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COMPUTER-ASSISTED PROOF OF A PERIODIC SOLUTION IN A NONLINEAR FEEDBACK DDE

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Abstract. In this paper, we rigorously prove the existence of a non-trivial periodic orbit for the nonlinear DDE: $x'(t) = -K\sin(x(t-1))$ for $K = 1.6$. We show that the equations for the Fourier coefficients have a solution by computing the local Brouwer degree. This degree can be computed by using a homotopy, and its validity can be proved by checking a finite number of inequalities. Checking these inequalities is done by a computer program.

1. Introduction

The equation:

$$x'(t) = -K\sin(x(t-1))$$

is an example of a nonlinear feedback delay differential equation (see [2]). It is used to model, for example, delay-lock loops in electronics (see [5]). Numerical simulations shows that for small $K < \pi/2$ the solutions tends to 0, for $\pi/2 < K < 5.1$ there is an attracting periodic orbit oscillating around 0, but for $K > 5.1$ there is chaos — the solutions jump by $\pm 2\pi$.

However, there are no rigorous proofs for such behavior. In [3] there is a proof of the chaos for large K, but after changing the sinus to a function close to a piecewise linear function. The [5] analyzes the eigenvalues of the linearization

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and provides an informal argument for the bifurcation in $K = \pi/2$ and for the creation of an attracting orbit, but this is not a proof.

In this paper we will strictly show that there is a periodic solution for $K = 1.6$. By a solution we mean a $\mathcal{C}^1$-function that satisfies the equation.

In the remainder of the introduction, we briefly outline our method. We will use the Fourier coefficients of the periodic orbit. First, we rescale the time to obtain a 2π period. We make the substitution: $\widetilde{x}(t) = x(t/\tau)$, where τ is a parameter. If τ is equal to $2\pi/T$ (where T is the period of the solution in the original equation), we will have a 2π-periodic solution of the new equation:

$$\begin{aligned}\widetilde{x}'(t) &= \frac{1}{\tau}x'\left(\frac{t}{\tau}\right) = -\frac{K}{\tau}\sin\left(x\left(\frac{t}{\tau}-1\right)\right)\\ &= -\frac{K}{\tau}\sin\left(x\left(\frac{t-\tau}{\tau}\right)\right) = -\frac{K}{\tau}\sin(\widetilde{x}(t-\tau)).\end{aligned}$$

By renaming $\widetilde{x}$ to x we obtain:

$$x'(t) = -\frac{K}{\tau}\sin(x(t-\tau)). \tag{1.1}$$

We will prove for this equation that there exists a τ from a small interval such that there exists a 2π-periodic orbit in a small neighbourhood of a specified function. Note that we do not obtain the exact value of the period (even if numerical solutions suggest it is 4, i.e. $\tau = \pi/2$), but treat τ as a variable.

To prove the existence of the orbit, we will use the method of self-consistent bounds that was introduced in the context of Kuramoto–Shivashinsky PDEs in [6] and [7]. When applied to the boundary value problem for ODEs or DDEs, this method is similar to the Cesari method introduced in [1] but does not require one of the conditions — see Section 2.4 in [7] for a comparison. Below we briefly describe the method.

We will derive from (1.1) the equations for the Fourier coefficients. Let us first write (1.1) with a Taylor expansion instead of the sinus:

$$x'(t) = -\frac{K}{\tau}\sum_{n=0}^{\infty}\frac{(-1)^k}{(2k+1)!}[x(t-\tau)]^{2k+1}.$$

The equations on the Fourier coefficients will look similar:

$$inc_n = -\frac{K}{\tau}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}(c^{*(2k+1)})_n e^{-in\tau} \quad \text{for all } n \in \mathbb{Z}$$

where $(c^{*(2k+1)})_n$ means convoluting the sequence c with itself $2k+1$ times and then taking the n-th coefficient. The operation of convolution and this notation is introduced in details in Section 2.

We will show that the following function has a non-trivial zero:

$$F(\tau, c) = \left\{ in\tau e^{in\tau} c_n + K \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(c^{*(2k+1)} \right)_n \right\}_{n=-\infty}^{\infty}$$

where the domain is:

$$X_\beta := \left\{ x\colon \mathbb{Z} \to \mathbb{C} \;\middle|\; |x_n| \le \frac{\beta}{(|n|+1)^2} \text{ and } x_n = \overline{x_{-n}} \text{ for all } n \in \mathbb{Z} \right\}$$

for some $\beta > 0$. On X_β we consider the product topology (which is equivalent to component-wise convergence). It turns out that on such a domain the sums in the definition of F are convergent, F is continuous, and each sequence $c \in X_\beta$ corresponds to a real-valued continuous function. Moreover, if $F(c) = 0$ then the function is $\mathcal{C}^1$.

We will search for a zero in the neighbourhood of an approximate solution, obtained from numerical simulations. We will denote this approximation by $\widehat{c}$. For $0 \le n \le 5$ it is equal to:

n	$\widehat{c}_n$
0	0
1	$-0.1521000000 - 0.1163508047i$
2	0
3	$0.0001123121 - 0.0002746107i$
4	0
5	$-0.0000008173 - 0.0000001014i$

TABLE 1

For $n > 5$ the $\widehat{c}_n$ is zero, for $n < 0$ we have $\widehat{c}_n = \overline{\widehat{c}_{-n}}$. The variable τ in the approximation is: $\widehat{\tau} = 1.570796$.

For each $l \in \mathbb{N}$, let us define the Galerkin projection P_l and immersion Q_l:

$$P_l\colon \mathbb{C}^{\mathbb{Z}} \ni c \to (c_0, \ldots, c_l) \in \mathbb{R} \times \mathbb{C}^l,$$
$$Q_l\colon \mathbb{R} \times \mathbb{C}^l \ni (c_0, \ldots, c_l) \to (\ldots, 0, 0, \overline{c_{-l}}, \ldots, c_0, \ldots, c_l, 0, 0, \ldots) \in \mathbb{C}^{\mathbb{Z}}.$$

Let us note that we need only the non-negative terms in the finite space, as the negative terms can be obtained by conjugation. Also as $c_0 = \overline{c_{-0}}$, we have $c_0 \in \mathbb{R}$. From the compactness of X_β, it will be easy to show that:

LEMMA 4.1. *Let $\beta > 0$, $l_0 > 0$, $\underline{\tau}, \overline{\tau} \in \mathbb{R}$ be fixed. If for each $l > l_0$ there is a $c^l \in X_\beta$ and $\tau_l \in [\underline{\tau}, \overline{\tau}]$ such that $P_l F(\tau_l, c^l) = 0$ then there exists* *a* $(c^0, \tau_0) \in X_\beta \times [\underline{\tau}, \overline{\tau}]$ *such that* $F(\tau_0, c^0) = 0$. *Moreover, if all the* c^l *are in a closed set* D *then* $c^0 \in D$.

Thus, it is enough to show that there is a zero for every Galerkin projection of F:

$$\widetilde{F}_l \colon \mathbb{R} \times (\mathbb{R} \times \mathbb{C}^l) \ni (\tau, c) \to P_l F(\tau, Q_l(c)) \in \mathbb{R} \times \mathbb{C}^l.$$

This will allow us to use a finite-dimensional topological method to prove the existence of a zero of F. Let us note that the real dimension of the domain is $2l+2$ while that of the image is $2l+1$. Thus, if there is a zero, one can expect a 1-dimensional manifold of zeros. This manifold can be easily identified — if $x(\,\cdot\,)$ is a solution then $x(\,\cdot\, + \phi)$ is also a solution. This means that if $\widetilde{F}_l(\tau, c_0, \dots, c_l) = 0$ then $\widetilde{F}_l(\tau, c_0, e^{i\phi} c_1, \dots, e^{il\phi} c_l) = 0$, as can be easily checked. To use the topological method, we want to have the zero isolated — we will limit the domain assuming that $c_1 \in \widehat{c}_1 + \mathbb{R}$. Of course, finding a zero in a limited domain implies a zero of the full system. By F_l we will denote the $\widetilde{F}_l$ limited to the smaller domain.

The before-mentioned method to prove the existence of a zero of F_l is the local Brouwer degree (introduced e.g. in [4]). Let us denote by $\deg(F_l, U, x)$ the degree of x on U. It is known that if the degree is non-zero then there exists $y \in U$ such that $F_l(y) = x$. We will use a neighbourhood of $\widehat{c}$ as U and $x = 0$.

To compute the degree, we will use the homotopy invariance of the local Brouwer degree. As our homotopy H_l, we will use a linear deformation of F_l into a function G_l that contains the most important terms of F_l (that is not strictly a linearization of F_l but it is close to it):

$$H_l(h, \tau, c) = hF_l(\tau, c) + (1-h)G_l(\tau, c).$$

It will be easy to show that the degree of G_l is non-zero. We will need to show that $0 \notin H_l([0;1]; \partial U)$. To show that, it is enough to show that the terms in G_l dominate the (mainly nonlinear) terms that are in F_l but not G_l, i.e. that $|G_l| > |F_l - G_l|$. That part of the proof is computer-assisted — the proofs of the estimates are given in this paper but computing the exact values and checking that the inequalities holds is done by a computer program using the CAPD package for rigorous interval arithmetics.

The program. This program is written in C++ and can be downloaded from `http://www.im.uj.edu.pl/MikolajZalewski/dl/delay-sin.tgz`. The rounding-mode changing code required by the interval arithmetic is system-dependent and has been checked to work on PCs (both 32-bit and 64-bit) on both Windows (compiled with cygwin) and Linux (compiled with gcc). It should also work on SPARC and Mac OS X, although that has not been tested. Using other CPUs or compilers might require modifications to the rounding code.

The constants used in the theorems — the K, β_1, β_2, $\Delta\tau$, $\widehat{\tau}$ and $\widehat{c}_n$ — are represented in the program as small intervals containing the values. Thus the theorems are true for values as written in the paper, even if they are not representable as IEEE floating point numbers.

2. Fourier coefficients

We will use the following notation: if $\{x_n\}_{n=-\infty}^{\infty}$ and $\{y_n\}_{n=-\infty}^{\infty}$ are sequences with complex values then by $x * y$ we will denote the convolution of the sequences:

$$x * y := \left\{ \sum_{k=-\infty}^{\infty} x_k y_{n-k} \right\}_{n=-\infty}^{\infty}.$$

It is well known that if the sum converges, the operation of convolution is associative. We will also use the notation $(\,\cdot\,)_n$ for the n-th coefficient of the sequence in brackets, e.g. $(x * y)_n = \sum_{k=-\infty}^{\infty} x_k y_{n-k}$. Also:

$$x^{*k} := \underbrace{x * \ldots * x}_{k \text{ times}} \quad (\text{for } k \geq 1), \qquad (x^{*0})_n := \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

Of course, the sum in the definition of the convolution may be not convergent, so we will limit our attention to a domain where the sum will be always convergent. The following lemma holds:

Lemma 2.1. *If x and y are such that there exists $\alpha \geq 2$ and $\beta_1, \beta_2 > 0$ such that*

$$|x_n| \leq \frac{\beta_1}{(|n|+1)^\alpha}, \quad |y_n| \leq \frac{\beta_2}{(|n|+1)^\alpha}, \quad \textit{for all } n,$$

then $(x * y)_n$ is convergent, for each n, and

$$|(x * y)_n| \leq C \frac{\beta_1 \beta_2}{(|n|+1)^\alpha} \quad \textit{where } C = \frac{2(2\alpha+1)}{\alpha - 1}.$$

Proof. Let us assume $n \geq 0$ and let us estimate $|(x * y)_n|$:

$$\left| \sum_{k=-\infty}^{\infty} x_k y_{n-k} \right| \leq \left| \sum_{k=0}^{\infty} x_{-k} y_{n+k} \right| + \left| \sum_{k=1}^{n-1} x_k y_{n-k} \right| + \left| \sum_{k=0}^{\infty} x_{n+k} y_{-k} \right|,$$

$$\begin{aligned} \left| \sum_{k=1}^{n-1} x_k y_{n-k} \right| &\leq \beta_1 \beta_2 \sum_{k=1}^{n-1} \frac{1}{(k+1)^\alpha} \frac{1}{(n-k+1)^\alpha} \\ &\leq 2\beta_1\beta_2 \sum_{1 \leq k \leq \frac{n}{2}} \frac{1}{(k+1)^\alpha} \frac{1}{(n-k+1)^\alpha} \\ &\leq 2\beta_1\beta_2 \sum_{1 \leq k \leq \frac{n}{2}} \frac{1}{(k+1)^\alpha} \frac{1}{((n/2)+1)^\alpha} \end{aligned}$$

$$\leq 2\frac{\beta_1\beta_2 2^\alpha}{(n+2)^\alpha}\sum_{1\leq k\leq \frac{n}{2}}\frac{1}{(k+1)^\alpha} \leq 2\frac{\beta_1\beta_2 2^\alpha}{(n+2)^\alpha}\sum_{k\geq 1}\frac{1}{(k+1)^\alpha},$$

$$\left|\sum_{k=0}^{\infty} x_{-k}y_{n+k}\right| \leq \beta_1\beta_2\sum_{k=0}^{\infty}\frac{1}{(k+1)^\alpha}\frac{1}{(n+k)^\alpha+1} \leq \frac{\beta_1\beta_2}{(n+1)^\alpha}\sum_{k\geq 0}\frac{1}{(k+1)^\alpha}.$$

Analogously:

$$\left|\sum_{k=0}^{\infty} x_{n+k}y_{-k}\right| \leq \frac{\beta_1\beta_2}{(n+1)^\alpha}\sum_{k\geq 0}\frac{1}{(k+1)^\alpha}.$$

The sums $\sum_{k=k_0}^{\infty} 1/(k^\alpha+1)$ can be estimated by integrals:

$$\sum_{k=0}^{\infty}\frac{1}{(k+1)^\alpha} \leq 1+\int_0^\infty \frac{1}{(x+1)^\alpha}\,dx = \frac{\alpha}{\alpha-1},$$
$$\sum_{k=1}^{\infty}\frac{1}{(k+1)^\alpha} \leq \frac{1}{2^\alpha}+\int_1^\infty \frac{1}{(x+1)^\alpha}\,dx = \frac{\alpha+1}{2^\alpha(\alpha-1)}.$$

From that we obtain:

$$\left|\sum_{k=-\infty}^{\infty} x_k y_{n-k}\right| \leq \frac{\beta_1\beta_2}{(n+1)^\alpha}\left[2^{\alpha+1}\frac{\alpha+1}{2^\alpha(\alpha-1)}+2\frac{\alpha}{\alpha-1}\right] \leq \frac{\beta_1\beta_2}{(n+1)^\alpha}\frac{2(2\alpha+1)}{\alpha-1}.$$

The result for $n<0$ can be obtained by analogous estimations or by taking sequences $\widetilde{x}$, $\widetilde{y}$: $\widetilde{x}_n := x_{-n}$, $\widetilde{y}_n := y_{-n}$ and applying for them the result for $n>0$. □

With one exception, we will use this lemma for $\alpha=2$. For $\alpha=2$ we have $C=10$.

Observation 2.2. *If x is such that there exists $\alpha\geq 2$, $\beta>0$, for all n such that $|x_n|\leq \beta/(|n|+1)^\alpha$, then*

$$|(x^{*k})_n| \leq C^{k-1}\frac{\beta^k}{(|n|+1)^\alpha} \quad \textit{where } C=\frac{2(2\alpha+1)}{\alpha-1}.$$

Observation 2.3. *If x and y are such that $x_{-n}=\overline{x_n}$, $y_{-n}=\overline{y_n}$ then $(x*y)_{-n}=\overline{(x*y)_n}$.*

Thus, we have that if $x\in X_{\beta_1}$ and $y\in X_{\beta_2}$ then $x*y\in X_{C\beta_1\beta_2}$ (where X_β was defined in the introduction and C is from Lemma 2.1). It will be also useful to define a set of sequences from any X_β:

$$X := \bigcup_{\beta>0} X_\beta.$$

We will limit ourselves to the sequences in X. For them we have:

Lemma 2.4. *Let $c \in X$. Then $\sum_{n=-\infty}^{\infty} c_n e^{int}$ converges to a real-valued continuous function.*

Proof. The functions $\sum_{n=-N}^{N} c_n e^{int}$ are continuous and real-valued as if $c_n = \overline{c_{-n}}$ then $c_n e^{int} + c_{-n} e^{-int} \in \mathbb{R}$. They converge uniformly because

$$\sum_{n=-\infty}^{\infty} |c_n e^{int}| \leq \sum_{n=-\infty}^{\infty} \frac{\beta}{(|n|+1)^2} < \infty.$$

Thus, the limit is also real-valued and continuous. □

Let us note that $c \in X$ does not guarantee that the function is $\mathcal{C}^1$.

We use the convolution because of the following well known fact:

Lemma 2.5. *If $c \in X$ are the Fourier coefficients of $x(t)$, $d \in X$ are the coefficients of $y(t)$ then the Fourier coefficients of $x(t) \cdot y(t)$ are $c * d$. As a consequence, the coefficients of $x^n(t)$ are c^{*n}.*

Now, we can write the equation (1.1) on the Fourier coefficients.

Theorem 2.6. *Let τ be fixed and $x(t)\colon \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function with the Fourier coefficients $c \in X$. Then:*

(a) *for each $n \in \mathbb{Z}$ the sum on the right-hand side of the following equation converges:*

$$inc_n = -\frac{K}{\tau} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (c^{*(2k+1)})_n e^{-in\tau}, \tag{2.1}$$

(b) *$x(t)$ is a 2π-periodic solution of (1.1) if and only if c satisfies the equations (2.1).*

To prove the theorem, we will need some lemmas. As already mentioned, $c \in X$ does not imply that $x(t)$ is $\mathcal{C}^1$. However, if c is the solution of equation (2.1), we have the following lemma that will allow us to show that $x(t)$ is $\mathcal{C}^1$ (using this method one can show that that $x(t)$ is $\mathcal{C}^\infty$, but we don't need it):

Lemma 2.7. *If c satisfies equation (2.1) and $\beta > 0$, $\alpha \geq 2$ are such that $|c_n| \leq \beta/(|n|+1)^\alpha$ for all n, then there exists β' such that $|c_n| \leq \beta'/(|n|+1)^{\alpha+1}$ for all n.*

Proof. Let $C := 2(2\alpha+1)/(\alpha-1)$. We have that:

$$\left| -\frac{K}{\tau} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (c^{*(2k+1)})_n e^{-in\tau} \right| \leq \frac{K}{\tau} \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} |(c^{*(2k+1)})_n| \tag{2.2}$$
$$\leq \frac{K}{\tau} \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} C^{2k} \frac{\beta^{2k+1}}{(|n|+1)^\alpha} = \frac{K}{\tau} \frac{1}{C(|n|+1)^\alpha} \sinh(C\beta).$$

We also have $|inc_n| = |n||c_n|$. The two sides of the equation (2.1) must be equal hence we obtain:

$$\frac{K}{\tau}\frac{1}{C(|n|+1)^\alpha}\sinh(C\beta) \geq |n||c_n|.$$

Thus for $n \neq 0$ the $\beta' = (2K/(C\tau))\sinh(C\beta)$ satisfies the assertion. If it is not satisfied for $n = 0$, we can increase β'. □

Increasing α is important, as we have:

LEMMA 2.8. *Let c be a sequence of complex values satisfying, for some β, $|c_n| \leq \beta/(|n|+1)^3$ and let $c_n = \overline{c_{-n}}$. Then the sequence c is a sequence of Fourier coefficients of a real-valued $\mathcal{C}^1$ function $x(t)$.*

PROOF. The sequences $\sum_{k=-n}^{n} c_k e^{ikt}$ and $\sum_{k=-n}^{n} ikc_k e^{ikt}$ are real-valued, the second is the derivative of the first one, and are uniformly convergent as $n \to \infty$. Hence both converge to continuous function and the second is the derivative of the first one. Thus $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}$ is $\mathcal{C}^1$. □

PROOF OF THE THEOREM 2.6. (a) The convolutions are convergent because $x \in X$. From the equation (2.2) from the proof of Lemma 2.7, we have that if $x \in X_\beta$ then

$$\sum_{k=0}^{\infty}\left|\frac{(-1)^k}{(2k+1)!}(c^{*(2k+1)})_n e^{-in\tau}\right| \leq \frac{1}{C(|n|+1)^2}\sinh(C\beta) < \infty.$$

(b) Implication $\Rightarrow$: It is enough to show that $\{inc_n\}_{n=-\infty}^{\infty}$ are the Fourier coefficients of $x'(t)$ while

$$-\frac{K}{\tau}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}(c^{*(2k+1)})_n e^{-in\tau}$$

are the Fourier coefficients of $-(K/\tau)\sin x(t-\tau)$.

The first part can be obtained by integrating $\int_0^{2\pi} x'(t)e^{ikt}\,dt$ by parts.

As for the second, we have that

$$-\frac{K}{\tau}\sum_{k=0}^{N}\frac{(-1)^k}{(2k+1)!}(x(t-\tau))^{2k+1} \to -\frac{K}{\tau}\sin x(t-\tau) \quad \text{as } N \to \infty.$$

The Fourier coefficients of $x(t-\tau)$ are equal to $d := \{c_n e^{-in\tau}\}_{n=-\infty}^{\infty}$. Hence the coefficients of $x(t-\tau)^{2k+1}$ are $d^{*(2k+1)}$ which is equal to $c^{*(2k+1)}e^{-in\tau}$.

Hence, we have that the Fourier coefficients of

$$-\frac{K}{\tau}\sum_{k=0}^{N}\frac{(-1)^k}{(2k+1)!}(x(t-\tau))^{2k+1}$$

are equal to

$$\left\{-\frac{K}{\tau}\sum_{k=0}^{N}\frac{(-1)^k}{(2k+1)!}(c^{*(2k+1)})_n e^{-in\tau}\right\}_{n=-\infty}^{\infty}.$$

It has been shown that this sequence in convergent as $N\to\infty$, thus the n-th coefficient of $-(K/\tau)\sin x(t-\tau)$ is

$$-\frac{K}{\tau}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}(c^{*(2k+1)})_n e^{-in\tau}.$$

This ends the proof of this case.

Implication $\Leftarrow$: from Lemmas 2.7 and 2.8 we obtain that $x(t)$ is a $\mathcal{C}^1$-function. We know that

$$-\frac{K}{\tau}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}(c^{*(2k+1)})_n e^{-in\tau}$$

are the Fourier coefficients of $-(K/\tau)\sin x(t-\tau)$. They are equal to inc_n — the Fourier coefficients of $x'(t)$. So both $-(K/\tau)\sin x(t-\tau)$ and $x'(t)$ are continuous and 2π-periodic functions with equal Fourier coefficients. Hence the functions themselves are equal, and equation (1.1) is satisfied. $\square$

Let us define a function $F:\mathbb{R}\times X\to\mathbb{C}^{\mathbb{Z}}$ (with the product topology on $\mathbb{C}^{\mathbb{Z}}$) corresponding to equation (2.1):

$$F(\tau,c):=\left\{in\tau e^{in\tau}c_n+K\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}(c^{*(2k+1)})_n\right\}_{n=-\infty}^{\infty}.$$

Of course, having $F(\tau,c)=0$ is equivalent to the fact that τ, c satisfies equation (2.1). In the rest of the paper, we will show that F has a nontrivial zero. Let us note that $F(\tau,c)_n=\overline{F(\tau,c)_{-n}}$.

As we will use topological tools, we will want F to be continuous. First, let us note that the convolution is not continuous on the whole X (as written above, we use the product topology on X) but we have:

Lemma 2.9. *The operation of convolution is continuous on each X_β*

Proof. Let us fix some $x^0,y^0\in X_\beta$ and some δ. Let N_δ be large enough that, for $|n|>N_\delta$, if $x,y,x^0,y^0\in X_\beta$ then $|x_n-(x^0)_n|,|y_n-(y^0)_n|<\delta$. Let us take a neighbourhood U of (x^0,y^0) such that for each $(x,y)\in U$ we have, for all $|n|\le N_\delta$, $|x_n-(x^0)_n|,|y_n-(y^0)_n|<\delta$. Then, we will have:

$$|(x^0*y^0-x*y)_n|\le|(x^0*(y^0-y))_n|+|((x^0-x)*y)_n|,$$

$$|(x^0*(y^0-y))_n|\le\sum_{j=-\infty}^{\infty}|(x^0)_j|\cdot|(y^0)_{n-j}-y_{n-j}|\le\beta\delta\sum_{j=-\infty}^{\infty}\frac{1}{(|j|+1)^2}.$$

This tends to zero as $\delta\to 0$. After an analogous estimation for $|((x^0-x)*y)_n|$, we have that the convolution is continuous. $\square$

LEMMA 2.10. *The function F is continuous on each X_β.*

PROOF. The convolutions are continuous on each X_β, and the result of a convolution lays in a $X_{\beta'}$ for some β'. Thus $K\sum_{k=0}^{N}((-1)^k/(2k+1)!)(c^{*(2k+1)})_n$ is continuous for each N. From estimates as in equation (2.2), we have that, for each coefficient, this series converges uniformly as $N \to \infty$. Hence, we obtain that the limit $K\sum_{k=0}^{\infty}((-1)^k/(2k+1)!)(c^{*(2k+1)})_n$ is continuous.

The term $in\tau e^{in\tau}c_n$ is also continuous, so F is continuous. $\square$

3. Some estimates

As mentioned in the introduction, we will need some estimates to show that the inequality holds in the neighbourhood of the $\widehat{c}$. We will use two kinds of sets: $\widehat{c}+X_{\beta_2}$ that will be used to obtain finer estimates (as β_2 will be small) and X_{β_1} (where β_1 will be large enough to contain the whole set $\widehat{c}+X_{\beta_2}$) for some more rough but simpler ones.

In this section, we will only assume about $\widehat{c}$ that almost all coefficients are equal to zero. We will denote by Y_l the space of the possible values of $\widehat{c}$ — the set of sequences such that at most the elements $-l,\dots,l$ are non-zero:

$$Y_l := \{c \in X : \text{if } |n| > l \text{ then } c_n = 0 \text{ for all } n\}.$$

First, let us note two simple properties:

OBSERVATION 3.1. *If $c \in Y_l$ then $c^{*k} \in Y_{kl}$.*

LEMMA 3.2. *If $x, y \in X$ then*

$$(x+y)^{*n} = \sum_{k=0}^{n}\binom{n}{k}x^{*k} * y^{*(n-k)}.$$

PROOF. Let the sequence x corresponds to a function $f_x(t)$, and let y corresponds to a function $f_y(t)$. Then, the RHS corresponds to the function $(f_x(t)+f_y(t))^n$, while the LHS corresponds to $\sum_{k=0}^{n}\binom{n}{k}f_x(t)^k f_y(t)^{n-k}$. These functions are equal, so their Fourier coefficients are as well. $\square$

To estimate $(K/p!)(x^{*p})_n$ — an element of the sum in the equation (2.1) — for $x \in X_\beta$, it is enough to use Lemma 2.1. However, for the sets of the form $c+X_\beta$, we will use a more sophisticated estimates:

LEMMA 3.3. *Let $c \in Y_l$, $p > 1$, $\beta > 0$. Then, for any $x \in X_\beta$ and $C = 10$, we have:*

$$\left|\frac{K}{p!}(c+x)^{*p}_n\right| \le \frac{K}{p!}\left(|c^{*p}_n| + \sum_{k=0}^{p-1}\binom{p}{k}\sum_{j=-lk}^{lk}|c^{*k}_j|C^{p-k-1}\frac{\beta^{p-k}}{(|n-j|+1)^2}\right).$$

Proof. From Lemmas 2.1 and 3.2 we have:

$$\left|\frac{K}{p!}(x+c)^{*p}_n\right| \leq \frac{K}{p!}\sum_{k=0}^{p}\binom{p}{k}|(c^{*k}*x^{*(p-k)})_n| \leq \frac{K}{p!}\sum_{k=0}^{p}\binom{p}{k}\sum_{j=-\infty}^{\infty}|c^{*k}_j||x^{*(p-k)}_{n-j}|$$
$$\leq \frac{K}{p!}\left(|c^{*p}_n| + \sum_{k=0}^{p-1}\binom{p}{k}\sum_{j=-\infty}^{\infty}|c^{*k}_j|C^{p-k-1}\frac{\beta^{p-k}}{(|n-j|+1)^2}\right).$$

We have $c \in Y_l$ so $c^{*k}_j = 0$ for $j > kl$ what ends the proof. □

Of course, we can use the previous lemma only for a finite number of terms. To estimate the tail, we will use a weaker estimate by using a neighbourhood of the second type, applying Lemma 2.1 and summing the geometric sequence:

Lemma 3.4. *Let N be odd, $\beta \in [0; N/10]$. If $x \in X_\beta$ then*

$$\left|\left(K\sum_{k=(N-1)/2}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{*(2k+1)}\right)_n\right| \leq \frac{K\cdot(C\beta)^{N-1}}{N!(1-(C\beta/N)^2)}\cdot\frac{\beta}{(|n|+1)^2}$$

where $C = 10$.

Proof. Using Lemma 2.1 we have

$$\begin{aligned}
&\left|K\left(\sum_{k=(N-1)/2}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{*(2k+1)}\right)_n\right| \\
&\qquad\leq K\sum_{k=(N-1)/2}^{\infty}\frac{1}{N!N^{2k-(N-1)}}|(x^{*(2k+1)})_n| \\
&\qquad\leq \frac{K}{N!}\sum_{k=(N-1)/2}^{\infty}\frac{(C\beta)^{2k}}{N^{2k-(N-1)}}\cdot\frac{\beta}{(|n|+1)^2} \\
&\qquad= \frac{K(C\beta)^{N-1}}{N!}\left(\sum_{k=0}^{\infty}\left(\frac{C^2\beta^2}{N^2}\right)^k\right)\cdot\frac{\beta}{(|n|+1)^2} \\
&\qquad= \frac{K(C\beta)^{N-1}}{N!(1-C\beta/N)^2)}\cdot\frac{\beta}{(|n|+1)^2}.
\end{aligned}$$

The geometric sequence is convergent because if $\beta < N/10$ then $(C\beta/N)^2 < 1$.□

From Lemma 3.3 we can obtain an estimate that after multiplication by $(|n|+1)^2$ is independent of n. That will be used to estimate the term in all inequalities for high n's with one formula:

Lemma 3.5. *Let $\beta > 0$, $c \in Y_l$, $p > 1$, $N > pl$. Then for each $x \in X_\beta$, $n \geq N$ we have:*

$$\left|\frac{K}{p!}(x+c)^{(*p)}_n\right| \leq \frac{K}{p!}\left(\sum_{k=0}^{p-1}\binom{p}{k}\sum_{j=-lk}^{lk}|c^{*k}_j|(C\beta)^{p-k-1}\frac{(N+1)^2}{(N-|j|+1)^2}\right)\frac{\beta}{(|n|+1)^2}.$$

Proof. Note that if $n > pl$ then $c_n^{*p} = 0$ and

$$\frac{\beta}{(|n-j|+1)^2} = \frac{(n+1)^2}{(n-j+1)^2} \cdot \frac{\beta}{(n+1)^2} \leq \frac{(N+1)^2}{(N-|j|+1)^2} \cdot \frac{\beta}{(n+1)^2}$$

as $n \geq N > j$, then apply Lemma 3.3. □

For the first terms, we will need to have a better estimate than in Lemma 3.3, so we will regroup the terms (to understand why this regrouping helps, let us compare two estimates: $|1.1 \cdot x - 1 \cdot x| \leq 1.1|x| + 1|x| = 2.1|x|$ and $|1.1 \cdot x - 1 \cdot x| = |(1.1-1)x| = 0.1|x|$).

Lemma 3.6. *Let $x \in X_\beta$, $c \in Y_l$.*

$$\begin{aligned} in\tau e^{in\tau}(c_n + x_n) + K \sum_{k=0}^{3} \frac{(-1)^k}{(2k+1)!}(c+x)_n^{*(2k+1)} &= (in\tau e^{in\tau} + K)c_n - \frac{K}{3!}(c^{*3})_n + \frac{K}{5!}(c^{*5})_n - \frac{K}{7!}(c^{*7})_n \\ &\quad + (in\tau e^{in\tau} + K)x_n + \sum_{p=1}^{7} \sum_{j=-6l}^{6l} \gamma_{p,j} x_{n-j}^{*p} \end{aligned}$$

where

$$\gamma_{p,j} = -\frac{K}{3!}\binom{3}{p}(c^{*(3-p)})_j + \frac{K}{5!}\binom{5}{p}(c^{*(5-p)})_j - \frac{K}{7!}\binom{7}{p}(c^{*(7-p)})_j$$

and we assume $\binom{n}{k} = 0$ for $k > n$.

4. Proving the existence of the periodic solution

We will search for the orbit in the neighbourhood of $\widehat{c}$ and $\widehat{\tau}$ defined in the introduction. Let us define the sets and boundaries on which we will work:

$$X_1 := X_{\beta_1}, \quad X_2 := \widehat{c} + X_{\beta_2}, \quad X_3 := \{y \in X_2 : y_1 - \widehat{c}_1 \in \mathbb{R}\},$$
$$\underline{\tau} := \widehat{\tau} - \Delta\tau, \quad \overline{\tau} := \widehat{\tau} + \Delta\tau$$

where $\Delta\tau = 0.000001$, $\beta_2 = 0.0000002438$, $\beta_1 = 0.766763$.

We will prove that a solution exists in the set $[\underline{\tau}; \overline{\tau}] \times X_3$. The X_1 is the bigger but simpler neighbourhood, mentioned at the beginning of Section 3 — the β_1, β_2 are such that $X_2 \subset X_1$. Obviously, we have $X_3 \subset X_2$.

As mentioned in the introduction, we will use the Galerkin projections of F with the condition $y_1 - \widehat{c}_1 \in \mathbb{R}$. This condition make the dimensions of the domain and the image equal, what allows us to use the Brouwer local degree. Hence the condition in X_3. The P_l and Q_l were defined in the introduction as the projection and immersion of the finite-dimensional space. Let us first prove the lemma stated in the introduction.

Lemma 4.1. *Let* $\beta > 0$*,* $l_0 > 0$*,* $\underline{\tau}, \overline{\tau} \in \mathbb{R}$ *be fixed. If for each* $l > l_0$ *there is a* $c^l \in X_\beta$ *and* $\tau_l \in [\underline{\tau}, \overline{\tau}]$ *such that* $P_l F(\tau_l, c^l) = 0$ *then there exists* $(\tau_0, c^0) \in [\underline{\tau}, \overline{\tau}] \times X_\beta$ *such that* $F(\tau_0, c^0) = 0$*. Moreover, if all the* c^l *are in a closed set* D *then* $c^0 \in D$*.*

Proof. From the compactness of $[\underline{\tau}, \overline{\tau}] \times X_\beta$, there exists a subsequence l_k such that c^{l_k} converges to a limit c^0 and τ_{l_k} converges to τ_0. Let us fix $n \geq 0$ and note that, for $l > n$, if $P_l(F(\tau, c)) = 0$ then $F(\tau, c)_n = 0$. Thus, from the continuity of F, we have

$$F(\tau_0, c^0)_n = \lim_{k\to\infty} F(\tau_{l_k}, c^{l_k})_n = \lim_{k\to\infty} P_{l_k} F(\tau_{l_k}, c^{l_k})_n = 0.$$

For $n < 0$ we have

$$F(\tau_0, c^0)_n = \lim_{k\to\infty} F(\tau_{l_k}, c^{l_k})_n = \overline{\lim_{k\to\infty} F(\tau_{l_k}, c^{l_k})_{-n}} = 0.$$

Thus $F(\tau_0, c^0) = 0$.

The last assertion follows from the fact that c^0 is then a limit of a sequence in the closed set D. $\square$

Before defining the homotopy, let us introduce some auxiliary notations. Let us denote: $f_n(\tau) := in\tau e^{in\tau}$. By L_n we will denote the linear part of f_n: $L_n(\tau) := f_n'(\widehat{\tau})(\tau - \widehat{\tau}) = (ine^{in\widehat{\tau}} - n^2 e^{in\widehat{\tau}})(\tau - \widehat{\tau})$. By r_n we will denote the nonlinear part: $r_n(\tau) := f_n(\tau) - f_n(\widehat{\tau}) - L_n(\tau)$.

We will define the homotopy on the whole infinite-dimensional space

$$H\colon [0;1] \times [\underline{\tau}; \overline{\tau}] \times X_3 \to \mathbb{C}^{\mathbb{Z}}.$$

It will be a bit more convenient to prove that there are no zeros on the boundary for such a homotopy, and later use the Galerkin projections of it to deform F_l (that is the Galerkin projection of F — see introduction). As written in the introduction, the homotopy is a linear deformation of F to a nearly linear function G:

$$H(h, \tau, x) := hF(\tau, \widehat{c} + x) + (1 - h)G(\tau, x)$$

where G on the n-th coefficient is equal to:

$$G(\tau, x)_n := \begin{cases} (f_n(\tau) + K)x_n & \text{for } n \neq \pm 1, \\ (f_n(\widehat{\tau}) + K + \gamma_{1,0} + \gamma_{1,2n})x_n + L(\tau)\widehat{c}_n & \text{for } n = \pm 1. \end{cases}$$

Where $\gamma_{p,j}$ is from Lemma 3.6. The $x_{\pm 1} \in \mathbb{R}$ (from the definition of X_3), thus for $n = \pm 1$ we have $x_{n-2n} = \overline{x_n} = x_n$, and the term $\gamma_{1,2n} x_{n-2n}$ that appears in the definition of F is in fact a linear term with respect to x_n.

The G is not strictly a linearization of F, as it does not contain all the linear terms and it is not linear with respect to τ. However, it contains the most important terms — the rest will be shown to be small compared to them.

One can write explicit formulas for H. For $n \neq \pm 1$ we have:

$$H(h,\tau,x)_n := (f_n(\tau)+K)x_n + h\biggl((f_n(\tau)+K)\widehat{c}_n + K\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)!}((x+\widehat{c})^{*(2k+1)})_n\biggr)$$

and, for $n = \pm 1$,

$$\begin{aligned} H(t,\tau,x)_n :=& (f_n(\widehat{\tau}) + K + \gamma_{1,0} + \gamma_{1,2n})x_n + L_n(\tau)\widehat{c}_n \\ &+ h\biggl((f_n(\widehat{\tau})+K)\widehat{c}_n - \frac{K}{3!}(\widehat{c}^{*3})_n + \frac{K}{5!}(\widehat{c}^{*5})_n - \frac{K}{7!}(\widehat{c}^{*7})_n\biggr) \\ &+ h\sum_{-6\cdot 5\le j\le 6\cdot 5, j\neq 0,2n}\gamma_{1,j}x_{n-j} + h\sum_{p=2}^{7}\sum_{j=-6\cdot 5}^{6\cdot 5}\gamma_{p,j}x_{n-j}^{*p} \\ &+ hr_n(\tau)\widehat{c}_n + h(f_n(\tau) - f_n(\widehat{\tau}))x_n + hR(\widehat{c}+x). \end{aligned}$$

Where R is a short notation for:

$$R(y) := K\sum_{k=4}^{\infty}\frac{(-1)^k}{(2k+1)!}y^{*(2k+1)}.$$

To show that any Galerkin projection of H does not have a zero on the boundary of $[\underline{\tau};\overline{\tau}]\times P_l(X_3)$, it is enough to show that H does not have a zero on the boundary. More exactly:

THEOREM 4.2. *Let $x \in X_3$, $\tau \in [\underline{\tau},\overline{\tau}]$. Let x be such that there exists n such that $|x_n| = \beta_2/(|n|+1)^2$ or let $\tau \in \{\underline{\tau},\overline{\tau}\}$. Then $H(h,\tau,x) \neq 0$ for all $h \in [0;1]$.*

PROOF. The proof is computer assisted — the calculations are done by the program. The computation is as follow.

First, let us note that if $|x_n| = \beta_2/(|n|+1)^2$ for $n < 0$, then also $|x_{-n}| = |\overline{x_n}| = \beta_2/(|-n|+1)^2$, where $-n > 0$. Thus, it is enough to consider this condition for $n \geq 0$ (and the case $\tau \in \{\underline{\tau},\overline{\tau}\}$). We do it in two steps.

Step 1. Let us assume that $|x_n| = \beta_2/(|n|+1)^2$ for some $n \geq 0$, $n \neq 1$. We will show that $H(h,\tau,x)_n \neq 0$ (what obviously implies $H(h,\tau,x) \neq 0$). We have:

$$\begin{aligned} &|H(h,\tau,x)_n| \\ &= \biggl|(f_n(\tau)+K)x_n + h\biggl((f_n(\tau)+K)\widehat{c}_n + K\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)!}((x+\widehat{c})^{*(2k+1)})_n\biggr)\biggr| \\ &\geq |(f_n(\tau)+K)x_n| - \biggl|\biggl((f_n(\tau)+K)\widehat{c}_n + K\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)!}((x+\widehat{c})^{*(2k+1)})_n\biggr)\biggr|. \end{aligned}$$

Thus, it is enough to show:

$$|(f_n(\tau)+K)x_n| > \biggl|(f_n(\tau)+K)\widehat{c}_n + K\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)!}((x+\widehat{c})^{*(2k+1)})_n\biggr|.$$

To prove it, we will use the estimates from the previous section to compute a lower bound of the LHS and an upper bound of the RHS. As it will require a lot of computations, we will use the program to compute the results. To overcome the problem of computers being able to represent naturally only a finite subset of $\mathbb{R}$, we will use the interval arithmetics. It will compute some small intervals where the mathematically strict results lies. This will be enough to be able to prove the inequality — if the left bound of LHS estimate interval will be bigger than the right bound of the RHS estimate interval, then the inequality will be proved.

The LHS will be estimated by:

$$|(f_n(\tau)+K)x_n| \geq ||f_n(\tau)|-K|\frac{\beta_2}{(|n|+1)^2} = |n\tau - K|\frac{\beta_2}{(|n|+1)^2}. \tag{4.1}$$

The τ can be any number from the interval $[\underline{\tau};\overline{\tau}]$. We can take advantage of the interval arithmetics and substitute the whole interval as τ. Hence, for any specified n, this estimate is computable by a program. We will use it for every $n < 225$ to obtain an interval containing a mathematically strict lower bound for the LHS.

Of course, our program cannot compute the bounds for each $n \in \mathbb{N}$ separately, thus for $n \geq 225$, we want to prove all the inequalities in some finite computations. Thus, we have two cases:

(a) For $n \geq 225$. Let us multiply the estimate for LHS by $(n+1)^2$. We have

$$|(f_n(\tau)+K)x_n|(n+1)^2 \geq |n\tau - K|\beta_2 \geq (225\tau - K)\beta_2$$

what, after substituting $[\underline{\tau};\overline{\tau}]$ for τ, gives us an estimate independent of n.

On the RHS, we have $(f_n(\tau)+K)\widehat{c}_n = 0$, because $\widehat{c}_n = 0$ for $n > 225$. Thus, we have only the sum

$$K\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)!}((x+\widehat{c})^{*(2k+1)})_n.$$

We have:

$$\begin{aligned}&\left|K\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)!}((x+\widehat{c})^{*(2k+1)})_n\right| \\ &\quad\leq \sum_{k=1}^{21}\frac{K}{(2k+1)!}|((x+\widehat{c})^{*(2k+1)})_n| + \left|K\sum_{k=22}^{\infty}\frac{(-1)^k}{(2k+1)!}((x+\widehat{c})^{*(2k+1)})_n\right|.\end{aligned}$$

The terms for $k = 1, \ldots, 21$ are estimated from Lemma 3.5 (with $N = 255$). For the sum for $k \geq 22$, we will use Lemma 3.4.

Both Lemmas 3.5 and 3.4 gives us estimates that after a multiplication by $(n+1)^2$ are independent of n. Of course, $|\mathrm{LHS}| > |\mathrm{RHS}|$ is equivalent to

$|\mathrm{LHS}|(n+1)^2 > |\mathrm{RHS}|(n+1)^2$, and the latter can be checked for every $n > 255$ by the program by checking just one inequality.

The program checks that inequality, and it is satisfied.

(b) For $n < 225$. In this case, the program computes a separate estimate for each $n \in \{0, 2, 3 \ldots, 224\}$ and checks that the inequalities are satisfied. We will also have to use some more sophisticated estimates.

Let us use Lemma 3.6 to group the terms of the RHS:

$$\begin{aligned}
&\left|(f_n(\tau) \ + K)\widehat{c}_n + K\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)!}((x+\widehat{c})^{*(2k+1)})_n\right| \\
&\quad= \bigg|\left((f_n(\tau)+K)\widehat{c}_n - \frac{K}{3!}(\widehat{c}^{*3})_n + \frac{K}{5!}(\widehat{c}^{*5})_n - \frac{K}{7!}(\widehat{c}^{*7})_n\right) \\
&\qquad + \sum_{p=1}^{7}\sum_{j=-6l}^{6l}\gamma_{p,j}x^{*k}_{n-j} + R(\widehat{c}+x)\bigg| \\
&\quad\le \left|(f_n(\tau)+K)\widehat{c}_n - \frac{K}{3!}(\widehat{c}^{*3})_n + \frac{K}{5!}(\widehat{c}^{*5})_n - \frac{K}{7!}(\widehat{c}^{*7})_n\right| \\
&\qquad + \left|\sum_{p=1}^{7}\sum_{j=-6l}^{6l}\gamma_{p,j}x^{*k}_{n-j}\right| + |R(\widehat{c}+x)| \\
&\quad\le |f_n(\tau) - f_n(\widehat{\tau})||\widehat{c}_n| + \left|(f_n(\widehat{\tau})+K)\widehat{c}_n - \frac{K}{3!}(\widehat{c}^{*3})_n + \frac{K}{5!}(\widehat{c}^{*5})_n - \frac{K}{7!}(\widehat{c}^{*7})_n\right| \\
&\qquad + \sum_{p=1}^{7}\sum_{j=-6l}^{6l}|\gamma_{p,j}|\frac{\beta_2(10\beta_2)^{p-1}}{(|n-j|+1)^2} + \sum_{k=4}^{21}\frac{K}{(2k+1)!}|((x+\widehat{c})^{*(2k+1)})_n| \\
&\qquad + \left|K\sum_{k=22}^{\infty}\frac{(-1)^k}{(2k+1)!}((x+\widehat{c})^{*(2k+1)})_n\right|.
\end{aligned}$$

For the first term, we will use the estimate:

$$\begin{aligned}
(4.2)\qquad |(f_n(\tau) - f_n(\widehat{\tau}))| &\le \max_{\tau\in[\underline{\tau},\overline{\tau}]}|f'(\tau)(\tau-\widehat{\tau})| \\
&\le \max_{\tau\in[\underline{\tau},\overline{\tau}]}|ine^{in\tau} - n^2\tau e^{in\tau}|\cdot|\Delta\tau| \le \max_{\tau\in[\underline{\tau},\overline{\tau}]}(n+n^2\tau)|\Delta\tau|.
\end{aligned}$$

To compute the maximum, it is enough to substitute $[\underline{\tau},\overline{\tau}]$ as τ and use the interval arithmetics — if we compute this expression for all possible τ, then the result interval will also contain the value of the expression for the maximal τ. Thus, the mathematically strict result will be in the interval.

The fact that this estimate grows quickly with n is irrelevant, as for $n > 5$ we have $\widehat{c}_n = 0$.

The $|(f_n(\widehat{\tau})+K)\widehat{c}_n - (K/3!)(\widehat{c}^{*3})_n + (K/5!)(\widehat{c}^{*5})_n - (K/7!)(\widehat{c}^{*7})_n|$ can be directly computed and is small as it is the numerical solution that is close to zero.

The $\sum_{p=1}^{7}\sum_{j=-6l}^{6l}|\gamma_{p,j}|(\beta_2(10\beta_2)^{p-1}/(|n-j|+1)^2)$ can be directly computed and is small for $p>1$ as $10\beta_2$ is small. The terms for $p=1$ happens to be small and they are not a problem for the inequality to hold (if they were not, we could try to change the coordinates to diagonalize the linear part). We estimate $(K/(2k+1)!)|((x+\widehat{c})^{*(2k+1)})_n|$ for $k\in\{4,\dots,21\}$ from Lemma 3.3. To estimate $|K\sum_{k=22}^{\infty}((-1)^k/(2k+1)!)((x+\widehat{c})^{*(2k+1)})_n|$, we use Lemma 3.4. In both estimates we have $1/(2k+1)!$ with k large enough to make the result small.

Thus, we see that, unless the $\gamma_{1,\cdot}$ are big, the estimate of the RHS upper bound should be small. And the program checks that they are — it computes this upper bound of the RHS and the lower bound of the LHS from equation (4.1), compares them and for each n in $\{0,2,3,\dots,224\}$ finds that the inequality holds. This ends the case for $n\neq 1$.

n	LHS	RHS
0	$4.1792\cdot 10^{-7}$	$0.1687\cdot 10^{-7}$
2	$0.4474\cdot 10^{-7}$	$0.0958\cdot 10^{-7}$
3	$0.5080\cdot 10^{-7}$	$0.0827\cdot 10^{-7}$
4	$0.4893\cdot 10^{-7}$	$0.0160\cdot 10^{-7}$
5	$0.4537\cdot 10^{-7}$	$0.0118\cdot 10^{-7}$
6	$0.4171\cdot 10^{-7}$	$0.0070\cdot 10^{-7}$
7	$0.3834\cdot 10^{-7}$	$0.3465\cdot 10^{-7}$
8	$0.3536\cdot 10^{-7}$	$0.0040\cdot 10^{-7}$
9	$0.3274\cdot 10^{-7}$	$0.0036\cdot 10^{-7}$

TABLE 2. Estimates of the LHS and the RHS for small n. The estimates obtained by the program. The RHS is closest to the LHS for $n=7$, because we have arbitrarily chosen $\widehat{c}_7=0$. That makes the term $|(f_n(\tau)+K)\widehat{c}_n-(K/3!)(\widehat{c}^{*3})_n+(K/5!)(\widehat{c}^{*5})_n-(K/7!)(\widehat{c}^{*7})_n|$ for $n=7$ approximately equal $0.3411\cdot 10^{-7}$.

Step 2. We have two cases left: $\tau\in\{\underline{\tau},\overline{\tau}\}$ and $|x_1|=\beta_2/4$. We have two variables left but only one equation — the equation for $n=1$. However, this is a complex equation and the variables are real, so we will be able to show that in both cases $H(\tau,x)\neq 0$. Let us denote:

$$L_\tau := L_1(\tau)\widehat{c}_1,\qquad L_x := (f_n(\widehat{\tau})+K+\gamma_{1,0}+\gamma_{1,2})x_1,$$

$$N := (f_n(\widehat{\tau})+K)\widehat{c}_n-\frac{K}{3!}(\widehat{c}^{*3})_n+\frac{K}{5!}(\widehat{c}^{*5})_n-\frac{K}{7!}(\widehat{c}^{*7})_n$$
$$+\sum_{-6l\le j\le 6l, j\neq 0,2}\gamma_{p,j}x^{*k}_{n-j}+\sum_{p=2}^{7}\sum_{j=-6l}^{6l}\gamma_{p,j}x^{*k}_{n-j}+R(c)+r_n(\tau)\widehat{c}_n+(f(\tau)-f(\widehat{\tau}))x_n.$$

The L_τ, L_x, N depend on τ, x, but to make the notation short, we skip it. They are chosen such that $G(\tau,x)_1 = L_\tau + L_x$, $F(\tau,x)_1 = L_\tau + L_x + N$ and $H(h,\tau,x)_1 = L_\tau + L_x + hN$.

If $\tau \in \{\underline{\tau}, \overline{\tau}\}$, then we will show that:

$$|L_\tau| > |L_x| + |N| \tag{4.3}$$

what will give $H(h,\tau,x)_1 \neq 0$. The $|L_\tau|$ can be computed:

$$|L_1(\tau)\widehat{c}_1| = |in - n^2| \cdot \Delta\tau \cdot |\widehat{c}_n||_{n=1} = \sqrt{2}\Delta\tau \cdot |\widehat{c}_1|.$$

We estimate the terms in $|N|$ like for $n \neq 1$, with the exception of the new terms: $|(f(\tau) - f(\widehat{\tau}))x_n|$ and $|r_n(\tau)\widehat{c}_n|$. For the first one we use inequality (4.2). For the second we use the estimate:

$$\begin{aligned}|r_n(\tau)\widehat{c}_n| &\leq \max_{\tau\in[\underline{\tau};\overline{\tau}]} \left|\frac{1}{2} f_1''(\tau)(\tau - \widehat{\tau})^2\right| \cdot |\widehat{c}_n| \\ &= \frac{|\widehat{c}_n|}{2} \max_{\tau\in[\underline{\tau};\overline{\tau}]} |-(2n^2 + in^3\tau)e^{in\tau}| = \frac{|\widehat{c}_n|}{2} \max_{\tau\in[\underline{\tau};\overline{\tau}]} |2 + i\tau|.\end{aligned}$$

Like for equation (4.2), we compute the maximum by substituting $[\underline{\tau};\overline{\tau}]$ for τ and computing all the possible values.

Having all these estimates our program checks that the inequality (4.3) is satisfied.

In the case $|x_1| = \beta_2/4$, the inequality $|L_x| > |L_\tau| + |N|$ is obviously false. To prove that there is no zero, we will need to use the fact that $\tau \in \mathbb{R}$ and $x_1 \in \mathbb{R}$. On Figure 1, we sketch how the sets of possible values of L_τ and $L_x + N$ look like on the complex plane. We see that they should not intersect, i.e. $0 \notin L_\tau - (L_x + N)$. That, for an estimate that is symmetric with respect to 0, is equivalent to $0 \notin L_\tau + (L_x + N)$.

Formally, we will show that

$$\begin{aligned}&\left\{ \tan[\arg(L_\tau)] : (\tau,x) \in [\underline{\tau},\overline{\tau}] \times X_3,\ x_1 = \pm\frac{\beta_2}{4}\right\} \\ &\qquad\qquad \cap \left\{ \tan[\arg(L_x + N)] : (\tau,x) \in [\underline{\tau},\overline{\tau}] \times X_3,\ x_1 = \pm\frac{\beta_2}{4}\right\} = \emptyset\end{aligned}$$

(where arg is the complex number argument). This implies that if we take an a from the first set and a b from the second, then $a+b \neq 0$. Thus $0 \neq L_\tau + L_x + N = H(h,\tau,x)_1$.

The $\arg(L_\tau)$ is easy to compute as this is $\tan\arg((ie^{i\widehat{\tau}} - e^{i\widehat{\tau}})\widehat{c}_1)$, and $\tan\arg z$ for a complex number z can be computed as $\Im z/\Re z$. To estimate the other set, we will use the estimate for N from the previous point. Let us denote by λ the right end of the interval containing the upper bound for N. Then we have $|N| \leq \lambda$. Thus, $L_x + N \in \pm(f_n(\widehat{\tau}) + K + \gamma_{1,0} + \gamma_{1,2})(\beta_2/4) + [-\lambda;\lambda] + [-\lambda;\lambda]i$ and using the interval arithmetics we can find $\tan\arg(L_x + N)$.

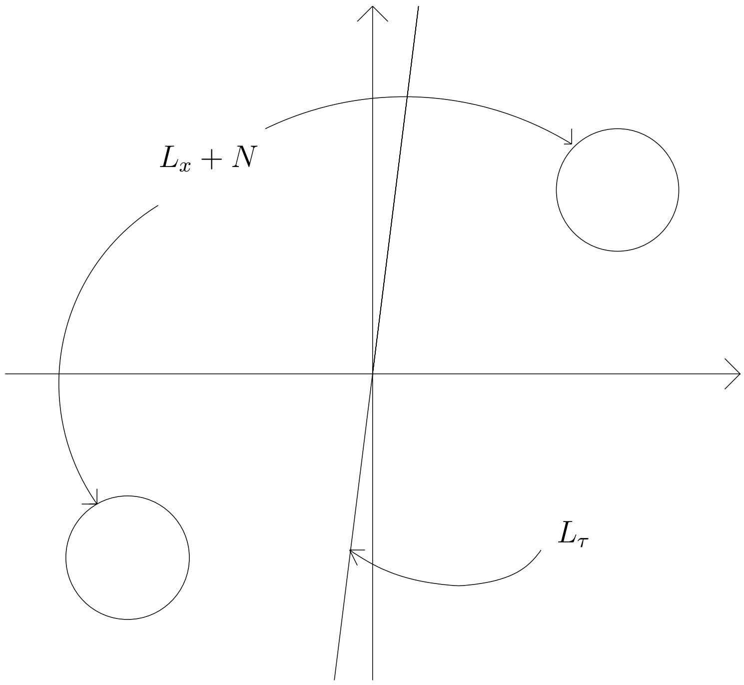

FIGURE 1. The possible values of L_τ and $L_x + N$

The program checks that these two sets are disjoint, and this ends the proof of the theorem. □

To finish the proof of the existence of the orbit, let us define a second homotopy:

$$H^L(h,\tau,x) := \begin{cases} (f_n(\widehat{\tau}+h(\tau-\widehat{\tau}))+K)x_n & \text{for } n \neq \pm 1, \\ (f_n(\widehat{\tau})+K+\gamma_{1,0}+\gamma_{1,2n})x_n + L(\tau)c_1 & \text{for } n = \pm 1. \end{cases}$$

It deforms G into:

$$G^L(h,\tau,x) := \begin{cases} (f_n(\widehat{\tau})+K)x_n & \text{for } n \neq \pm 1, \\ (f_n(\widehat{\tau})+K+\gamma_{1,0}+\gamma_{1,2n})x_n + L(\tau)c_1 & \text{for } n = \pm 1. \end{cases}$$

LEMMA 4.3. *If* $(\tau,x) \in [\underline{\tau},\overline{\tau}] \times X_3$ *such that* $\tau \in \{\underline{\tau},\overline{\tau}\}$ *or there exists* n *such that* $|x_n| = \beta_2/(|n|+1)^2$ *then* $H^L(h,x,\tau) \neq 0$.

PROOF. Let h, x, τ be such that $H^L(h,x,\tau) = 0$. Let $n \neq \pm 1$. The $\widehat{\tau}+h(\tau-\widehat{\tau}) \in [\underline{\tau};\overline{\tau}]$ thus $|f_n(\widehat{\tau}+h(\tau-\widehat{\tau}))+K|$ can be estimated as in equation (4.1). For each such value we have proven that it is strictly greater than an RHS≥ 0. Hence $|f_n(\widehat{\tau}+h(\tau-\widehat{\tau}))+K| > 0$ and, if $H^L(h,\tau,x) = 0$, then $x_n = 0$.

Thus we have $x_n = 0$ for each $n \neq \pm 1$. But then $H^L(h,\tau,x) = G(\tau,x) = H(0,\tau,x)$ and H has no zeros on the boundary. □

OBSERVATION 4.4. *The Galerkin projection of H, i.e. $H_l(\tau, y_0, \dots, y_l) := P_l H(\tau, Q_l(y_0, \dots, y_l))$ (for $(y_0, \dots, y_n) \in \mathbb{R} \times \mathbb{R} \times \mathbb{C}^{n-1}$) is a homotopy from $F^l(\tau, c + \cdot\,)$ to the projection of G. Analogously, the projection of H^L is a homotopy from G to G^L. There are no zeros on the boundaries for these homotopies.*

OBSERVATION 4.5. *The degree* $\deg(F_l, [\underline{\tau}, \overline{\tau}] \times P_l(X_3), 0)$ *is well defined.*

PROOF. The homotopy H_l for $h = 1$ have no zeros on the boundary and this is F_l. □

LEMMA 4.6. $\deg(F_l, [\underline{\tau}, \overline{\tau}] \times P_l(X_3), 0) \neq 0$

PROOF. We know that the degree of F_l is equal to the degree of the Galerkin projection of G^L — let us denote it by G_l^L. Function G_l^L is linear. If the determinant of the differential were zero then there would be a zero on the boundary of any neighbourhood of $(\widehat{\tau}, 0)$. Hence, the determinant is non-zero. That means that the degree is ± 1, i.e. non-zero. □

THEOREM 4.7. *Equation* (1.1) *has a periodic solution for some* $\tau \in [\underline{\tau}, \overline{\tau}]$ *whose Fourier coefficients are in the set* X_3.

PROOF. From Lemma 4.6 we know that the local Brouwer degree is non-zero, thus each F_l has a zero in X_3. From Lemma 4.1 we have that F has a zero in X_3. From Theorem 2.6 we obtain that in X_3 there is a solution of the equation. □

5. Conclusions

In this paper, I rigorously proved the existence of a periodic orbit for $K = 1.6$. I was not able to show that the period is 4, although the numerical simulations suggests that. We needed the τ as a variable in the proof for the image and the domain to have the same dimension.

The value $K = 1.6$ has been chosen, because it is easiest to find the $\widehat{c}_n$ values for an attracting orbit. However, using the Newton method, it should be possible to find an approximation of an orbit which is not attracting (from the numerical simulations, it seems to happen for $K > 5.11$). For larger K, the values of $\widehat{c}_n$ may not decrease as fast, so we may need to diagonalize the first coefficients of F_l, as mentioned in the proof.

Of course, the sinus function is periodic, so there exist infinitely many such orbit that differ by $2k\pi$ ($k \in \mathbb{Z}$). As mentioned, for $K > 5.11$ it seems that these periodic orbits stop to be attracting. The orbits from numerical simulations jump by $\pm 2\pi$, from one periodic orbit to another. This suggests heteroclinic connections and chaos but proving it would require some new ideas.

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References

[1] L. Cesari, *Functional analysis and Galerkin methods*, Michigan Math. J. **11** (1964), 383–414.

[2] O. Diekmann, S. van Gils, S. Lunel and H. Walther, *Delay Equations*, Springer, 1995.

[3] B. Lani-Wayda and R. Srzednicki, *A generalized Lefschetz fixed point theorem and symbolic dynamics in delay equations*, Ergodic Theory Dynam. Systems **22** (2002), 1215–1232.

[4] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Springer, 1983.

[5] W. Wischert, W. Wunderlin, A. Pelster, M. Olivier and J. Groslambert, *Delay-induced instabilities in nonlinear feedback systems*, Phys. Rev. E **49** (1994), 203–219.

[6] P. Zgliczyński, *Rigorous numerics for dissipative partial differential equations* II. *Periodic orbit for the Kuramoto–Sivashinsky equation*, Found. Comput. Math. (2004), 157–195.

[7] P. Zgliczyński and K. Mischaikow, *Rigorous numerics for partial differential equations: the Kuramoto–Sivashinsky equation*, Found. Comput. Math. **1** (2001), 255–288.

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