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FIXED POINT RESULTS FOR GENERALIZED φ-CONTRACTION ON A SET WITH TWO METRICS

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Abstract. The aim of this paper is to present fixed point theorems for multivalued operators $T: X \to P(X)$, on a nonempty set X with two metrics d and ϱ, satisfying the following generalized φ-contraction condition:

$$H_\varrho(T(x), T(y)) \leq \varphi(M^T(x,y)), \quad \text{for every } x, y \in X,$$

where

$$M^T(x,y) := \max\{\varrho(x,y), D_\varrho(x,T(x)), D_\varrho(y,T(y)), 2^{-1}[D_\varrho(x,T(y)) + D_\varrho(y,T(x))]\}.$$

1. Introduction

In this paper we will give some local and global fixed point results for multivalued generalized $\varphi-$contractions on a set with two metrics. The multivalued operator $T: Y \to P_{\rm cl}(X)$, $Y \subset X$ will satisfy a generalized φ-contraction condition of the following type:

$$H_\varrho(T(x), T(y)) \leq \varphi(M^T(x,y)), \quad \text{for every } x, y \in X,$$

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where

$$M^T(x,y) := \max\{\varrho(x,y), D_\varrho(x,T(x)), D_\varrho(y,T(y)), \\ 2^{-1}[D_\varrho(x,T(y)) + D_\varrho(y,T(x))]\}.$$

Our results extend and generalize some similar theorems given by Agarwal–Dshalalow–O'Regan in [1] for the case of a space endowed with a metric, as well as, the results given in Lazăr–O'Regan–Petruşel [3] for the case of Ćirić type multivalued operator.

2. Notations

Let us consider the following families of subsets of a metric space (X,ϱ):

$$\begin{aligned} P(X) &:= \{Y \in \mathcal{P}(X) \mid Y \neq \emptyset\}; \\ P_{\rm b}(X) &:= \{Y \in P(X) \mid Y \text{ is bounded}\}; \\ P_{\rm cl}(X) &:= \{Y \in P(X) \mid Y \text{ is closed}\}. \end{aligned}$$

If d is another metric on X we will denote by $\overline{B}^d_\varrho(x_0,r)$ the closure of $B_\varrho(x_0,r)$ in (X,d), where $B_\varrho(x_0,r) := \{x \in X \mid \varrho(x_0,x) < r\}$. Let us define the gap functional between the sets A and B in the metric space (X,ϱ) as:

$$D_\varrho\colon P(X)\times P(X) \to \mathbb{R}_+ \cup \{\infty\}, \quad D_\varrho(A,B) = \inf\{\varrho(a,b) \mid a \in A,\ b \in B\}$$

(in particular, if $x_0 \in X$ then $D_\varrho(x_0,B) := D_\varrho(\{x_0\},B)$) and the (generalized) Pompeiu–Hausdorff functional as:

$$H_\varrho\colon P(X)\times P(X) \to \mathbb{R}_+ \cup \{\infty\}, \\ H_\varrho(A,B) = \max\Big\{\sup_{a\in A} D_\varrho(a,B), \sup_{b\in B} D_\varrho(A,b)\Big\}.$$

Let (X,ϱ) be a metric space. If $T\colon X \to P(X)$ is a multivalued operator, then $x \in X$ is called fixed point for T if and only if $x \in T(x)$. The set $\operatorname{Fix} T := \{x \in X \mid x \in T(x)\}$ is called the fixed point set of T, while $\operatorname{SFix} T = \{x \in X \mid \{x\} = Tx\}$ is called the strict fixed point set of T. The operator T is closed if its graphic is a closed set in $X \times X$. For $x, y \in X$ let us denote:

$$M^T_\varrho(x,y) = \max\{\varrho(x,y), D_\varrho(x,T(x)), D_\varrho(y,T(y)), \\ 2^{-1}[D_\varrho(x,T(y)) + D_\varrho(y,T(x))]\}.$$

3. Main results

The starting point of our research was the recently given result, a multivalued version of Maia's fixed point theorem for multivalued contractions, in [5] by A. Petruşel and I. A. Rus.

Theorem 3.1 (A. Petruşel, I. A. Rus [5]). *Let X be a nonempty set, d and ϱ two metrics on X and $T\colon X \to P(X)$ be a multivalued operator. We suppose that:*

(i) *(X,d) is a complete metric space;*
(ii) *there exists $c > 0$ such that $d(x,y) \le c \cdot \varrho(x,y)$, for each $x, y \in X$;*
(iii) *$T\colon (X,d) \to (P(X), H_d)$ is closed;*
(iv) *there exists $\alpha \in [0,1[$ such that $H_\varrho(T(x),T(y)) \le \alpha \cdot \varrho(x,y)$, for each $x, y \in X$.*

Then we have:

(a) $\operatorname{Fix} T \neq \emptyset$;
(b) *for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that:*
(b1) *$x_0 = x$, $x_1 = y$;*
(b2) *$x_{n+1} \in T(x_n)$, $n \in \mathbb{N}$;*
(b3) *$x_n \xrightarrow{d} x^* \in T(x^*)$, as $n \to \infty$.*

Our first main result is a local version of Ćirić's theorem ([2]) for generalized φ-contractions on a set with two metrics.

Theorem 3.2. *Let X be a nonempty set, $x_0 \in X$ and $r > 0$. Suppose that d and ϱ are two metrics on X and $F\colon \overline{B}^d_\varrho(x_0,r) \to P(X)$ is a multivalued operator. We suppose that:*

(a) *(X,d) is a complete metric space;*
(b) *there exists $c > 0$ such that $d(x,y) \le c \cdot \varrho(x,y)$, for each $x, y \in X$;*
(c) (c1) *if $d \neq \varrho$ then $F\colon \overline{B}^d_\varrho(x_0,r) \to P(X^d)$ is closed;*
(c2) *if $d = \varrho$ then $F\colon \overline{B}^d_\varrho(x_0,r) \to P_{\mathrm{cl}}(X^d)$;*
(d) *there exists a continuous function $\varphi\colon [0,\infty) \to [0,\infty)$, with $\varphi(t) < t$, for every $t > 0$, $\varphi(0) = 0$ and φ is nondecreasing on $(0,r]$ such that:*

$$H_\varrho(F(x),F(y)) \le \varphi(M^F_\varrho(x,y)), \tag{3.1}$$

for every $x, y \in \overline{B}^d_\varrho(x_0,r)$, with strict inequality if $M^F_\varrho(x,y) \neq 0$.

Also assume that:

$$D_\varrho(x_0, F(x_0)) < r - \varphi(r); \tag{3.2}$$

$$\sum_{i=0}^{\infty} \varphi^i(t) < \infty, \quad \text{for } t \in (0, r - \varphi(r)]; \tag{3.3}$$

$$\sum_{i=1}^{\infty} \varphi^i(r - \varphi(r)) \le \varphi(r). \tag{3.4}$$

Then F has a fixed point.

Proof. If $M_\varrho^F(x,y) = 0$ for some $x, y \in \overline{B}_\varrho^d(x_0, r)$ then by $D_\varrho(x, F(x)) \leq M_\varrho^F(x,y)$ we get that $D_\varrho(x, F(x)) = 0$ and thus $x \in \overline{F(x)}^\varrho \subseteq \overline{F(x)}^d = F(x)$. From (3.2) we may choose $x_1 \in F(x_0)$ with

$$\varrho(x_0, x_1) < r - \varphi(r) \tag{3.5}$$

then $x_1 \in \overline{B}_\varrho^d(x_0, r)$. We may assume $M_\varrho^F(x_0, x_1) \neq 0$, since otherwise x_1 is a fixed point, so the proof is complete. If $M_\varrho^F \neq 0$ then from (3.1) we have that $H_\varrho(F(x_0), F(x_1)) < \varphi(M_\varrho^F(x_0, x_1))$. We may choose $\varepsilon > 0$ with

$$H_\varrho(F(x_0), F(x_1)) + \varepsilon \leq \varphi(M_\varrho^F(x_0, x_1)).$$

Next we choose $x_2 \in F(x_1)$ so that

$$\varrho(x_1, x_2) \leq H_\varrho(F(x_0), F(x_1)) + \varepsilon.$$

It follows that $\varrho(x_1, x_2) \leq \varphi(M_\varrho^F(x_0, x_1))$. We want to show that

$$\varrho(x_1, x_2) \leq \varphi(\varrho(x_0, x_1)). \tag{3.6}$$

We have

$$\begin{aligned}\varrho(x_1, x_2) \leq \varphi(\max\{&\varrho(x_0, x_1), D_\varrho(x_0, F(x_0)), D_\varrho(x_1, F(x_1)),\\ &2^{-1}[D_\varrho(x_0, F(x_1)) + D_\varrho(x_1, F(x_0))]\}).\end{aligned}$$

Let

$$\begin{aligned}\gamma_1 = \max\{&\varrho(x_0, x_1), D_\varrho(x_0, F(x_0)), D_\varrho(x_1, F(x_1)),\\ &2^{-1}[D_\varrho(x_0, F(x_1)) + D_\varrho(x_1, F(x_0))]\}\end{aligned}$$

If $\gamma_1 = \varrho(x_0, x_1)$ then $\varrho(x_1, x_2) \leq \varphi(\varrho(x_0, x_1))$. If $\gamma_1 = D_\varrho(x_0, F(x_0))$ then, since $D_\varrho(x_0, F(x_0)) \leq \varrho(x_0, x_1)$ we have that $\varrho(x_1, x_2) \leq \varphi(\varrho(x_0, x_1))$. If $\gamma_1 = D_\varrho(x_1, F(x_1))$ then if $\gamma_1 \neq 0$, since $x_2 \in F(x_1)$ then $\varrho(x_1, x_2) \leq \varphi(D_\varrho(x_1, F(x_1))) < D_\varrho(x_1, F(x_1)) \leq \varrho(x_1, x_2)$ which is a contradiction. Then we have that $\gamma_1 = 0 = D_\varrho(x_1, F(x_1))$. Thus $\varrho(x_1, x_2) \leq \varphi(\gamma_1) = \varphi(0) = 0$ and (3.5) is true. If $\gamma_1 = 2^{-1}[D_\varrho(x_0, F(x_1)) + D_\varrho(x_1, F(x_0))]$ then:

- if $\gamma_1 = 0$ then $\varrho(x_1, x_2) \leq \varphi(\gamma_1) = \varphi(0) = 0$ implies that (3.6) is true;
- if $\gamma_1 \neq 0$ then

$$\begin{aligned}\varrho(x_1, x_2) &\leq \varphi(\gamma_1) < \gamma_1 = \frac{1}{2}[D_\varrho(x_0, F(x_1)) + D_\varrho(x_1, F(x_0))]\\ &\leq \frac{1}{2}\varrho(x_0, x_2) \leq \frac{1}{2}\varrho(x_0, x_1) + \frac{1}{2}\varrho(x_1, x_2) \Rightarrow \varrho(x_1, x_2) < \varrho(x_0, x_1).\end{aligned}$$

Then

$$\begin{aligned}\gamma_1 &= \frac{1}{2}[D_\varrho(x_0, F(x_1)) + D_\varrho(x_1, F(x_0))] \leq \frac{1}{2}\varrho(x_0, x_2)\\ &\leq \frac{1}{2}\varrho(x_0, x_1) + \frac{1}{2}(\varrho(x_1, x_2)) < \frac{1}{2}\varrho(x_0, x_1) + \frac{1}{2}\varrho(x_0, x_1) = \varrho(x_0, x_1)\end{aligned}$$

which is a contradiction with the definition of γ_1.

We have that (3.6) is true in all cases. Notice that $x_2 \in \overline{B}^d_\varrho(x_0, r)$ since

$$\begin{aligned}\varrho(x_0, x_2) &\le \varrho(x_0, x_1) + \varrho(x_1, x_2) \le \varrho(x_0, x_1) + \varphi(\varrho(x_0, x_1))\\ &< [r - \varphi(r)] + \varphi(r - \varphi(r)) \le r - \phi(r) + \varphi(r) = r.\end{aligned}$$

We may assume that $M^F_\varrho(x_1, x_2) \neq 0$ since otherwise we are finished. Now choose $\delta > 0$ such that

$$H(F(x_1), F(x_2)) + \delta \le \varphi(M^F_\varrho(x_1, x_2))$$

and choose $x_3 \in F(x_2)$ so that

$$\varrho(x_2, x_3) \le H(F(x_1), F(x_2)) + \delta.$$

Thus $\varrho(x_2, x_3) \le \varphi(M^F_\varrho(x_1, x_2))$. We now show that

$$\varrho(x_2, x_3) \le \varphi(\varrho(x_1, x_2)) \le \varphi^2(\varrho(x_0, x_1)) \tag{3.7}$$

Indeed, we can notice that

$$\begin{aligned}\varrho(x_2, x_3) \le \varphi(\max\{&\varrho(x_1, x_2), D_\varrho(x_1, F(x_1)), D_\varrho(x_2, F(x_2)),\\ &2^{-1}[D_\varrho(x_1, F(x_2)) + D_\varrho(x_2, F(x_1))]\}).\end{aligned}$$

Let

$$\begin{aligned}\gamma_2 = \max\{&\varrho(x_1, x_2), D_\varrho(x_1, F(x_1)), D_\varrho(x_2, F(x_2)),\\ &2^{-1}[D_\varrho(x_1, F(x_2)) + D_\varrho(x_2, F(x_1))]\}.\end{aligned}$$

If $\gamma_2 = \varrho(x_1, x_2)$ then $\varrho(x_2, x_3) \le \varphi(\varrho(x_1, x_2)) \le \varphi^2(\varrho(x_0, x_1))$, so (3.7) is true. If $\gamma_2 = D_\varrho(x_1, F(x_1))$ then, since $D_\varrho(x_1, F(x_1)) \le \varrho(x_1, x_2)$, (3.7) is true again. If $\gamma_2 = D_\varrho(x_2, F(x_2))$ and $\gamma_2 \neq 0$ then, since $x_3 \in F(x_2)$, we will have the following inequalities

$$\varrho(x_2, x_3) \le \varphi(\gamma_2) < \gamma_2 = D_\varrho(x_2, F(x_2)) \le \varrho(x_2, x_3),$$

which is a contradiction. Thus in this case $\gamma_2 = D_\varrho(x_2, F(x_2)) = 0$ so $\varrho(x_2, x_3) \le \varphi(\gamma_2) = \varphi(0) = 0$ and (3.7) is true. Suppose that $\gamma_2 = 2^{-1}[D_\varrho(x_1, F(x_2)) + D_\varrho(x_2, F(x_1))]$. If $\gamma_2 = 0$ then $\varrho(x_2, x_3) \le \varphi(\gamma_2) = \varphi(0) = 0$ thus (3.7) is immediate.

If $\gamma_2 \neq 0$ then

$$\begin{aligned}\varrho(x_2, x_3) &\le \varphi(\gamma_2) < \gamma_2 = \frac{1}{2}[D_\varrho(x_1, F(x_2)) + D_\varrho(x_2, F(x_1))]\\ &\le \varrho(x_1, x_3) \le \frac{1}{2}\varrho(x_1, x_2) + \frac{1}{2}\varrho(x_2, x_3)\end{aligned}$$

so $2^{-1}\varrho(x_2,x_3) \le 2^{-1}\varrho(x_1,x_2)$. Thus

$$\begin{aligned}\gamma_2 &= \frac{1}{2}[D_\varrho(x_1,F(x_2)) + D_\varrho(x_2,F(x_1))]\\ &\le \frac{1}{2}\varrho(x_1,x_3) \le \frac{1}{2}\varrho(x_1,x_2) + \frac{1}{2}\varrho(x_2,x_3) < \varrho(x_1,x_2),\end{aligned}$$

which contradicts the definition of γ_2. Thus in all cases (3.7) is true. Notice again that $x_3 \in \overline{B}^d_\varrho(x_0,r)$, since (3.4) implies

$$\begin{aligned}\varrho(x_0,x_3) &\le \varrho(x_0,x_1) + \varrho(x_1,x_2) + \varrho(x_2,x_3)\\ &\le [r-\varphi(r)] + \varphi(\varrho(x_0,x_1)) + \varphi^2(\varrho(x_0,x_1))\\ &< [r-\varphi(r)] + \varphi(r-\varphi(r)) + \varphi^2(r-\varphi(r))\\ &\le r + \bigg[\sum_{i=1}^{\infty}\varphi^j(r-\varphi(r)) - \varphi(r)\bigg] \le r.\end{aligned}$$

Proceeding inductively we obtain $x_{n+1} \in F(x_n)$ for $n \in \{3,4,\dots\}$ such that

$$\varrho(x_{n+1},x_n) \le \varphi(M^F_\varrho(x_{n-1},x_n)).$$

We assumed without loss of generality that $M^F_\varrho(x_{n-1},x_n) \ne 0$. Thus

$$\varrho(x_n,x_{n+1}) \le \varphi(\varrho(x_{n-1},x_n)) \le \varphi^n(\varrho(x_0,x_1)) \tag{3.8}$$

and $x_{n+1} \in \overline{B}^d_\varrho(x_0,r)$ for $n \in \{3,4,\dots\}$. We want to prove that $\{x_n\}$ is a Cauchy sequence. Notice that (3.8) implies

$$\begin{aligned}\varrho(x_{n+p},x_n) &\le \varrho(x_{n+p},x_{n+p-1}) + \ldots + \varrho(x_{n+1},x_n)\\ &\le \varphi^{n+p-1}(\varrho(x_0,x_1)) + \ldots + \varphi^n(\varrho(x_0,x_1)) \le \sum_{j=n}^{\infty}\varphi^j(\varrho(x_0,x_1)),\end{aligned}$$

thus (3.3) guarantees that $\{x_n\}$ is a ϱ-Cauchy sequence. From (b) we have that $\{x_n\}$ is a d-Cauchy sequence too. Denote by $x \in B^d_\varrho(x_0,r)$ the limit of the sequence. We can now separate two cases:

- if $d \ne \varrho$ we have from (a) and (c1) that $d(x_n,x) \to 0$, as $n \to \infty$, where $x \in \operatorname{Fix} F$. So we have $\operatorname{Fix} F \ne \emptyset$ and the proof is complete.
- if $d = \varrho$ we have that there exists $x \in \overline{B}^d_\varrho(x_0,r)$ with $x_n \to x$ when $n \to \infty$. It only remains to show that $x \in F(x)$.

$$\begin{aligned}D_\varrho(x,F(x)) &\le \varrho(x,x_n) + D_\varrho(x_n,F(x))\\ &\le \varrho(x,x_n) + H_\varrho(F(x_{n-1}),F(x))\\ &\le \varrho(x,x_n) + \varphi(\max\{\varrho(x_1,x_2), D_\varrho(x_1,F(x_1)),\\ &\quad D_\varrho(x_2,F(x_2)), 2^{-1}[D_\varrho(x_1,F(x_2)) + D_\varrho(x_2,F(x_1))]\}).\end{aligned}$$

Since

$$\begin{aligned} D_\varrho(x, F(x_{n-1})) &\leq \varrho(x, x_n) \to 0, \\ D_\varrho(x_{n-1}, F(x_{n-1})) &\leq \varrho(x_{n-1}, x_n) \to 0, \\ |D_\varrho(x_{n-1}, F(x)) - D_\varrho(x, F(x))| &\leq \varrho(x_{n-1}, x) \to 0, \end{aligned}$$

as $n \to \infty$, we get by letting $n \to \infty$ that

$$D_\varrho(x, F(x)) = 0 + \varphi(\max\{0, 0, D_\varrho(x, F(x)), 2^{-1} D_\varrho(x, F(x))\}).$$

Thus $D_\varrho(x, F(x)) = 0$, so $x \in \overline{F(x)} = F(x)$. The proof is now complete. □

Remark 3.3. It is known that if X is a Banach space, then a fixed point theorem for $T: \overline{B}(x_0, r) \to P_{\mathrm{cl}}(X)$ induces domain invariance theorems for the field associated to T (i.e. $G(x) = x - T(x)$) see [4], as well as, homotopy theorems for multivalued operators (see [3]). From this point of view, it is an open question to obtain such consequences for our multivalued generalized φ-contractions. For a homotopy result see Theorem 3.7.

We continue the section with a global version of Ćirić's theorem ([2]) for generalized φ-contractions on a set with two metrics.

Theorem 3.4. *Let X be a nonempty set, $r > 0$. Suppose that d and ϱ are two metrics on X and $F: X \to P(X)$ is a multivalued operator. We suppose that:*

(a) *(X, d) is a complete metric space;*
(b) *there exists $c > 0$ such that $d(x, y) \leq c \cdot \varrho(x, y)$, for each $x, y \in X$;*
(c) *if $d \neq \varrho$ then $F: X^d \to P(X^d)$ is closed;*
if $d = \varrho$ then $F: X^d \to P_{\mathrm{cl}}(X^d)$;
(d) *there exists a continuous function $\varphi: [0, \infty) \to [0, \infty)$, with $\varphi(t) < t$, for every $t > 0$, $\varphi(0) = 0$ and φ is nondecreasing on $(0, r]$ such that:*

$$H_\varrho(F(x), F(y)) \leq \varphi(M_\varrho^F(x, y)), \tag{3.9}$$

for every $x, y \in X$, with strict inequality if $M_\varrho^F(x, y) \neq 0$.

Also assume that:

$$\sum_{i=0}^{\infty} \varphi^i(t) < \infty, \quad \textit{for } t \in (0, r]. \tag{3.10}$$

Then F has a fixed point.

Proof. We claim that we can choose $x_0 \in X$ and $x_1 \in F(x_0)$ such that:

$$\varrho(x_1, x_0) < r. \tag{3.11}$$

If (3.11) is true then, as in the proof of theorem Theorem 3.2, we can choose $x_{n+1} \in F(x_n)$ for $n \in \{1, 2, \dots\}$ with $\varrho(x_n, x_{n+1}) \leq \varphi(M_\varrho^F(x_n, x_{n+1})) \leq$

$\varphi^n(\varrho(x_0, x_1))$. The same reasonings guarantees that $\{x_n\}$ is a d-Cauchy sequence, so there exists $x \in X$ with $d(x_n, x) \to 0$, as $n \to \infty$. Thus in Theorem 3.2 we have that $x \in F(x)$. It remains to show (3.11). If we are in the case when φ is nondecreasing on $(0, \infty)$ then (3.11) is satisfied. We can observe that (3.11) is immediate if we could show

$$\inf_{x \in X} D_\varrho(x, F(x)) = 0 \tag{3.12}$$

since if (3.12) is true then there exists $x \in X$ with $D_\varrho(x, F(x)) < r$, so there exists $y \in F(x)$ with $\varrho(x, y) < r$. Suppose that (3.12) is false, i.e. suppose

$$\inf_{x \in X} D_\varrho(x, F(x)) = \delta. \tag{3.13}$$

Since $\varphi(\delta) < \delta$ and φ is continuous we have that there exists $\varepsilon > 0$ such that

$$\varphi(t) < \delta \quad \text{for } t \in [\delta, \delta + \varepsilon). \tag{3.14}$$

We can choose $v \in X$ such that $\delta \le D_\varrho(v, F(v)) < \delta + \varepsilon$. Then there exists $y \in F(v)$ such that

$$\delta \le \varrho(v, y) < \delta + \varepsilon. \tag{3.15}$$

Thus

$$\begin{aligned} D_\varrho(y, F(y)) \le H_\varrho(F(v), F(y)) \le \varphi(\max\{&\varrho(v, y), D_\varrho(v, F(v)), \\ &D_\varrho(y, F(y)), 2^{-1}[D_\varrho(v, F(y)) + D_\varrho(y, F(v))]\}). \end{aligned}$$

Let

$$\gamma = \max\{\varrho(v, y), D_\varrho(v, F(v)), D_\varrho(y, F(y)), 2^{-1}[D_\varrho(v, F(y)) + D_\varrho(y, F(v))]\}.$$

If $\gamma = \varrho(v, y)$ then (3.14) and (3.15) yields

$$D_\varrho(y, F(y)) \le \varphi(\varrho(v, y)) < \delta.$$

If $\gamma = D_\varrho(v, F(v))$ then (3.14) and (3.15) also yields

$$D_\varrho(y, F(y)) \le \varphi(D_\varrho(v, F(v))) < \delta.$$

If $\gamma = D_\varrho(y, F(y))$ then $\gamma = 0$ since if $\gamma \ne 0$ then

$$D_\varrho(y, F(y)) \le \varphi(D_\varrho(y, F(y))) < D_\varrho(y, F(y)),$$

which is a contradiction.

If $\gamma = 2^{-1}[D_\varrho(v, F(y)) + D_\varrho(y, F(v))]$ and $\gamma \ne 0$ then

$$\begin{aligned} D_\varrho(y, F(y)) &\le \varphi(\gamma) = \gamma = \frac{1}{2}[D_\varrho(v, F(y)) + D_\varrho(y, F(v))] \\ &\le 2^{-1}[\varrho(v, y) + D_\varrho(y, F(y)) + 0] = 2^{-1}[\varrho(v, y) + D_\varrho(y, F(y))] \end{aligned}$$

so $2^{-1} \cdot D_\varrho(y, F(y)) \le 2^{-1} \cdot \varrho(y, v)$. Thus

$$\begin{aligned}\gamma &= \frac{1}{2}[D_\varrho(v, F(y)) + D_\varrho(y, F(v))] \\ &\le \frac{1}{2}[\varrho(v, y) + D_\varrho(y, F(y))] < \frac{1}{2} \cdot \varrho(y, v) + \frac{1}{2} \cdot \varrho(y, v) = \varrho(y, v),\end{aligned}$$

which contradicts the definition of γ. So we have proved that in this case $\gamma = 0$ which implies $D_\varrho(y, F(y)) \le \varphi(\gamma) = \varphi(0) = 0$. We can notice that in all cases we have $D_\varrho(y, F(y)) \le \delta$ which contradicts (3.13). Thus (3.12) is true. □

REMARK 3.5. Some examples of functions φ are:

$$\begin{aligned}\varphi(t) &= at, &&\text{for } a \in [0, 1), \\ \varphi(t) &= \frac{t}{1+t}, &&\text{for } t \in \mathbb{R}_+.\end{aligned}$$

Hence, our previous results generalise and extend theorems from [1], [3], [5].

In the following we will give a data dependence theorem.

THEOREM 3.6. *Let X be a nonempty set. Suppose that d and ϱ are two metrics on X and $T, F: X \to P(X)$ are two multivalued operators. We suppose that:*

(a) *(X, d) is a complete metric space;*
(b) *there exists $c > 0$ such that $d(x, y) \le c \cdot \varrho(x, y)$, for each $x, y \in X$;*
(c) *if $d \ne \varrho$ then $T, F: X \to P(X^d)$ are closed;*
if $d = \varrho$ then $T, F: X \to P_{\mathrm{cl}}(X^d)$;
(d) *there exists a continuous function $\varphi: [0, \infty) \to [0, \infty)$, with $\varphi(t) < t$, for every $t > 0$, $\varphi(0) = 0$ and φ is nondecreasing such that:*

$$H_\varrho(T(x), T(y)) \le \varphi(M_\varrho^T(x, y)), \qquad H_\varrho(F(x), F(y)) \le \varphi(M_\varrho^F(x, y)),$$

for every $x, y \in X$, with strict inequality if $M(x, y) \ne 0$;
(e) *Also assume that:*

$$a(t) := \sum_{i=0}^{\infty} \varphi^i(t) < \infty,$$

and a is continuous on $(0, +\infty)$;
(f) *there exists $\eta > 0$ such that*

$$H_\varrho(T(x), F(x)) \le \eta, \quad \text{for every } x \in X. \tag{3.16}$$

Then $H_d(\operatorname{Fix} T, \operatorname{Fix} F) \le c \cdot a(\eta)$.

PROOF. Let $x_0 \in FixT$ be arbitrary chosen. We will prove that there exists $y^* \in \operatorname{Fix} F$ such that $d(x_0, y^*) \le c \cdot a(\eta)$. From Theorem 3.2 we can choose a Cauchy sequence $\{y_n\}$ starting from $y_0 = x_0$ and $y_n \xrightarrow{d} y^*$, as $n \to \infty$, $y^* \in F(y^*)$ with

$$\varrho(y_{n+p}, y_n) \leq \sum_{i=n}^{\infty} \varphi^i(\varrho(y_0, y_1)).$$

Thus we have that

$$d(y_{n+p}, y_n) \leq c \cdot \varrho(y_{n+p}, y_n) \leq c \cdot \sum_{i=1}^{\infty} \varphi^i(\varrho(y_0, y_1)).$$

Since $y_0 = x_0 \in \operatorname{Fix} T$ we have that $y_0 \in T(y_0)$. Thus from (3.16) for $x = y_0$ and for every $q > 1$ we have that there exists $y_1 \in F(y_0)$ such that

$$\varrho(y_0, y_1) \leq q \cdot H_\varrho(T(y_0), F(y_0)) \leq q \cdot \eta.$$

Since $\{y_n\}$ is a Cauchy sequence we have that $d(y_{n+p}, y_n) \to d(y^*, y_n)$, as $p \to \infty$, so we have the following inequality:

$$d(y^*, y_n) \leq c \cdot \sum_{i=0}^{\infty} \varphi^i(\varrho(y_0, y_1)) \leq c \cdot \sum_{i=0}^{\infty} \varphi^i(q \cdot \eta) = c \cdot a(q\eta).$$

For $n = 0$ we have that $d(y^*, y_0) \leq c \cdot a(q \cdot \eta)$. Letting $q \to 1$ we get that $d(y^*, y_0) \leq c \cdot a(\eta)$. By a similar procedure we obtain that for each $x_0 \in \operatorname{Fix} F$ there exists $x^* \in \operatorname{Fix} T$ such that $d(x_0, x^*) \leq ca(\eta)$. The proof is complete. □

In what follows we will obtain a homotopy result via Zorn's Lemma.

Theorem 3.7. *Let (X, d) be a complete metric space and ϱ another metric on X such that there exists $c > 0$ with $d(x, y) \leq c \cdot \varrho(x, y)$, for each $x, y \in X$. Let U be an open subset of (X, ϱ), V be a closed subset of (X, d) with $U \subset V$ and $r_0 > 0$. Let $G: V \times [0, 1] \to P(X)$ be a multivalued operator such that the following conditions are satisfied:*

(a) *$x \notin G(x, t)$, for each $x \in V \setminus U$ and each $t \in [0, 1]$;*
(b) *there exists $r_0 > 0$ and a continuous function $\varphi: [0, \infty) \to [0, \infty)$, with $\varphi(z) < z$, for every $z > 0$, $\varphi(0) = 0$ and φ is nondecreasing on $(0, r_0]$ such that:*

$$H_\varrho(G(x, t), G(y, t)) \leq \varphi(M_\varrho^{G(\,\cdot\,,t)}(x, y)),$$

for every $x, y \in X$ with strict inequality if $M_\varrho^{G(\,\cdot\,,t)}(x, y) \neq 0$;
(c) *there exists a continuous increasing function $\psi: [0, 1] \to \mathbb{R}$ such that*

$$H_\varrho(G(x, t), G(y, t)) \leq |\psi(t) - \psi(s)|,$$

for all $t, s \in [0, 1]$ and each $x \in V$;
(d) *$G: V^d \times [0, 1] \to P(X^d)$ is closed;*
(e) *$\phi: [0, \infty) \to [0, \infty)$ is strictly increasing (here $\phi(x) = x - \varphi(x)$);*
(f) *$\phi^{-1}(a) + \phi^{-1}(b) \leq \phi^{-1}(a + b)$, for $a \geq 0$, $b \geq 0$;*

(g) $\sum_{i=0}^{\infty} \varphi^i(t) < \infty$, *for* $t \in (0, r_0 - \varphi(r_0)]$;
(h) $\sum_{i=1}^{\infty} \varphi^i(r_0 - \varphi(r_0)) \leq \varphi(r_0)$.

Then $G(\,\cdot\,, 0)$ *has a fixed point if and only if* $G(\,\cdot\,, 1)$ *has a fixed point.*

PROOF. Suppose $G(\,\cdot\,, 0)$ has a fixed point z. Thus from (a) we have that $z \in U$. Define

$$Q = \{(t, x) \in [0, 1] \times U \mid x \in G(x, t)\}.$$

We can notice that $Q \neq \emptyset$, since $(0, z) \in Q$.

We will consider on Q a partial order defined as follows $(t, x) \leq (s, y)$ if and only if $t \leq s$ and $\varrho(x, y) \leq \phi^{-1}(2[\psi(s) - \psi(t)])$. Let M be a totally ordered subset of Q and consider $t^* = \sup\{t \mid (t, x) \in M\}$. Consider a sequence $(t_n, x_n)_{n\in\mathbb{N}^*} \subset M$ such that $(t_n, x_n) \leq (t_{n+1}, x_{n+1})$ and $t_n \to t^*$ as $n \to \infty$. Then $\varrho(x_m, x_n) \leq \phi^{-1}(2[\psi(t_m) - \psi(t_n)])$, for each $m, n \in \mathbb{N}^*$, $m > n$.

When $m, n \to \infty$ we obtain $\varrho(x_m, x_n) \to 0$, thus $(x_n)_{n\in\mathbb{N}^*}$ is ϱ-Cauchy. So $(x_n)_{n\in\mathbb{N}^*}$ is d-Cauchy too. We will denote by $x^* \in (X, d)$ its limit. Since $x_n \in G(x_n, t_n)$, $n \in \mathbb{N}^*$ and condition (d) we have that $x^* \in G(x^*, t^*)$. Also, from (a) we have that $x^* \in U$. Hence $(t^*, x^*) \in Q$. Since M is totally ordered we get $(t, x) \leq (t^*, x^*)$, for each $(t, x) \in M$. Thus (t^*, x^*) is an upper bound of M. By applying Zorn's Lemma we obtain that Q admits a maximal element $(t_0, x_0) \in Q$.

We will prove that $t_0 = 1$. Suppose that $t_0 < 1$. Choose $r > 0$ with $r \leq r_0$ and $t \in\,]t_0, 1]$ such that $B_\varrho(x_0, r) \subset U$, and $r := \phi^{-1}(2[\psi(t) - \psi(t_0)])$. Then from condition (c) we have

$$\begin{aligned} D_\varrho(x_0, G(x_0, t)) &\leq D_\varrho(x_0, G(x_0, t_0)) + H_\varrho(G(x_0, t_0), G(x_0, t)) \\ &\leq |\psi(t_0) - \psi(t)| \leq \frac{\phi(r)}{2} < \phi(r) = r - \varphi(r). \end{aligned}$$

Since $\overline{B}^d_\varrho(x_0, r) \subset V$, the multivalued operator $G(\,\cdot\,, t)\colon \overline{B}^d_\varrho(x_0, r) \to P(X^d)$ satisfies for all $t \in [0, 1]$ the assumptions of Theorem 3.2. Hence, for all $t \in [0, 1]$ there exists $x \in \overline{B}^d_\varrho(x_0, r)$ such that $x \in G(x, t)$. Thus $(t, x) \in Q$. Since $t_0 < t$ and $\varrho(x_0, x) \leq r = \phi^{-1}(2[\psi(t) - \psi(t_0)])$ we obtain that $(t_0, x_0) < (t, x)$. This contradicts the maximality of (t_0, x_0). For the reverse implication, just put $t := 1 - t$. □

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