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RETRACTING BALL ONTO SPHERE IN $BC_0(\mathbb{R})$

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Abstract. In infinite dimensional Banach spaces the unit sphere is a lipschitzian retract of the unit ball. We use the space of continuous functions vanishing at a point to provide an example of such retraction having relatively small Lipschitz constant.

1. Introduction

Let $(X, \|\|)$ be an infinite dimensional Banach space with the unit ball B and the unit sphere S. Since the works of Nowak [8] and Benyamini and Sternfeld [2] it is known that S is a lipschitzian retract of B. It means that there exists a mapping (a retraction) $R: B \to S$ satisfying, with a certain constant $k > 0$, the Lipschitz condition

$$\|Rx - Ry\| \leq k\|x - y\| \tag{1.1}$$

for all $x, y \in B$ and such that $Rx = x$ for all $x \in S$. Obviously, the above is not true for spaces of finite dimension due to the Brouwer's Non Retraction Theorem. There is an interesting question. What is the infimum of all k admitting existence of a retraction $R: B \to S$ satisfying the Lipschitz condition (1.1) with constant k?

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More precisely, the investigation is going on to find or evaluate the *optimal retraction constant* $k_0(X)$ defined by:

$$k_0(X) = \inf\{k : \text{ there exists a retraction } R\colon B \to S \text{ satisfying (1.1)}\}.$$

At present the exact value of $k_0(X)$ is not known for any single Banach space. Various evaluations can be found in books of Goebel and Kirk [4] and Goebel [3] and papers cited there. Obviously, constant $k_0(X)$ can not be to small. The universal known bound from below is $k_0(X) \geq 3$ but probably it is not sharp. For some spaces there are better estimates e.g. $k_0(X) > 3$ for uniformly convex spaces, $k_0(l_1) \geq 4$ and $k_0(H) > 4.5$ for Hilbert space. There were several approaches to give a reasonable universal estimate from above. All of them are based on individual constructions and tricks. It is a general feeling that spaces can differ by the value of $k_0(X)$ depending on the regularity of the norm geometry.

For several years the best known estimate from above was for $L_1(0,1)$ (see [3]). Together with a general estimation from below, we have

$$3 \leq k_0(L_1(0,1)) \leq 9.43\ldots$$

Very recently M. Annoni and E. Casini [1] obtained better evaluation for l_1. Together with known bound from below, we have

$$4 \leq k_0(l_1) \leq 8.$$

Immediately, the same estimate has been extended for $L_1(0,1)$ and few other spaces [6].

An interesting situation is observed for spaces with uniform norm. The best known estimate for the space of continuous functions is (see [3])

$$3 \leq k_0(C[0,1]) \leq 4(1+\sqrt{2})^2 = 23.31\ldots$$

Added in the proof: Author get a better estimation: $k_0(C[0,1]) < 14.93$ in his master's degree thesis. But for subspace $C_0[0,1]$ consisting of all the functions vanishing at zero the best published estimate is (see [5])

$$3 \leq k_0(C_0[0,1]) \leq 12.$$

This was improved by the very recent result [7] stating that

$$3 \leq k_0(C_0[0,1]) \leq 7.$$

The aim of this note is to present a construction for the space $BC_0(\mathbb{R})$ of all bounded continuous functions vanishing at zero which improves the estimates presented above. Then we extend this construction to a much wider class of spaces.

2. The case of $BC_0(\mathbb{R})$

Let us start with the space $BC_0(\mathbb{R})$ of all bounded continuous functions on $\mathbb{R}$ vanishing at zero and furnished with the standard uniform norm $\|f\| = \sup\{|f(t)| : t \in \mathbb{R}\}$. For our construction we shall need two simple special functions. First function is $\alpha\colon \mathbb{R} \to [-1,1]$,

$$\alpha(t) = \begin{cases} -1 & \text{for } t < -1, \\ t & \text{for } -1 \le t \le 1, \\ 1 & \text{for } t > 1. \end{cases}$$

Function α generates the truncation retraction Q of the whole space $BC_0(\mathbb{R})$ onto its unit ball B,

$$Qf(t) = \alpha(f(t)) = \max\{-1, \min\{1, f(t)\}\}.$$

Obviously Q satisfies the Lipschitz condition (1.1) with the constant $k = 1$

$$\|Qf - Qg\| \le \|f - g\| \tag{2.1}$$

and for each f such that $\|f\| > 1$ we have

$$\|Qf\| = 1. \tag{2.2}$$

Also for any $r \ge 0$ it generates the truncation Q_r on the ball $B(r)$ with center at zero and radius r,

$$Q_r f = \begin{cases} rQ((1/r)f) & \text{if } r > 0, \\ 0 & \text{if } r = 0. \end{cases}$$

Moreover, for any $r_1 \ge 0, r_2 \ge 0$, we have

$$\|Q_{r_1} f - Q_{r_2} g\| \le \max\{|r_1 - r_2|, \|f - g\|\}. \tag{2.3}$$

The second simple function to be used in the construction is $\Lambda\colon [0,\infty) \to [0,1]$

$$\Lambda(t) = \begin{cases} 3t & \text{for } 0 \le t \le 1/3, \\ 2 - 3t & \text{for } 1/3 < t \le 2/3, \\ 0 & \text{for } t > 2/3. \end{cases}$$

It is clear that Λ satisfies for all $s, t \in [0,\infty)$ the Lipschitz condition

$$|\Lambda(s) - \Lambda(t)| \le 3|s - t|. \tag{2.4}$$

The function Λ can be used to define a mapping $T\colon BC_0(\mathbb{R}) \to B$ by putting for each $f \in BC_0(\mathbb{R})$

$$Tf(t) = \Lambda\left(|f(t)| + \frac{|t|}{1 + |t|}\right). \tag{2.5}$$

In view of (2.4), for all $f, g \in BC_0(\mathbb{R})$ we have

$$\|Tf - Tg\| \leq 3\|f - g\|. \tag{2.6}$$

Moreover, for each $f \in BC_0(\mathbb{R})$ there exists a point t_1 such that

$$|f(t_1)| + \frac{|t_1|}{1+|t_1|} = \frac{1}{3} \quad \text{and} \quad Tf(t_1) = 1.$$

Hence

$$\|f - Tf\| \geq |Tf(t_1)| - |f(t_1)| = 1 - \left(\frac{1}{3} - \frac{|t_1|}{1+|t_1|}\right) = \frac{2}{3} + \frac{|t_1|}{1+|t_1|} \geq \frac{2}{3}. \tag{2.7}$$

In the next step let us define a mapping $F\colon B((2+\sqrt{2})/3) \to BC_0(\mathbb{R})$

$$Ff = \begin{cases} f - Tf & \text{if } \|f\| \leq 2/3, \\ f - Q_{3(1-\|f\|)}Tf & \text{if } 2/3 \leq \|f\| \leq 1, \\ (4 - 3\|f\|)f & \text{if } 1 \leq \|f\| \leq (2+\sqrt{2})/3. \end{cases}$$

The radius $(2+\sqrt{2})/3$ has been selected via certain process of optimization. We skip the detailes.

Observe that if $\|f\| = 2/3$ both formulas give the same result. The same holds if $\|f\| = 1$.

Let us prove that mapping F satisfies the Lipschitz condition with constant 4.

- In view of (2.6) for all f, g with $\|f\| \leq 2/3$ and $\|g\| \leq 2/3$ we have

$$\begin{aligned} \|Ff - Fg\| &= \|(f - Tf) - (g - Tg)\| \leq \|f - g\| + \|Tf - Tg\| \\ &\leq \|f - g\| + 3\|f - g\| = 4\|f - g\|; \end{aligned}$$

- In view of (2.3) and (2.6) for all f, g with $2/3 \leq \|f\| \leq 1$ and $2/3 \leq \|g\| \leq 1$ we have

$$\begin{aligned} \|Ff - Fg\| &= \|(f - Q_{3(1-\|f\|)}Tf) - (g - Q_{3(1-\|g\|)}Tg)\| \\ &\leq \|f - g\| + \|Q_{3(1-\|f\|)}Tf - Q_{3(1-\|g\|)}Tg\| \\ &\leq \|f - g\| + \max\{|3(1-\|f\|) - 3(1-\|g\|)|, \|Tf - Tg\|\} \\ &\leq \|f - g\| + \max\{3|\|f\| - \|g\||, 3\|f - g\|\} = 4\|f - g\|; \end{aligned}$$

- Without loss of generality, we can assume that $1 \leq \|g\| \leq \|f\| \leq (2+\sqrt{2})/3$,

$$\begin{aligned} \|Ff - Fg\| &= \|(4 - 3\|f\|)f - (4 - 3\|g\|)g\| \\ &\leq \|(4 - 3\|f\|)(f - g)\| + \|(4 - 3\|f\|)g - (4 - 3\|g\|)g\| \\ &\leq (4 - 3\|f\|)\|f - g\| + 3\|g\|(\|f\| - \|g\|) \\ &\leq (4 - 3\|f\| + 3\|g\|)\|f - g\| \leq 4\|f - g\|. \end{aligned}$$

Finally, the standard reasoning shows that for all $f, g \in B((2+\sqrt{2})/3)$ we have

$$\|Ff - Fg\| \le 4\|f - g\|. \tag{2.8}$$

Let us prove now that for each $f \in B((2+\sqrt{2})/3)$ we have

$$\|Ff\| \ge \frac{2}{3}. \tag{2.9}$$

In view of (2.7), $\|Ff\| = \|f - Tf\| \ge 2/3$ for all f with $\|f\| \le 2/3$. The same holds for all f with $2/3 \le \|f\| \le 1$. Indeed, if f attains its norm at a point $\bar{t}$, $\|f\| = |f(\bar{t})| \ge 2/3$, then using the fact that $\Lambda(|f(\bar{t})| + |\bar{t}|/(1+|\bar{t}|)) = 0$, we have

$$\begin{aligned}\|Ff\| &= \|f - Q_{3(1-\|f\|)}Tf\| \ge |f(\bar{t}) - Q_{3(1-\|f\|)}Tf(\bar{t})| \\ &\ge |f(\bar{t})| - \left|Q_{3(1-\|f\|)}\Lambda\left(|f(\bar{t})| + \frac{|\bar{t}|}{1+|\bar{t}|}\right)\right| = \|f\| \ge \frac{2}{3}.\end{aligned}$$

Since functions attaining their norm form the dense set in B we conclude that

$$\|Ff\| \ge \frac{2}{3} \quad \text{for each } f \in B.$$

If $1 \le \|f\| \le (2+\sqrt{2})/3$ then

$$\|Ff\| = \|(4 - 3\|f\|)f\| = (4 - 3\|f\|)\|f\| \ge \frac{2}{3},$$

and inequality (2.9) is proved.

Observe also that for each f with $\|f\| = \frac{2+\sqrt{2}}{3}$ we have

$$Ff = (2-\sqrt{2})f. \tag{2.10}$$

Let us define now a mapping $\widetilde{F}\colon B \to BC_0(\mathbb{R})$ by putting for each $f \in B$

$$\widetilde{F}f = \frac{3}{2+\sqrt{2}}F\left(\frac{2+\sqrt{2}}{3}f\right)$$

In view of (2.8)–(2.10)

- for all $f, g \in B$ we have

$$\|\widetilde{F}f - \widetilde{F}g\| \le 4\|f - g\|; \tag{2.11}$$

- for each $f \in B$ we have

$$\|\widetilde{F}f\| \ge \frac{2}{2+\sqrt{2}}; \tag{2.12}$$

- for each $f \in S$ we have

$$\widetilde{F}f = \frac{2}{2+\sqrt{2}}f. \tag{2.13}$$

Putting together (2.1), (2.2), (2.11)–(2.13) we can now define the retraction $R\colon B \to S$ as

$$Rf = Q\biggl(\frac{2+\sqrt{2}}{2}\widetilde{F}f\biggr)$$

and observe that for all $f,g \in B$ we have

$$\begin{aligned}\|Rf - Rg\| &= \biggl\|Q\biggl(\frac{2+\sqrt{2}}{2}\widetilde{F}f\biggr) - Q\biggl(\frac{2+\sqrt{2}}{2}\widetilde{F}g\biggr)\biggr\| \\ &\le \frac{2+\sqrt{2}}{2}\|\widetilde{F}f - \widetilde{F}g\| \le 4\biggl(\frac{2+\sqrt{2}}{2}\biggr)\|f-g\| = 2(2+\sqrt{2})\|f-g\|.\end{aligned}$$

What we have shown can be formulated as

$$k_0(BC_0(\mathbb{R})) \le 2(2+\sqrt{2}) < 6.83.$$

3. Possibility of generalization

Presented construction can be repeated with minor changes and applied to a much wider class of spaces. Suppose (M,d) is a connected metric space consisting of more than one point and let $z \in M$ be a selected point. Consider the space $BC_z(M)$ of all bounded continuous functions $f\colon M \to \mathbb{R}$ vanishing at z, $f(z) = 0$, with the standard uniform norm $\|f\| = \sup\{|f(x)| : x \in M\}$.

If M is an unbounded then the following modification of the formula (2.5),

$$Tf(x) = \Lambda\biggl(|f(x)| + \frac{d(x,z)}{1+d(x,z)}\biggr)$$

allows to carry on the proof with only technical changes.

For bounded space M, the same holds. It is enough to put

$$m = \sup\{d(x,z) : x \in M\}$$

and modify (2.5) by

$$Tf(x) = \Lambda\biggl(|f(x)| + \frac{d(x,z)}{m}\biggr).$$

All the above allows us to conclude with the theorem,

Theorem 3.1. *If (M,d) is a connected metric space consisting of more than one point and $z \in M$ is a given point, then*

$$k_0(BC_z(M)) \le 2(2+\sqrt{2}) < 6.83.$$

The above proof combined tricks known from [5] and [7].

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