasing in $(-\infty, -\Lambda)$ and $f(-t, x, y) \geq 0$ for almost every $t \geq \Lambda$, every $x \in \mathcal{I}$ and $y \leq 0$, we have that if $u_n \in C^1(I_n)$ is a solution of problem (P_n^*) then $u_n'(t) \geq 0$, for every $t \in [-n, -\Lambda]$. In fact, assume by contradiction $u_n'(\widetilde{t}) < 0$ for some $\widetilde{t} \in [-n, -\Lambda]$. Since $u_n(-n) = \alpha(-n)$ and $\alpha(-n) \leq \alpha(t) \leq u_n(t)$ in $[-n, -\Lambda]$, put $t^* := \inf\{t \in [-n, \widetilde{t}] : u_n'(\tau) < 0 \text{ in } [t, \widetilde{t}]\}$, we have $-n < t^*$ with $u_n'(t^*) = 0$. By the assumptions we have $(a(u_n(t))u_n'(t))' \geq 0$ for almost every $t \in [t^*, \widetilde{t}]$, that is $a(u_n)u_n'$ is increasing in $[t^*, \widetilde{t}]$, with $a(u_n(t^*))u_n'(t^*) = 0$, a contradiction with the choice of $\widetilde{t}$.

Hence, the function x obtained by Lemma 2.2 is the pointwise limit in $(-\infty, \Lambda)$ of a sequence of increasing functions and this implies that also x is increasing. The same argument holds if we suppose β is increasing in $(\Lambda, +\infty)$ and $f(t, x, y) \leq 0$ for almost every $t \geq \Lambda$, every $x \in \mathcal{I}$ and $y \leq 0$. $\square$

The following result differs from the previous one since herein we require the validity of the Nagumo condition just on a compact subset of $\mathbb{R}$, but impose a sign condition on the right-hand side f, outside this set.

Theorem 3.3. *Let the assumptions* (H1) *and* (H2) *of Theorem* 3.2 *hold, with α increasing in $(-\infty, -L)$, β increasing in (L, ∞), for some constant $L \in \mathbb{R}$. Moreover, assume that*

(H4) *there exists a constant $H > \nu/(2L)$, a continuous function $\theta: \mathbb{R}^+ \to \mathbb{R}^+$ and a function $\lambda \in L^p([-L, L])$ with $1 \leq p \leq \infty$, such that* (3.2) *holds and* (3.3) *is satisfied just for $|t| \leq L$.*

Finally, suppose that

$$\begin{cases} f(t, x, y) \leq 0 \\ f(-t, x, y) \geq 0 \end{cases} \quad \text{for a.e. } t \geq L, \ \text{every } x \in \mathcal{I}, \ y \in \mathbb{R}. \tag{3.5}$$

Then, there exists a definitively increasing function $x \in C^1(\mathbb{R})$, with $a(x)x' \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$, such that

$$\begin{cases} (a(x(t))x'(t))' = f(t, x(t), x'(t)) & \text{a.e. } t \in \mathbb{R}, \\ \alpha(t) \leq x(t) \leq \beta(t) & \text{for every } t \in \mathbb{R}, \\ \alpha(-\infty) \leq x(-\infty), \quad x(+\infty) \leq \beta(+\infty). \end{cases}$$

Proof. The proof is same to that of Theorem 3.2. It differs only in Step 2, when proving that $|x'(t)| \le (M/m)C$ for every $t \in I_n$ for every solution $x \in W^{1,1}(I_n)$ of problem (P_n^*), $n \ge L$.

Indeed, here inequality (3.3) holds in $[-L, L]$. So, we obtain that $|x'(t)| \le C \le (M/m)C$ for every $t \in [-L, L]$ and from assumption (3.5) we deduce, as in the Step 4 of the proof of Theorem 3.2, that $x'(t) \ge 0$ for every $t \in I_n \setminus [-L, L]$. In order to prove that $x'(t) \le (M/m)C$ for every $t \in I_n \setminus [-L, L]$, observe that $(a(x'(t))x'(t))' = f(t, x(t), Q_x(t))$, almost everywhere on $I_n \setminus [-L, L]$ and so from (3.5) we have $(a(x'(t))x'(t))' \le 0$, almost everywhere on $[L, n]$, implying that

$$a(x'(t))x'(t) \le (a(x'(L))x'(L) \le MC, \quad \text{for every } t \in [L, n]$$

hence $x'(t) \le (M/m)C$. The same argument holds in $[-n, -L]$.

Steps 3 and 4 proceed as in the previous proof. □

Remark 3.4. In view of the proof of the previous results (see (3.4)), note that condition (3.2) can be weakened as follows

$$\int_{HM}^{\infty} \frac{r^{1-1/p}}{\theta(r)}\, dr > (M\nu)^{1-1/p} \|\lambda\|_p$$

where $\|\lambda\|_p$ is intended in $\mathbb{R}$ for Theorem 3.2, in $[-L, L]$ for Theorem 3.3.

4. Comparison-type results

The key tool of the existence results stated in the previous section is the existence of a well-ordered pair of upper and lower solutions. Usually it is rather easy to find a pair of upper and lower solutions, but unfortunately, in general they are not well-ordered. This is strictly linked to the monotonicity property of the right-hand side, the presence of the non-linear differential operator a and the unboundedness of the domain.

In this section we investigate such a matter, establishing some conditions ensuring that any pair of lower and upper-solutions of equation (E) is well-ordered.

Throughout the section $f: \mathbb{R}^3 \to \mathbb{R}$ is a Carathéodory function, $a: \mathbb{R} \to \mathbb{R}$ is a positive, continuous function and $\alpha, \beta \in C^1(\mathbb{R})$ are a lower and an upper solution of the equation (E) as in the Definition 3.1, from which we can deduce the existence, in $\mathbb{R}$, of the limits $\alpha(-\infty), \alpha(+\infty), \beta(-\infty), \beta(+\infty)$.

However, the following result are stated for a generic open interval $J = (c, d) \subseteq \mathbb{R}$, bounded or unbounded. Moreover, in what follows

$$\mathcal{I} = [\min\{\inf \alpha, \inf \beta\}, \max\{\sup \alpha, \sup \beta\}].$$

Theorem 4.1. *Let $J = (c, d) \subseteq \mathbb{R}$ be a given open interval. Suppose that there exists a function $h \in C^1(J)$ with $h'(t) \le 0$ for every $t \in J$, such that*

$$f(t, x, y_1) - f(t, x, y_2) \ge h(t)(y_1 - y_2) \tag{4.1}$$

for almost every $t \in J$, every $x \in \mathcal{I}$, $y_1, y_2 \in \mathbb{R}$. Moreover, assume that

$$x \mapsto f(t, x, y) \text{ is increasing in } \mathcal{I} \text{ for a.e. } t \in J \text{ and every } y \in \mathbb{R}; \tag{4.2}$$

and there exists $k > 0$ such that

$$|a(x_1) - a(x_2)| \le k|x_1 - x_2| \quad \text{for every } x_1, x_2 \in \mathcal{I}. \tag{4.3}$$

Then, if $\alpha(c) < \beta(c)$ and $\alpha(d) < \beta(d)$, we have $\alpha(t) \le \beta(t)$ for every $t \in J$.

Proof. Assume, by contradiction, the existence of an interval $I = [t_0, t_1] \subset J$ such that $\alpha(t_0) = \beta(t_0), \alpha(t_1) = \beta(t_1)$ and $\alpha(t) > \beta(t)$, for all $t \in I$.

Put $\rho := \max_{t \in I}(\alpha(t) - \beta(t)) > 0$, let us fix $0 < \varepsilon < \rho$ and define the continuous functions $\gamma_\varepsilon, \Gamma_\varepsilon : \mathbb{R} \to [0, +\infty)$ by

$$\gamma_\varepsilon(r) := \frac{1}{k^2}\left(\frac{1}{\varepsilon} - \frac{1}{r}\right)^+ \quad \text{and} \quad \Gamma_\varepsilon(r) := \int_\varepsilon^r \gamma_\varepsilon(s)\, ds.$$

Of course, $\gamma_\varepsilon(r) = \Gamma_\varepsilon(r) = 0$ for every $r \le \varepsilon$.

Moreover, set $T_\varepsilon := \{t \in I : \alpha(t) - \beta(t) > \varepsilon\}$, the set T_ε has positive measure and is the union of at most countably many open intervals.

From the definition of lower and upper solutions and from assumptions (4.1) and (4.2) it follows:

$$\begin{aligned}(a(\alpha(t))\alpha'(t))' - (a(\beta(t))\beta'(t))' &\ge f(t, \alpha(t), \alpha'(t)) - f(t, \beta(t), \beta'(t)) \\ &\ge f(t, \beta(t), \alpha'(t)) - f(t, \beta(t), \beta'(t)) \ge h(t)(\alpha'(t) - \beta'(t))\end{aligned}$$

almost everywhere on T_ε, so multiplying by $\gamma_\varepsilon(\alpha(t) - \beta(t)) \ge 0$ and integrating on T_ε, we have

$$\begin{aligned}&\int_{T_\varepsilon} (a(\alpha(t))\alpha'(t) - a(\beta(t))\beta'(t))'\gamma_\varepsilon(\alpha(t) - \beta(t))\, dt \\ &\qquad\qquad \ge \int_{T_\varepsilon} h(t)(\alpha'(t) - \beta'(t))\gamma_\varepsilon(\alpha(t) - \beta(t))\, dt.\end{aligned} \tag{4.4}$$

Using integration by parts in each connected component of T_ε, we obtain

$$\begin{aligned}&\int_{T_\varepsilon} (a(\alpha(t))\alpha'(t) - a(\beta(t))\beta'(t))'\gamma_\varepsilon(\alpha(t) - \beta(t))\mathrm{d}t \\ &\qquad = -\int_{T_\varepsilon} (a(\alpha(t))\alpha'(t) - a(\beta(t))\beta'(t))\gamma_\varepsilon'(\alpha(t) - \beta(t))(\alpha'(t) - \beta'(t))\, dt,\end{aligned}$$

and

$$\int_{T_\varepsilon} h(t)(\alpha'(t) - \beta'(t))\gamma_\varepsilon(\alpha(t) - \beta(t))\, dt = -\int_{T_\varepsilon} h'(t)\Gamma_\varepsilon(\alpha(t) - \beta(t))\, dt \ge 0.$$

Hence, from (4.4) we deduce that

$$\int_{T_\varepsilon} (a(\alpha(t))\alpha'(t) - a(\beta(t))\beta'(t))\gamma'_\varepsilon(\alpha(t)-\beta(t))(\alpha'(t)-\beta'(t))\,dt \le 0, \tag{4.5}$$

and so

$$\begin{aligned} &\int_{T_\varepsilon} a(\alpha(t))(\alpha'(t)-\beta'(t))\gamma'_\varepsilon(\alpha(t)-\beta(t))(\alpha'(t)-\beta'(t))\,dt \\ &\qquad + \int_{T_\varepsilon} (a(\alpha(t))-a(\beta(t)))\beta'(t)\gamma'_\varepsilon(\alpha(t)-\beta(t))(\alpha'(t)-\beta'(t))\,dt \le 0. \end{aligned} \tag{4.6}$$

We consider the two integrals separately. Firstly,

$$\begin{aligned} &\int_{T_\varepsilon} a(\alpha(t))(\alpha'(t)-\beta'(t))^2\gamma'_\varepsilon(\alpha(t)-\beta(t))\,dt \\ &\qquad \ge \frac{\widehat{m}}{k^2}\int_{T_\varepsilon}\left[\frac{\alpha'(t)-\beta'(t)}{\alpha(t)-\beta(t)}\right]^2 dt > 0, \end{aligned} \tag{4.7}$$

where $\widehat{m} = \inf_{x\in\mathcal{I}} a(x) > 0$. Thus, the second integral in (4.6) is negative and using Hölder inequality and (4.3), we deduce

$$\begin{aligned} &\left|\int_{T_\varepsilon} (a(\alpha(t))-a(\beta(t)))\beta'(t)\gamma'_\varepsilon(\alpha(t)-\beta(t))(\alpha'(t)-\beta'(t))\,dt\right| \\ &\quad \le \int_{T_\varepsilon} k|\alpha(t)-\beta(t)||\beta'(t)|\frac{1}{k^2(\alpha(t)-\beta(t))^2}|\alpha'(t)-\beta'(t)|\,dt \\ &\quad \le \frac{1}{k}\parallel \beta' \parallel_{L^2(I)}\left(\int_{T_\varepsilon}\left|\frac{\alpha'(t)-\beta'(t)}{\alpha(t)-\beta(t)}\right|^2 dt\right)^{1/2}. \end{aligned} \tag{4.8}$$

Therefore, from (4.6)–(4.8), we deduce that

$$\left(\int_{T_\varepsilon}\left[\frac{\alpha'(t)-\beta'(t)}{\alpha(t)-\beta(t)}\right]^2 dt\right)^{1/2} \le \frac{k}{\widehat{m}}\|\beta'\|_{L^2(I)}. \tag{4.9}$$

Put

$$\phi_\varepsilon(t) := \begin{cases} \displaystyle\int_\varepsilon^{\alpha(t)-\beta(t)} \frac{1}{s}\,ds & \text{for } t \in T_\varepsilon, \\ 0 & \text{for } t \notin T_\varepsilon. \end{cases}$$

The function ϕ_ε is absolutely continuous so it is differentiable for almost every $t \in \mathbb{R}$.

Notice that put $H_\varepsilon := \{t \in I : \alpha(t)-\beta(t) = \varepsilon\}$, we have that $\phi'_\varepsilon(t) = 0$ at every cluster point of H_ε where the derivative exists. So, we get $\phi'_\varepsilon(t) = 0$ for almost every $t \in H_\varepsilon$. Moreover, for every t such that $\alpha(t) < \beta(t)+\varepsilon$ we have that $\phi_\varepsilon(t)$ is locally identically null, so $\phi'_\varepsilon(t) = 0$ whenever $\alpha(t) < \beta(t)+\varepsilon$.

Summarizing, we deduce

$$\phi'_\varepsilon(t) := \begin{cases} \dfrac{\alpha'(t)-\beta'(t)}{\alpha(t)-\beta(t)} & \text{for every } t \in T_\varepsilon, \\ 0 & \text{for a.e. } t \notin T_\varepsilon, \end{cases}$$

and inequality (4.9) can be written as follows:

$$\|\phi'_\varepsilon\|_{L^2(I)} \le \frac{k}{\widehat{m}}\|\beta'\|_{L^2(I)}.$$

Hence, since $\phi_\varepsilon \in W_0^{1,2}(I)$, invoking the Poincaré inequality we infer that

$$\int_I \phi_\varepsilon^2(t)\,dt \le \widehat{C},$$

for some $\widehat{C} > 0$; but taking the limit as $\varepsilon \searrow 0$, the left-hand side diverges to $+\infty$, a contradiction. □

Remark 4.2. In view of the proof just developed, one can easily verify that if $c \in \mathbb{R}$ (respectively $d \in \mathbb{R}$), then the result holds even if $\alpha(c) \le \beta(c)$ ($\alpha(d) \le \beta(d)$).

Weak inequalities at the boundary conditions can be assumed also when the operator a is constant, as the following Corollary states.

Corollary 4.3. *Let $a(t) \equiv 1$ and assume that conditions* (4.1) *and* (4.2) *holds true. Then, if $\alpha(c) \le \beta(c)$ and $\alpha(d) \le \beta(d)$, we have $\alpha(t) \le \beta(t)$ for every $t \in J$.*

Proof. Fixed $\varepsilon > 0$, let us consider the function $\alpha_\varepsilon(t) := \alpha(t) - \varepsilon$. As it is easy to see, by assumption (4.2) also α_ε is a lower-solution and satisfies $\alpha_\varepsilon(c) < \beta(c)$, $\alpha_\varepsilon(d) < \beta(d)$. Hence, by applying Theorem 4.1 we deduce that $\alpha_\varepsilon(t) \le \beta(t)$ for every $t \in J$ and the assertion follows by the arbitrariness of $\varepsilon > 0$. □

When a is not constant, the following comparison results with non-strict inequalities at the boundary hold.

Theorem 4.4. *Under the same assumptions of Theorem* 4.1 *with $h(t) \ge 0$ for every $t \in J$, suppose in addition that the operator a is decreasing in $\mathcal{I}$ and at least one of the functions α and β be increasing in J. Then, if $\alpha(c) < \beta(c)$ and $\alpha(d) \le \beta(d)$, we have $\alpha(t) \le \beta(t)$, for every $t \in J$.*

Proof. Put $\eta(t) := \alpha(t) - \beta(t)$, assume, by contradiction, $\eta(\bar{t}) > 0$ for some $\bar{t} \in J$. Set

$$\tau_0 := \inf\{t : \eta(s) > 0 \text{ in } [t,\bar{t}]\}, \quad \tau_1 := \sup\{t : \eta(s) > 0 \text{ in } [\bar{t},t]\} \le +\infty.$$

By the boundary conditions, we have $\tau_0 > c$ and $\eta(\tau_0) = 0$. Let $t_1 \in (\tau_0, \tau_1)$ be such that $\eta'(t_1) = 0$.

Fixed a positive real $\varepsilon < \eta(t_1)$, let γ_ε and Γ_ε be as in Theorem 4.1. Moreover, set $T_\varepsilon := \{t \in [\tau_0, t_1) : \eta(t) > \varepsilon\}$, of course T_ε is an open set such that

$$\sup T_\varepsilon = t_1 \quad \text{and} \quad \eta(t) = \varepsilon \quad \text{for every } t \in \mathrm{Bd}(T_\varepsilon) \setminus \{t_1\} \tag{4.10}$$

since $\eta(\tau_0) = 0$ and $\eta(t_1) > \varepsilon$.

From the definition of lower and upper solutions and assumptions (4.1) and (4.2), we get

$$(a(\alpha(t))\alpha'(t))' - (a(\beta(t))\beta'(t))' \geq h(t)\eta'(t) \quad \text{a.e. on } [\tau_0, t_1].$$

Let us multiply now by $\gamma_\varepsilon(\eta(t)) \geq 0$ and integrate on T_ε. Recalling that $h(t) \geq 0$, $h'(t) \leq 0$, by (4.10), the monotonicity assumptions on a and α, β, and the choice of t_1, we have

$$\begin{aligned}
&\int_{T_\varepsilon} (a(\alpha(t))\alpha'(t) - a(\beta(t))\beta'(t))\gamma'_\varepsilon(\eta(t))\eta'(t)\,dt \\
&\leq \int_{T_\varepsilon} (a(\alpha(t))\alpha'(t) - a(\beta(t))\beta'(t))\gamma'_\varepsilon(\eta(t))\eta'(t)\mathrm{d}t \\
&\quad - (a(\alpha(t_1))\alpha'(t_1) - a(\beta(t_1))\beta'(t_1))\gamma_\varepsilon(\eta(t_1)) \\
&= - \int_{T_\varepsilon} (a(\alpha(t))\alpha'(t) - a(\beta(t))\beta'(t))'\gamma_\varepsilon(\eta(t))\,dt \\
&\leq - \int_{T_\varepsilon} h(t)\eta'(t)\gamma_\varepsilon(\eta(t)) = -h(t_1)\Gamma_\varepsilon(\eta(t_1)) + \int_{T_\varepsilon} h'(t)\Gamma_\varepsilon(\eta(t))\,dt \leq 0,
\end{aligned}$$

that is (4.5). Now proceeding as in Theorem 4.1 we obtain that

$$\left(\int_{T_\varepsilon} \left[\frac{\eta'(t)}{\eta(t)}\right]^2 dt\right)^{1/2} \leq \frac{k}{\widehat{m}}\|\beta'\|_{L^2([\tau_0,t_1])}.$$

By the Hölder inequality we have

$$\int_{T_\varepsilon} \left|\frac{\eta'(t)}{\eta(t)}\right| dt \leq \frac{k}{\widehat{m}}\|\beta'\|_{L^2([\tau_0,t_1])}\sqrt{|T_\varepsilon|} \leq \frac{k}{\widehat{m}}\|\beta'\|_{L^2([\tau_0,t_1])}\sqrt{t_1 - \tau_0},$$

a contradiction since the integral on the left side diverges to $+\infty$ as $\varepsilon \to 0$. $\square$

In view of the proof of the previous theorem, we can invert the monotonicity assumptions on the operator a and the functions α, β, obtaining the following result.

Theorem 4.4'. *Let a be increasing in $\mathcal{I}$ and let at least one of the functions α and β be decreasing in $\mathbb{R}$. Suppose that all the other assumptions of Theorem* 4.4 *are satisfied. Then the same assertion of Theorem* 4.4 *holds.*

When the first inequality of the boundary conditions is weak, the following two results hold, whose proof is similar.

Theorem 4.5. *Under the same assumptions of Theorem* 4.1 *with $h(t) \leq 0$ for every $t \in J$, suppose in addition that a is decreasing in $\mathcal{I}$ and at least one of the functions α and β be increasing in J. Then, if $\alpha(c) \leq \beta(c)$ and $\alpha(d) < \beta(d)$, we have $\alpha(t) \leq \beta(t)$ for every $t \in J$.*

Theorem 4.5'. *Let a be increasing in $\mathcal{I}$ and let at least one of the functions α and β be decreasing in $\mathbb{R}$. Suppose that all the other assumptions of Theorem 4.5 are satisfied. Then the same assertion of Theorem 4.5 holds.*

We conclude the section with a comparison result concerning semilinear equations (a constant) where the right-hand side $f(t,x,y)$ is not increasing in the variable x.

Theorem 4.6. *Suppose that there exist a constant $\ell \geq 0$ and a function $h \in C^1(J)$ with $h'(t) \leq -2\ell$, such that*

$$x \mapsto f(t,x,y) + \ell x \text{ is increasing in } \mathcal{I}, \text{ for a.e. } t \in J, \text{ every } y \in \mathbb{R};$$

$$f(t,x,y_1) - f(t,x,y_2) \geq h(t)(y_1 - y_2), \tag{4.11}$$

for a.e. $t \in J$, every $y_1, y_2 \in \mathbb{R}$ and every $x \in \mathcal{I}$. Then if $\alpha(c) < \beta(c)$ and $\alpha(d) < \beta(d)$, we have $\alpha(t) \leq \beta(t)$, for every $t \in J$.

Proof. We proceed again by contradiction and suppose the existence of a compact interval $[t_1,t_2]$, such that $\alpha(t_1) = \beta(t_1), \alpha(t_2) = \beta(t_2)$ and $\alpha(t) > \beta(t)$, for every $t \in [t_1,t_2]$. Using the definitions of lower and upper solutions, from our assumptions we obtain

$$\alpha''(t) - \beta''(t) + \ell(\alpha(t) - \beta(t)) \geq h(t)(\alpha'(t) - \beta'(t)) \quad \text{a.e. on } [t_1,t_2].$$

Multiplying the previous inequality for $\alpha(t) - \beta(t) > 0$, and integrating on $[t_1,t_2]$, by (4.11) we obtain

$$\begin{aligned} -\int_{t_1}^{t_2} (\alpha'(t) - \beta'(t))^2\,dt + \ell \int_{t_1}^{t_2} (\alpha(t) - \beta(t))^2\,dt \\ \geq -\int_{t_1}^{t_2} h'(t)\frac{(\alpha(t) - \beta(t))^2}{2}\,dt \geq \ell \int_{t_1}^{t_2} (\alpha(t) - \beta(t))^2\,dt. \end{aligned}$$

Hence,

$$\int_{t_1}^{t_2} (\alpha'(t) - \beta'(t))^2\,dt \leq 0,$$

implying $\alpha(t) = \beta(t)$ in $[t_1,t_2]$, a contradiction. □

Remark 4.7. We observe that, if one knows that both α and β are increasing (or decreasing) in $\mathbb{R}$, in all results of this section we can modify the conditions on $f(t,x,y)$ by assuming that they hold only for $y \geq 0$ (or for $y \leq 0$).

5. Applications

In this section we present some classes of equations to which it is possible to apply our results for proving the existence of a pair of well-ordered lower and upper solutions and consequently the existence of a bounded solution.

Of course, the simplest method for finding upper and lower-solutions consists in considering solutions of known equations. To this aim, let us firstly consider the following linear equation with constant coefficients

$$u'' - cu' - \rho u + k(t) = 0 \tag{5.1}$$

where $c \in \mathbb{R}$, $\rho > 0$ and $k \in L^1_{\mathrm{loc}}(\mathbb{R})$ is bounded and increasing.

Put

$$u(t) := \frac{1}{\alpha_2 - \alpha_1}\left[e^{\alpha_2 t}\int_t^{\infty} k(s)e^{-\alpha_2 s}\,ds + e^{\alpha_1 t}\int_{-\infty}^{t} k(s)e^{-\alpha_1 s}\,ds\right],$$

where $\alpha_1 < 0 < \alpha_2$ are the roots of the algebraic equation $x^2 - cx - \rho = 0$. Simple calculations show that $u \in C^1(\mathbb{R})$, with $u' \in W^{1,1}(\mathbb{R})$, and u is a solution to (5.1). Moreover, u is increasing, bounded, with $u(-\infty) = k(-\infty)/\rho$ and $u(+\infty) = k(+\infty)/\rho$.

Consider now the wider class of equations having linear right-hand side with non-constant coefficients:

$$(a(u)u')' = c(t)u' + \rho(t)u - k(t). \tag{5.2}$$

where $c \in L^\infty(\mathbb{R}) \cap C^1(\mathbb{R})$, $\rho, k \in L^\infty(\mathbb{R})$.

The next result concerns the existence of bounded solutions to (5.2).

Corollary 5.1. *Suppose that $c(t)$ is decreasing, $k(t)$ is non-negative and increasing and* $\inf \rho(t) > 0$. *Moreover, let a be a locally Lipschitz continuous function such that*

$$\int_0^{+\infty} a(s)\,ds = +\infty. \tag{5.3}$$

Then equation (5.2) *admits a bounded solution $u \in C^1(\mathbb{R})$, with $a(u)u' \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$. Moreover, if in addition we assume ρ decreasing, then the solution u is definitively monotone. Finally, if also the operator a is monotone* (*decreasing or increasing*), *then the solution is monotone increasing.*

Proof. Put $A(y) := \int_0^y a(s)\,ds$, of course A is a strictly increasing C^1-function and also its inverse A^{-1} is C^1 and strictly increasing. Put $\rho_1 := \inf \rho(t)$, and fixed a constant $\varepsilon > 0$, set

$$\widetilde{I} := \left[0, \frac{k(+\infty)}{\rho_1}\right], \quad \widetilde{I}_\varepsilon := \left[0, \frac{k(+\infty)+\varepsilon}{\rho_1}\right].$$

Moreover, set

$$\begin{aligned}
\rho_2 &:= \sup \rho(t), \\
\mu_1 &:= \min_{s \in \widetilde{I}_\varepsilon} a(s) > 0, & \mu_2 &:= \max_{s \in \widetilde{I}_\varepsilon} a(s), \\
\lambda_1 &:= \min_{s \in A^{-1}(\mu_2 \widetilde{I}_\varepsilon)} a(s) > 0, & \lambda_2 &:= \max_{s \in A^{-1}(\mu_2 \widetilde{I}_\varepsilon)} a(s)
\end{aligned}$$

where $\mu_2\widetilde{I}_\varepsilon := \{y : \mu_2 y \in \widetilde{I}_\varepsilon\}$, and finally let

$$d_1 := \begin{cases} \lambda_1 & \text{if } c_1 < 0, \\ \lambda_2 & \text{if } c_1 \geq 0, \end{cases} \quad \text{and} \quad d_2 := \begin{cases} \lambda_1 & \text{if } c_2 \geq 0, \\ \lambda_2 & \text{if } c_2 < 0, \end{cases}$$

where $c_1 := \inf c(t)$ and $c_2 := \sup c(t)$.

Let us consider the following linear equations with constant coefficients

$$u'' = \frac{c_2}{d_2}u' + \frac{\rho_2}{\mu_1}u - k(t), \tag{5.4}$$

$$u'' = \frac{c_1}{d_1}u' + \frac{\rho_1}{\mu_2}u - k(t) - \varepsilon. \tag{5.5}$$

By what observed at the beginning of the present section, there exists a pair of C^1 bounded, increasing functions α, β, respectively solutions to (5.4) and (5.5), with $\alpha', \beta' \in W^{1,1}(\mathbb{R})$, such that

$$\alpha(\pm\infty) = \frac{\mu_1 k(\pm\infty)}{\rho_2}, \quad \beta(\pm\infty) = \frac{\mu_2(k(\pm\infty) + \varepsilon)}{\rho_1}.$$

So, in particular $\alpha(-\infty) < \beta(-\infty)$, $\alpha(+\infty) < \beta(+\infty)$. Set

$$\widetilde{\alpha}(t) := A^{-1}(\alpha(t)) \quad \text{and} \quad \widetilde{\beta}(t) := A^{-1}(\beta(t)).$$

By (5.3) the previous definition is well-posed; the functions $\widetilde{\alpha}$ and $\widetilde{\beta}$ are bounded, increasing C^1-functions, with

$$a(\widetilde{\alpha}(t))\widetilde{\alpha}'(t) = (A(\widetilde{\alpha}(t))' = \alpha'(t), \quad a(\widetilde{\beta}(t))\widetilde{\beta}'(t) = (A(\widetilde{\beta}(t))' = \beta'(t)$$

in $W^{1,1}(\mathbb{R})$. Moreover, since $\alpha(t)/\mu_1, \beta(t)/\mu_2 \in \widetilde{I}_\varepsilon$ for every $t \in \mathbb{R}$, we have

$$A\left(\frac{\alpha(t)}{\mu_1}\right) \geq \alpha(t) \quad \text{and} \quad A\left(\frac{\beta(t)}{\mu_2}\right) \leq \beta(t),$$

that is

$$\frac{\alpha(t)}{\mu_1} \geq A^{-1}(\alpha(t)) = \widetilde{\alpha}(t), \quad \frac{\beta(t)}{\mu_2} \leq A^{-1}(\beta(t)) = \widetilde{\beta}(t) \quad \text{for every } t \in \mathbb{R}.$$

Since $\beta(t) \in \mu_2\widetilde{I}_\varepsilon$ then $\widetilde{\beta}(t) \in A^{-1}(\mu_2\widetilde{I}_\varepsilon)$ and we get

$$\begin{aligned} (a(\widetilde{\beta}(t))\widetilde{\beta}'(t))' &= (A(\widetilde{\beta}(t)))'' = \beta''(t)\frac{c_1}{d_1}\beta'(t) + \frac{\rho_1}{\mu_2}\beta(t) - k(t) - \varepsilon \\ &\leq \frac{c_1}{d_1}a(\widetilde{\beta}(t))\widetilde{\beta}'(t) + \rho_1\widetilde{\beta}(t) - k(t) \leq c_1\widetilde{\beta}'(t) + \rho_1\widetilde{\beta}(t) - k(t) \\ &\leq c(t)\widetilde{\beta}'(t) + \rho(t)\widetilde{\beta}(t) - k(t), \end{aligned}$$

for almost every $t \in \mathbb{R}$. Hence, $\widetilde{\beta}$ is an upper-solution for equation (5.2). Similarly, one can prove that $\widetilde{\alpha}$ is a lower-solution. Being

$$\begin{aligned} \widetilde{\alpha}(-\infty) &= A^{-1}(\alpha(-\infty)) < A^{-1}(\beta(-\infty)) = \widetilde{\beta}(-\infty), \\ \widetilde{\alpha}(+\infty) &= A^{-1}(\alpha(+\infty)) < A^{-1}(\beta(+\infty)) = \widetilde{\beta}(+\infty) \end{aligned}$$

we can apply Theorem 4.1 to deduce that $\widetilde{\alpha}(t) \le \widetilde{\beta}(t)$ for every $t \in \mathbb{R}$. Moreover, being

$$\mathcal{I} = [\inf \widetilde{\alpha}(t), \sup \widetilde{\beta}(t)] \left[A^{-1}\left(\frac{\mu_1}{\rho_2} k(-\infty)\right), A^{-1}\left(\frac{\mu_2}{\rho_1}(k(+\infty)+\varepsilon)\right)\right],$$

put

$$L := A^{-1}\left(\frac{\mu_2}{\rho_1}(k(+\infty)+\varepsilon)\right),$$

we have

$$|f(t,x,y)| \le |c(t)||y| + \rho_2 L + k(+\infty), \quad \text{a.e. on } \mathbb{R}, \text{ for every } x \in \mathcal{I},\ y \in \mathbb{R}.$$

So, set $\eta_s(t) := |c(t)|s + \rho_2 L + k(+\infty)$, $\theta(r) := r$ and

$$\lambda(t) := \frac{1}{m}(|c(t)| + \rho_2 L + k(+\infty)),$$

where $m = \min_{s \in \mathcal{I}} a(s)$, also conditions (H2)–(H3) of Theorem 3.2 are satisfied (with $p = +\infty$ and $H > 1$). Therefore, equation (5.2) admits a bounded solution $u \in C^1(\mathbb{R})$, with $u' \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$ such that $\widetilde{\alpha}(t) \le u(t) \le \widetilde{\beta}(t)$, for all $t \in \mathbb{R}$.

Notice now that if ρ is decreasing then no bounded interval $[t_1, t_2]$ can exist satisfying $u'(t_1) = u'(t_2) = 0$ and $u'(t) < 0$ in (t_1, t_2). Indeed, in this case we would have

$$0 \le A(u(t))''_{|t=t_2} = \rho(t_2)u(t_2) - k(t_2) < \rho(t_1)u(t_1) - k(t_1) = A(u(t))''_{|t=t_1} \le 0,$$

a contradiction. Therefore, if the solution is strictly decreasing in some interval, such a interval must be unbounded. This implies that the solution is definitively monotone (increasing or decreasing).

Finally, assume that also the operator a is monotone (increasing or decreasing). Firstly note that if u is decreasing in a right half-line, then there exists the limit $u'(+\infty) = 0$. Indeed, otherwise we would have $L := \liminf_{t\to+\infty} u'(t) < 0$ and $\limsup_{t\to+\infty} u'(t) = 0$. Choose $\ell \in (L, 0)$. If a is decreasing, we could find two divergent increasing sequences $(t_n)_n$, $(T_n)_n$, such that $t_n < T_n$, $u'(T_n) = \ell$, $u'(t_n) = \ell/2$, and $u'(t) \in (\ell, \ell/2)$ for every $t \in (t_n, T_n)$. Since u' has definitively constant sign and is summable in the right half-lines, then $T_n - t_n \to 0$ as $n \to +\infty$. Moreover, the function $t \mapsto c(t)u'(t) + \rho(t)u(t) - k(t)$ is bounded in the set $\bigcup_{n\in\mathbb{N}}[t_n, T_n]$, so there exists a positive constant H such that $|(a(u(t))u'(t))'| \le H$ for every $t \in \bigcup_{n\in\mathbb{N}}[t_n, T_n]$.

Put $m := a(\sup u)$, we get

$$\begin{aligned} &m\frac{\ell}{2} = m(u'(T_n) - u'(t_n)) \ge a(u(T_n))(u'(T_n) - u'(t_n)) \\ &\ge a(u(T_n))u'(T_n) - a(u(t_n))u'(t_n) = \int_{t_n}^{T_n} (a(u(t))u'(t))'\,dt \ge -H(T_n - t_n), \end{aligned}$$

a contradiction since $\ell < 0$ and $T_n - t_n \to 0$.

The reasoning in the case a is increasing is analogous (it suffices to choose t_n, T_n in such a way that $u'(t_n) = \ell$, $u'(T_n) = \ell/2$). Therefore, there exists the limit $u'(+\infty) = 0$. Hence, there exists also the limit $\lambda^+ := \lim_{t\to+\infty} A(u(t))''$ and it results $\lambda^+ \geq 0$. Indeed, if $\lambda^+ < 0$ then the function $a(u(t))u'(t)$ should be definitively decreasing, with $u'(+\infty) = 0$ and $u'(t) < 0$, a contradiction.

Similarly, if u is decreasing in a left half-line, one can prove that there exists $u'(-\infty) = 0$ and there exists also the limit $\lambda^- := \lim_{t\to-\infty} A(u(t))'' \leq 0$. Therefore, if u is strictly decreasing in some unbounded (maximal) interval $(\tau_1, \tau_2) \subseteq \mathbb{R}$, we would have

$$0 \leq A(u(t))''_{|t=\tau_2} = \rho(\tau_2)u(\tau_2) - k(\tau_2) < \rho(\tau_1)u(\tau_1) - k(\tau_1) = A(u(t))''_{|t=\tau_1} \leq 0,$$

a contradiction.

Summarizing, when ρ is decreasing and a is monotone (increasing or decreasing), then the solution u is monotone increasing. $\square$

Remark 5.2. In view of the previous proof, notice that the addition of the constant $-\varepsilon$ in equation (5.5) only serves to guarantee that $\alpha(-\infty) < \beta(-\infty)$, in order to apply Theorem 4.1. One does not need this if $k(-\infty) \neq 0$ or in the situations where Theorem 4.5 is applicable, or if a is constant (since in this last case one applies Corollary 4.3). Actually, note that if a and ρ are both constant then $\alpha(\pm\infty) = \beta(\pm\infty)$ and so there exist $u(-\infty) = k(-\infty)/\rho$ and $u(+\infty)k(+\infty)/\rho$.

Moreover, note that when $k(-\infty) = 0$ we have $\widetilde{\alpha}(-\infty) = 0$, $\widetilde{\beta}(-\infty) = A^{-1}(\varepsilon\mu_2/\rho_1)$ with $\varepsilon > 0$ arbitrarily small. So, we can find monotone bounded solutions of (5.2) with $u(-\infty)$ arbitrarily small.

Finally, consider the general equation

$$(a(u)u')' = f(t, u, u') \tag{5.6}$$

where $a: \mathbb{R} \to \mathbb{R}$ is a positive, continuous function and $f: \mathbb{R}^3 \to \mathbb{R}$ is a Carathéodory function.

Corollary 5.3. *Assume that*

$$c_1(t)y + \rho_1(t)x - k_1(t) \leq f(t, x, y) \leq c_2(t)y + \rho_2(t)x - k_2(t) \tag{5.7}$$

for almost every $t \in \mathbb{R}$, and every $x, y \geq 0$, where c_i, ρ_i, k_i, $i = 1, 2$, satisfy all the assumptions of Corollary 5.1. *Moreover, let a be a locally Lipschitz continuous function, satisfying* (5.3). *Then equation* (5.6) *admits a bounded solution $u \in C^1(\mathbb{R})$, with $a(u)u' \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$.*

Proof. We give just a sketch of the proof, which is quite similar to that developed for Corollary 5.1.

Proceeding as in the previous proof, one can find a lower solution $\widetilde{\alpha}$ of the equation $(a(u)u')' = c_2(t)u' + \rho_2(t)u - k_2(t)$ considering the analogous of equation (5.4), and an upper solution $\widetilde{\beta}$ of the other equation $(a(u)u')' c_1(t)u' + \rho_1(t)u - k_1(t)$ considering the analogous of equation (5.5). The strict inequalities between them at $\pm\infty$ is ensured by the relations $k_1(t) \geq k_2(t)$, $\rho_1(t) \leq \rho_2(t)$ for every $t \in \mathbb{R}$, deriving by the assumption (5.7) respectively taken for $x = y = 0$ and for $y = 0$, $x \to +\infty$.

Of course, $\widetilde{\beta}$ is an upper solution also for the equation $(a(u)u')' = c_2(t)u' + \rho_2(t)u - k_2(t)$, for which Theorem 4.1 is applicable. Hence, $\widetilde{\alpha}(t) \leq \widetilde{\beta}(t)$ for every $t \in \mathbb{R}$ and then assumption (H1) of Theorem 3.2 holds true for the general right-hand side $f(t, x, y)$.

Finally, since

$$|f(t, x, y)| \leq (|c_1(t)| + |c_2(t)|)|y| + (|\rho_1(t)| + |\rho_2(t)|)|x| + |k_1(t)| + |k_2(t)|$$

similarly to what we done in the proof of Corollary 5.1, it is possible to show that also assumptions (H2), (H3) of Theorem 3.2 are satisfied. $\square$

Example 5.4. Let a be a generic positive, locally Lipschitz continuous function and let

$$f(t, x, y) := \sqrt{(c(t)y + \rho(t)x)^2 + 1} - k(t),$$

where c, ρ, k are bounded non-negative functions, satisfying the assumptions of Corollary 5.1 and such that $k(t) \geq 1$ for every $t \in \mathbb{R}$.

Then, being $|\xi| \leq \sqrt{1 + \xi^2} \leq 1 + |\xi|$ for every $\xi \in \mathbb{R}$, condition (5.7) is satisfied for $c_1(t) = c_2(t) := c(t)$, $\rho_1(t) = \rho_2(t) := \rho(t)$, $k_1(t) := k(t)$ and $k_2(t) := k(t) - 1$. Therefore, the equation

$$(a(u)u')' = c(t)y + \sqrt{(c(t)y + \rho(t)x)^2 + 1} - k(t)$$

admits a bounded solution $u \in C^1(\mathbb{R})$, with $a(u)u' \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$.

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