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MULTIPLE NONTRIVIAL SOLUTIONS OF NEUMANN p-LAPLACIAN SYSTEMS

Dumitru Motreanu — Kanishka Perera

Abstract. We obtain multiple nontrivial solutions of Neumann p-Laplacian systems via Morse theory.

1. Introduction

We consider the problem:

$$\begin{cases} -\Delta_{p_i} u_i + \theta_i(x)|u_i|^{p_i-2}u_i = F_{u_i}(x,u) & \text{in } \Omega, \\ \dfrac{\partial u_i}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \quad i = 1, \dots, m \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with C^2-boundary $\partial\Omega$, each $p_i \in (1,\infty)$, $\Delta_{p_i} u_i = \operatorname{div}(|\nabla u_i|^{p_i-2}\nabla u_i)$ is the p_i-Laplacian of u_i, $\theta_i \in L^\infty(\Omega)$ is ≥ 0 almost everywhere and $\neq 0$, $F \in C^1(\Omega \times \mathbb{R}^m, \mathbb{R})$ with $F(\,\cdot\,,0) = 0$, $u = (u_1, \dots, u_m)$, and $\partial/\partial n$ is the exterior normal derivative on $\partial\Omega$. We assume that the nonlinearities F_{u_i} satisfy the subcritical growth conditions

$$|F_{u_i}(x,u)| \leq C\bigg(\sum_{j=1}^m |u_j|^{r_{ij}-1} + 1\bigg) \quad \text{for all } (x,u) \in \Omega \times \mathbb{R}^m \tag{1.2}$$

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for some $C > 0$ and $r_{ij} \in (1, 1 + p_j^*/(p_i^*)')$. Here

$$p_i^* = \begin{cases} \dfrac{Np_i}{N - p_i} & \text{if } p_i < N, \\ \infty & \text{if } p_i \geq N, \end{cases}$$

is the critical exponent for the Sobolev imbedding $W^{1,p_i}(\Omega) \hookrightarrow L^r(\Omega)$ and $(p_i^*)' = p_i^*/(p_i^* - 1)$ is the Hölder conjugate of p_i^*. Let

$$C_n^1(\overline{\Omega}) = \left\{ u \in C^1(\overline{\Omega}) : \frac{\partial u_i}{\partial n} = 0 \text{ on } \partial\Omega \right\}$$

and let $W_n^{1,p_i}(\Omega)$ be its completion with respect to the $W^{1,p_i}(\Omega)$-norm. By Lemma 3.1 of Barletta and Papageorgiou [1],

$$\|u_i\|_i = \left(\int_\Omega |\nabla u_i|^{p_i} + \theta_i(x)|u_i|^{p_i} \right)^{1/p_i}$$

defines an equivalent norm on $W_n^{1,p_i}(\Omega)$. We recall that a weak solution of the system (1.1) is any $u \in W = W_n^{1,p_1}(\Omega) \times \ldots \times W_n^{1,p_m}(\Omega)$ such that

$$\int_\Omega |\nabla u_i|^{p_i-2} \nabla u_i \cdot \nabla v_i + \theta_i(x)|u_i|^{p_i-2} u_i v_i = \int_\Omega F_{u_i}(x, u) v_i$$

for all $v_i \in W_n^{1,p_i}(\Omega)$, $i = 1, \ldots, m$. They coincide with the critical points of the C^1-functional

$$\Phi(u) = \int_\Omega \sum_{i=1}^m \frac{1}{p_i} (|\nabla u_i|^{p_i} + \theta_i(x)|u_i|^{p_i}) - F(x, u), \quad u \in W,$$

and are in $C_n^1(\overline{\Omega}) \times \ldots \times C_n^1(\overline{\Omega})$ by nonlinear regularity theory. The purpose of this paper is to obtain multiple nontrivial weak solutions using Morse theory. We refer to Barletta and Papageorgiou [1], Filippakis, Gasiński, and Papageorgiou [3], Marano and Motreanu [5], Motreanu and Papageorgiou [6], Ricceri [8], and Wu and Tan [9] for related multiplicity results on Neumann p-Laplacian problems and to the monograph of Perera, Agarwal, and O'Regan [7] for some recent developments in quasilinear problems via Morse theory.

We assume that $u = 0$ is a solution of (1.1) and the behavior of F near zero is given by

$$F(x, u) = \lambda J(x, u) + G(x, u) \tag{1.3}$$

where $\lambda \in \mathbb{R}$,

$$J(x, u) = V(x)|u_1|^{r_1} \ldots |u_m|^{r_m}$$

with $r_i \in (1, p_i)$ and $r_1/p_1 + \ldots + r_m/p_m = 1$, $V \in L^\infty(\Omega)$ is a (possibly indefinite) weight function, and G is a higher-order term:

$$|G(x, u)| \leq C \sum_{i=1}^m |u_i|^{s_i} \quad \text{for all } (x, u) \in \Omega \times \mathbb{R}^m, \tag{1.4}$$

for some $s_i \in (p_i, p_i^*)$.

The associated eigenvalue problem

$$(1.5)\qquad \begin{cases} -\Delta_{p_i} u_i + \theta_i(x)|u_i|^{p_i-2}u_i = \lambda J_{u_i}(x,u) & \text{in } \Omega, \\ \dfrac{\partial u_i}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \quad i = 1, \ldots, m,$$

has nondecreasing (resp. nonincreasing) and unbounded sequences of positive (resp. negative) variational eigenvalues $(\lambda_k^{\pm})$ when $V > 0$ (resp. < 0) on sets of positive measure (see Section 2). When $V \le 0$ (resp. ≥ 0) almost everywhere we set $\lambda_1^{\pm} = \pm\infty$ for convenience.

We also assume that

$$(1.6)\qquad \limsup_{|u|\to\infty} \frac{F(x,u)}{\displaystyle\sum_{i=1}^m |u_i|^{p_i}} \le 0 \quad \text{uniformly in } x \in \Omega.$$

Our main result is

Theorem 1.1. *Assume* (1.2)–(1.4) *and* (1.6). *If* $\lambda_{k+1}^- < \lambda < \lambda_k^-$ *or* $\lambda_k^+ < \lambda < \lambda_{k+1}^+$ *for some* $k \ge 1$, *then* (1.1) *has at least two nontrivial solutions.*

Let Φ be a C^1-functional defined on a real Banach space W. We recall that in Morse theory the local behavior of Φ near an isolated critical point u_0 is described by the sequence of critical groups

$$C^q(\Phi, u_0) = H^q(\Phi^c \cap U, \Phi^c \cap U \setminus \{u_0\}), \quad q \ge 0$$

where $c = \Phi(u_0)$ is the corresponding critical value, Φ^c is the sublevel set $\{u \in W : \Phi(u) \le c\}$, U is a neighbourhood of u_0 containing no other critical points, and H denotes Alexander–Spanier cohomology with $\mathbb{Z}_2$-coefficients (see e.g. Chang [2]). We also recall that Φ satisfies the Palais–Smale compactness condition (PS) if every sequence $(u^j) \subset W$ such that

$$(\Phi(u^j)) \text{ is bounded}, \quad \Phi'(u^j) \to 0,$$

called a (PS) sequence, has a convergent subsequence. We will prove Theorem 1.1 using the following "three critical points theorem" of Liu [4].

Proposition 1.2. *Assume that* Φ *is bounded from below and satisfies* (PS). *If zero is an isolated critical point of* Φ *and* $C^k(\Phi, 0) \ne 0$ *for some* $k \ge 1$, *then* Φ *has at least two nontrivial critical points.*

We will show that $C^k(\Phi, 0) \ne 0$ under the hypotheses of Theorem 1.1 using some recent results of Perera, Agarwal, and O'Regan [7] on nontrivial critical groups in nonlinear eigenvalue problems and related perturbed systems, which we will recall in the next section.

2. Preliminaries

For $i = 1, \dots, m$, let $(W_i, \|\cdot\|_i)$ be a real reflexive Banach space with the dual $(W_i^*, \|\cdot\|_i^*)$ and the duality pairing $\langle \cdot, \cdot \rangle_i$. Then their product

$$W = W_1 \times \dots \times W_m = \{u = (u_1, \dots, u_m) : u_i \in W_i\}$$

is also a reflexive Banach space with the norm

$$\|u\| = \bigg(\sum_{i=1}^m \|u_i\|_i^2 \bigg)^{1/2}$$

and has the dual

$$W^* = W_1^* \times \dots \times W_m^* = \{L = (L_1, \dots, L_m) : L_i \in W_i^*\},$$

with the pairing

$$\langle L, u \rangle = \sum_{i=1}^m \langle L_i, u_i \rangle_i$$

and the dual norm

$$\|L\|^* = \bigg(\sum_{i=1}^m (\|L_i\|_i^*)^2 \bigg)^{1/2}.$$

Consider the system of operator equations

$$A_p u = F'(u) \tag{2.1}$$

in W^*, where $p = (p_1, \dots, p_m)$ with each $p_i \in (1, \infty)$,

$$A_p u = (A_{p_1} u_1, \dots, A_{p_m} u_m),$$

$A_{p_i} \in C(W_i, W_i^*)$ is:

(A$_{i1}$) $(p_i - 1)$-homogeneous and odd if

$$A_{p_i}(\alpha u_i) = |\alpha|^{p_i - 2} \alpha A_{p_i} u_i \quad \text{for all } u_i \in W_i,\ \alpha \in \mathbb{R},$$

(A$_{i2}$) uniformly positive if there exists $c_i > 0$ such that

$$\langle A_{p_i} u_i, u_i \rangle_i \geq c_i \|u_i\|_i^{p_i} \quad \text{for all } u_i \in W_i,$$

(A$_{i3}$) a potential operator if there is a functional $I_{p_i} \in C^1(W_i, \mathbb{R})$, called a potential for A_{p_i}, such that

$$I'_{p_i}(u_i) = A_{p_i} u_i \quad \text{for all } u_i \in W_i,$$

(A$_4$) A_p is of type (S) if for any sequence $(u^j) \subset W$,

$$u^j \rightharpoonup u,\ \langle A_p u^j, u^j - u \rangle \to 0 \ \Rightarrow\ u^j \to u,$$

and $F \in C^1(W, \mathbb{R})$ with $F' = (F_{u_1}, \dots, F_{u_m}) \colon W \to W^*$ compact and $F(0) = 0$.

Proposition 2.1 [7, Proposition 10.0.5]). *If each W_i is uniformly convex and*

$$\langle A_{p_i} u_i, v_i\rangle_i \le r_i \|u_i\|_i^{p_i-1}\|v_i\|_i, \quad \langle A_{p_i} u_i, u_i\rangle_i = r_i\|u_i\|_i^{p_i} \quad \text{for all } u_i, v_i \in W_i,$$

for some $r_i > 0$, then (A_4) *holds.*

By Proposition 1.0.2 of [7], A_p is also a potential operator and the potential I_p of A_p satisfying $I_p(0) = 0$ is given by

$$I_p(u) = \sum_{i=1}^{m} \frac{1}{p_i}\langle A_{p_i} u_i, u_i\rangle_i.$$

Now the solutions of the system (2.1) coincide with the critical points of the C^1-functional $\Phi(u) = I_p(u) - F(u)$, $u \in W$.

Proposition 2.2 ([7, Lemma 3.1.3]). *Every bounded* (PS) *sequence of* Φ *has a convergent subsequence.*

Unlike in the scalar case, here the functional I_p is not homogeneous except when $p_1 = \ldots = p_m$. However, I_p still has the following weaker property. Define a continuous flow on W by

$$\mathbb{R} \times W \to W, \quad (\alpha, u) \mapsto u_\alpha := (|\alpha|^{1/p_1 - 1}\alpha u_1, \ldots, |\alpha|^{1/p_m-1}\alpha u_m).$$

Then $I_p(u_\alpha) = |\alpha| I_p(u)$ by (A_{i1}). This suggests that the appropriate class of eigenvalue problems to study for the operator A_p are of the form

$$A_p u = \lambda J'(u) \tag{2.2}$$

where the functional $J \in C^1(W, \mathbb{R})$ satisfies

$$J(u_\alpha) = |\alpha| J(u) \quad \text{for all } \alpha \in \mathbb{R},\ u \in W \tag{2.3}$$

and J' is compact. Taking $\alpha = 0$ shows that $J(0) = 0$, and taking $\alpha = -1$ shows that J is even, so J' is odd, in particular, $J'(0) = 0$. Moreover, if u is an eigenvector associated with λ, then so is u_α for any $\alpha \neq 0$ (see [7, Proposition 10.1.2]).

Let $\mathcal{M} = \{u \in W : I_p(u) = 1\}$ and $\mathcal{M}^\pm = \{u \in \mathcal{M} : J(u) \gtrless 0\}$. Then $\mathcal{M} \subset W \setminus \{0\}$ is a bounded complete symmetric C^1-Finsler manifold radially homeomorphic to the unit sphere in W, $\mathcal{M}^\pm$ are symmetric open submanifolds of $\mathcal{M}$, and the positive (resp. negative) eigenvalues of (2.2) coincide with the critical values of the even functionals

$$\Psi^\pm(u) = \frac{1}{J(u)}, \quad u \in \mathcal{M}^\pm$$

(see Lemmas 10.1.4 and 10.1.5 of [7]). Denote by $\mathcal{F}^\pm$ the classes of symmetric subsets of $\mathcal{M}^\pm$ and by $i(M)$ the Fadell–Rabinowitz cohomological index of $M \in \mathcal{F}^\pm$. Then

$$\begin{aligned} \lambda_k^+ &:= \inf_{\substack{M \in \mathcal{F}^+ \\ i(M) \geq k}} \sup_{u \in M} \Psi^+(u), \quad 1 \leq k \leq i(\mathcal{M}^+), \\ \lambda_k^- &:= \sup_{\substack{M \in \mathcal{F}^- \\ i(M) \geq k}} \inf_{u \in M} \Psi^-(u), \quad 1 \leq k \leq i(\mathcal{M}^-), \end{aligned}$$

define nondecreasing (resp. nonincreasing) sequences of positive (resp. negative) eigenvalues of (2.2) that are unbounded when $i(\mathcal{M}^\pm) = \infty$ (see Theorems 10.1.8 and 10.1.9 of [7]). When $i(\mathcal{M}^\pm) = 0$ we set $\lambda_1^\pm = \pm\infty$ for convenience.

Returning to (2.1), suppose that $u = 0$ is a solution and the asymptotic behavior of F near zero is given by

$$F(u_\alpha) = \lambda J(u_\alpha) + o(\alpha) \quad \text{as } \alpha \searrow 0, \text{ uniformly in } u \in \mathcal{M}. \tag{2.4}$$

Proposition 2.3 ([7, Proposition 10.2.1]). *Assume* (A_{i1})–(A_{i3}), (A_4), $J \in C^1(W, \mathbb{R})$ *satisfies* (2.3), J' *and* F' *are compact,* (2.4) *holds, and zero is an isolated critical point of* Φ.

(a) *If* $\lambda_1^- < \lambda < \lambda_1^+$, *then* $C^q(\Phi, 0) \approx \delta_{q0}\mathbb{Z}_2$.
(b) *If* $\lambda_{k+1}^- < \lambda < \lambda_k^-$ *or* $\lambda_k^+ < \lambda < \lambda_{k+1}^+$, *then* $C^k(\Phi, 0) \neq 0$.

3. Proof of Theorem 1.1

First we verify that our problem fits into the operator setting of Section 2. Let $W_i = W_n^{1,p_i}(\Omega)$,

$$\langle A_{p_i} u_i, v_i \rangle_i = \int_\Omega |\nabla u_i|^{p_i - 2} \nabla u_i \cdot \nabla v_i + \theta_i(x) |u_i|^{p_i - 2} u_i v_i,$$

and

$$F(u) = \int_\Omega F(x, u).$$

Then (A_{i1}) is clear, $\langle A_{p_i} u_i, u_i \rangle_i = \|u_i\|_i^{p_i}$ in (A_{i2}), and (A_{i3}) holds with

$$I_{p_i}(u_i) = \frac{1}{p_i} \int_\Omega |\nabla u_i|^{p_i} + \theta_i(x) |u_i|^{p_i}.$$

By the Hölder inequalities for integrals and sums,

$$\begin{aligned} \langle A_{p_i} u_i, v_i \rangle_i \leq{} & \bigg(\int_\Omega |\nabla u_i|^{p_i} \bigg)^{1/p_i'} \bigg(\int_\Omega |\nabla v_i|^{p_i} \bigg)^{1/p_i} \\ & + \bigg(\int_\Omega \theta_i(x) |u_i|^{p_i} \bigg)^{1/p_i'} \bigg(\int_\Omega \theta_i(x) |v_i|^{p_i} \bigg)^{1/p_i} \end{aligned}$$

$$\leq \bigg(\int_\Omega |\nabla u_i|^{p_i} + \theta_i(x)|u_i|^{p_i}\bigg)^{1/p_i'} \bigg(\int_\Omega |\nabla v_i|^{p_i} + \theta_i(x)|v_i|^{p_i}\bigg)^{1/p_i}$$
$$= \|u_i\|_i^{p_i-1}\|v_i\|_i,$$

so (A_4) follows from Proposition 2.1. By the growth condition (1.2),

$$|\langle F'(u), v\rangle| = \bigg|\int_\Omega \sum_{i=1}^m F_{u_i}(x,u)v_i\bigg| \leq C\sum_{i=1}^m \bigg(\sum_{j=1}^m \|u_j\|^{r_{ij}-1}_{L^{(r_{ij}-1)(p_i^*)'}} + 1\bigg)\|v_i\|_i.$$

Since $(r_{ij}-1)(p_i^*)' < p_j^*$ and hence the imbedding $W_n^{1,p_j}(\Omega) \hookrightarrow L^{(r_{ij}-1)(p_i^*)'}(\Omega)$ is compact, the compactness of F' follows. We have

$$|J_{u_i}(x,u)| = r_i|V(x)||u_1|^{r_1}\dots|u_i|^{r_i-1}\dots|u_m|^{r_m} \leq C\sum_{j=1}^m |u_j|^{p_j/p_i'}$$

since $r_1/p_1+\dots+(r_i-1)/p_i+\dots+r_m/p_m = 1-1/p_i = 1/p_i'$, and $p_j/p_i' < p_j^*/(p_i^*)'$, so the compactness of J' follows similarly.

Integrating (1.2) gives

$$|F(x,u)| \leq C\sum_{i=1}^m \bigg(\sum_{j=1}^m |u_j|^{r_{ij}-1} + 1\bigg)|u_i|. \tag{3.1}$$

By (1.6) and (3.1), for each $\varepsilon > 0$, there is a $C_\varepsilon > 0$ such that

$$F(x,u) \leq \varepsilon\sum_{i=1}^m |u_i|^{p_i} + C_\varepsilon$$

and hence

$$\Phi(u) \geq \sum_{i=1}^m \bigg(\frac{1}{p_i} - C\varepsilon\bigg)\|u_i\|_i^{p_i} - C_\varepsilon|\Omega|$$

where $|\Omega|$ is the Lebesgue measure of Ω. Taking ε sufficiently small, it follows that Φ is bounded from below and coercive. Then every (PS) sequence is bounded and hence Φ satisfies the (PS) condition by Proposition 2.2.

Turning to the eigenvalue problem (1.5), let

$$J(u) = \int_\Omega J(x,u), \qquad G(u) = \int_\Omega G(x,u).$$

Then

$$J(u_\alpha) = \int_\Omega V(x)|\alpha|^{r_1/p_1+\dots+r_m/p_m}|u_1|^{r_1}\dots|u_m|^{r_m} = |\alpha|J(u).$$

By (1.4),

$$|G(u_\alpha)| \leq C\sum_{i=1}^m |\alpha|^{s_i/p_i}\|u_i\|_i^{s_i},$$

so (2.4) also holds. Applying Proposition 2.3, we have $C^k(\Phi,0) \neq 0$.

Proposition 1.2 now gives the result. □

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Dumitru Motreanu
Département de Mathématiques
Université de Perpignan
66860 Perpignan, FRANCE

E-mail address: motreanu@univ-perp.fr

Kanishka Perera
Department of Mathematical Sciences
Florida Institute of Technology
Melbourne, FL 32901, USA

E-mail address: kperera@fit.edu