Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 33, 2009, 275–283

PERIODIC SOLUTIONS FOR A NEUTRAL DIFFERENTIAL EQUATION WITH VARIABLE PARAMETER

Bo Du — Jianxin Zhao — Weigao Ge

Abstract. By means of Mawhin's continuation theorem, we present some sufficient conditions which guarantee the existence of at least one T-periodic solution for a first-order neutral equation with variable parameter. The interest is that the coefficient c is not a constant, which is different from the corresponding ones of past work.

1. Introduction

This paper is devoted to using Mawhin's continuation theorem to investigate the existence of periodic solutions for a first-order neutral equation with variable parameter as follows:

$$(x(t) - c(t)x(t-\tau))' + g(x(t-\gamma(t))) = e(t), \tag{1.1}$$

where $g, e, \gamma \in C(\mathbb{R}, \mathbb{R})$ with $e(t) = e(t+T)$ and $\gamma(t) = \gamma(t+T)$; $c \in C^1(\mathbb{R}, \mathbb{R})$ with $|c(t)| \neq 1$ and $c(t+T) = c(t)$; T, τ are given constants with $T > 0$.

In recent years, neutral functional differential equations (NFDEs) have been extensively studied by many researchers. In [4]–[6], Lu and Ge studied the

2000 *Mathematics Subject Classification.* 34B15, 34B13.
Key words and phrases. Neutral, variable parameter, periodic solution, Mawhin's continuation theorem.
Supported by National Natural Science Foundation of P.R. China (No:10671012).

following NFDEs:

$$\frac{d}{dt}(u(t) - ku(t-\tau)) = g_1(u(t)) + g_2(u(t-\tau_1)) + p(t),$$
$$(x(t) + cx(t-r))'' + f(x'(t)) + g(x(t-\tau(t))) = p(t),$$

$$\frac{d^2}{dt^2}(u(t) - ku(t-\tau)) = f(u(t))u'(t) + \alpha(t)g(u(t)) + \sum_{j=1}^{n} \beta_j(t)g(u(t-\gamma_j(t))) + p(t).$$

In [7] Enrico Serra studied a kind of NFDE in the following form:

$$x'(t) + ax'(t-\tau) = f(t, x(t)).$$

In [3] Liu considered the following first-order neutral functional differential equation:

$$(u(t) + Bu(t-\tau))' = g_1(t, u(t)) - g_2(t, u(t-\tau_1)) + p(t).$$

However, to the best of our knowledge, there are few results on the existence of periodic solutions to first-order neutral equations for the case of a variable $c(t)$. Recently, we obtained the properties of the neutral operator $A: C_T \to C_T$, $[Ax](t) = x(t) - c(t)x(t-\tau)$ in [6]. In this paper, we will obtain the existence of periodic solutions to equation (1.1) by using the properties of the operator A and Mawhin's continuation theorem.

2. Preliminary

In this section, we give some lemmas which will be used in this paper. Let

$$c_0 = \max_{t\in[0,T]} |c(t)|, \quad \sigma = \min_{t\in[0,T]} |c(t)|, \quad c_1 = \max_{t\in[0,T]} |c'(t)|,$$
$$C_T = \{x \mid x \in C(\mathbb{R}, \mathbb{R}),\ x(t+T) \equiv x(t),\ \text{for all } t \in \mathbb{R}\}$$

with the norm

$$|\varphi|_0 = \max_{t\in[0,T]} |\varphi(t)|, \quad \text{for all } \varphi \in C_T$$

and

$$C_T^1 = \{x \mid x \in C^1(\mathbb{R}, \mathbb{R}),\ x(t+T) \equiv x(t),\ \text{for all } t \in \mathbb{R}\}$$

with the norm

$$||\varphi|| = \max_{t\in[0,T]} \{|\varphi|_0, |\varphi'|_0\}, \quad \text{for all } \varphi \in C_T^1.$$

Clearly, C_T and C_T^1 are Banach spaces.

Define linear operator:

$$A: C_T \to C_T, \quad [Ax](t) = x(t) - c(t)x(t-\tau), \quad \text{for all } t \in \mathbb{R}.$$

Lemma 2.1 ([1]). *If $|c(t)| \neq 1$, then operator A has continuous inverse A^{-1} on C_T, satisfying:*

$$\text{(a)}\qquad [A^{-1}f](t) = \begin{cases} f(t) + \displaystyle\sum_{j=1}^{\infty}\prod_{i=1}^{j} c(t-(i-1)\tau)f(t-j\tau), \\ \qquad\qquad\qquad\qquad c_0 < 1, \ \text{for all } f \in C_T, \\ -\dfrac{f(t+\tau)}{c(t+\tau)} - \displaystyle\sum_{j=1}^{\infty}\prod_{i=1}^{j+1} \frac{1}{c(t+i\tau)} f(t+j\tau+\tau), \\ \qquad\qquad\qquad\qquad \sigma > 1, \ \text{for all } f \in C_T. \end{cases}$$

$$\text{(b)}\qquad \int_0^T |[A^{-1}f](t)|\,dt \le \begin{cases} \dfrac{1}{1-c_0}\displaystyle\int_0^T |f(t)|\,dt, & c_0 < 1, \ \text{for all } f \in C_T, \\ \dfrac{1}{\sigma-1}\displaystyle\int_0^T |f(t)|\,dt, & \sigma > 1, \ \text{for all } f \in C_T. \end{cases}$$

Let X and Y be real Banach spaces and let $L\colon D(L) \subset X \to Y$ be a Fredholm operator with index zero, here $D(L)$ denotes the domain of L. This means that ImL is closed in Y and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$. If L is a Fredholm operator with index zero, then there exist continuous projectors $P\colon X \to X$, $Q\colon Y \to Y$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I-Q)$ and $L_{D(L)\cap \operatorname{Ker} P}\colon (I-P)X \to \operatorname{Im} L$ is invertible. Denote by K_p the inverse of L_P.

Let Ω be an open bounded subset of X, a map $N\colon \overline{\Omega} \to Y$ is said to be L-compact in $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and the operator $K_p(I-Q)N(\overline{\Omega})$ is relatively compact. We first recall the famous Mawhin's continuation theorem.

Lemma 2.2 ([2]). *Suppose that X and Y are two Banach spaces and $L\colon D(L) \subset X \to Y$ is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded set and $N\colon \overline{\Omega} \to Y$ is L-compact on $\overline{\Omega}$. If all the following conditions hold:*

(a) *$Lx \neq \lambda Nx$, for all $x \in \partial\Omega \cap D(L)$, for all $\lambda \in (0,1)$,*
(b) *$Nx \notin \operatorname{Im} L$, for all $x \in \partial\Omega \cap \operatorname{Ker} L$,*
(c) *$\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$,*

where $J\colon \operatorname{Im} Q \to \operatorname{Ker} L$ is an isomorphism. Then equation $Lx = Nx$ has a solution on $\overline{\Omega} \cap D(L)$.

Define a linear operator

$$L\colon D(L) \subset C_T \to C_T, \quad Lx = (Ax)',$$

where $D(L) = \{x \mid x \in C_T^1\}$, and a nonlinear operator

$$N\colon C_T \to C_T, \quad Nx = -g(x(t-\gamma(t))) + e(t).$$

It is easy to see

$$\operatorname{Im} L = \left\{ y \;\middle|\; y \in C_T,\ \int_0^T y(s)\,ds = 0 \right\}.$$

Since for all $x \in \operatorname{Ker} L$, $(x(t) - c(t)x(t-\tau))' = 0$, we have

$$x(t) - c(t)x(t-\tau) = 1. \tag{2.1}$$

Let $\varphi(t)$ be the unique T-periodic solution of (2.1), then $\varphi(t) \neq 0$ and

$$\operatorname{Ker} L = \{a\varphi(t),\ a \in \mathbb{R}\}.$$

So $\operatorname{Im} L$ is closed in C_T and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 1$. Then the operator L is a Fredholm operator with index zero. Define continuous projectors

$$P\colon C_T \to \operatorname{Ker} L, \quad (Px)(t) = \frac{\int_0^T x(t)\varphi(t)\,dt}{\int_0^T \varphi^2(t)\,dt}\,\varphi(t)$$

and

$$Q\colon C_T \to C_T/\operatorname{Im} L, \quad Qy = \frac{1}{T}\int_0^T y(s)\,ds.$$

Hence,

$$\operatorname{Im} P = \operatorname{Ker} L \quad \text{and} \quad \operatorname{Ker} Q = \operatorname{Im} L.$$

Set operators

$$L_P = L|_{D(L)\cap \operatorname{Ker} P}\colon D(L) \cap \operatorname{Ker} P \to \operatorname{Im} L$$

and

$$L_P^{-1} = K_p\colon \operatorname{Im} L \to D(L) \cap \operatorname{Ker} P.$$

Since

$$K_p\colon \operatorname{Im} L \subset C_T \to D(L) \cap \operatorname{Ker} P \subset C_T^1$$

is an embedding operator, so K_p is a completely continuous operator; on the other hand, by the definitions of Q and N, it is clear that $QN(\overline{\Omega})$ is bounded. Hence nonlinear operator N is L-compact on $\overline{\Omega}$.

3. Existence of periodic solution for equation (1.1)

Theorem 3.1. *Suppose that $\int_0^T e(s)\,ds = 0$, $\int_0^T \varphi^2(s)\,ds \neq 0$, $|c(t)| \neq 1$ for all $t \in \mathbb{R}$, and there exist constants $d > 0$ and $r \geq 0$ such that*

- (H1) *$xg(x) > 0$, whenever $|x| > d$;*
- (H2) *$\lim_{|x|\to\infty} |g(x)|/|x| \leq r \in [0,\infty)$.*

Then equation (1.1) has at least one T-periodic solution, if

$$\max\left\{ \frac{c_1 T}{1-c_0}, \frac{Tr}{1-c_0-c_1T} \right\} < 1 \quad \textit{for } c_0 < \frac{1}{2},$$

$$\max\left\{ \frac{c_1 T}{\sigma-1}, \frac{Tr}{\sigma-1-c_1T} \right\} < 1 \quad \textit{for } \sigma > 1.$$

PROOF. We complete the proof by three steps.

Step 1. Let $\Omega_1 = \{x \in D(L) : Lx = \lambda Nx,\ \lambda \in (0,1)\}$. We show that Ω_1 is a bounded set. For all $x \in \Omega_1$, $Lx = \lambda Nx$, i.e.

$$(Ax)'(t) = -\lambda g(x(t-\gamma(t))) + \lambda e(t). \tag{3.1}$$

Integrating both sides of (3.1) over $[0,T]$, we have

$$\int_0^T g(x(t-\gamma(t)))\,dt = 0.$$

From integral mean value theorem, there is a constant $\xi \in [0,T]$ such that $g(x(\xi-\gamma(\xi))) = 0$, from assumption (H1) we have $|x(\xi-\gamma(\xi))| \le d$. Because $x(t)$ is a T-periodic function, then there exists a constant $\xi^* \in [0,T]$ satisfying $\xi - \gamma(\xi) = \xi^* + kT$, $k \in Z$, then we have $|x(\xi^*)| \le d$. Hence

$$\begin{aligned} |x|_0 &= \max_{t\in[0,T]} \left| x(\xi^*) + \int_{\xi^*}^t x'(s)\,ds \right| \\ &\le |x(\xi^*)| + \int_0^T |x'(s)|\,ds \le d + \int_0^T |x'(s)|\,ds. \end{aligned} \tag{3.2}$$

From $[Ax](t) = x(t) - c(t)x(t-\tau)$, for all $x \in C_T^1$, we have

$$(Ax')(t) = (Ax)'(t) + c'(t)x(t-\tau),$$

then from Lemma 2.1 and (3.2), if $c_0 < 1/2$ we have

$$\begin{aligned} \int_0^T |x'(t)|\,dt &= \int_0^T |(A^{-1}Ax')(t)|\,dt \le \int_0^T \frac{|(Ax')(t)|}{1-c_0}\,dt \\ &= \int_0^T \frac{|(Ax)'(t) + c'(t)x(t-\tau)|}{1-c_0}\,dt \\ &\le \int_0^T \frac{|(Ax)'(t)|}{1-c_0}\,dt + \frac{c_1 T}{1-c_0}\left(d + \int_0^T |x'(t)|\,dt\right). \end{aligned}$$

In view of $c_1T/(1-c_0) < 1$, we have

$$\int_0^T |x'(t)|\,dt \le \int_0^T \frac{|(Ax)'(t)|}{1-c_0-c_1T}\,dt + \frac{c_1Td}{1-c_0-c_1T}. \tag{3.3}$$

On the other hand, by (3.1) we have

$$\begin{aligned} \int_0^T |(Ax)'(t)|\,dt &\le \int_0^T |g(x(t-\gamma(t)))|\,dt + \int_0^T |e(t)|\,dt \\ &\le \int_0^T |g(x(t-\gamma(t)))|\,dt + T|e|_0. \end{aligned} \tag{3.4}$$

Now we consider $\int_0^T |g(x(t-\gamma(t)))|\,dt$. Let

$$F(z) = \frac{T(r+z)}{1-c_0-c_1T}, \quad z \in [0,\infty).$$

From $Tr/(1-c_0-c_1T) < 1$, we have $F(0) < 1$. Since $F(z)$ is continuous on $[0,\infty)$, so there exists a constant $\delta > 0$ such that

$$F(z) = \frac{T(r+z)}{1-c_0-c_1T} < 1, \quad z \in (0,\delta].$$

Choosing $\varepsilon_1 = \delta_1/2 > 0$, we have

$$\frac{T(r+\varepsilon_1)}{1-c_0-c_1T} < 1. \tag{3.5}$$

Similarly, there exists a constant $\varepsilon_2 > 0$ such that

$$\frac{T(r+\varepsilon_2)}{\sigma-1-c_1T} < 1. \tag{3.6}$$

For such a constant ε_1, in view of assumption (H2), we obtain that there exists a constant $\rho > 0$ such that

$$|g(x)| \le (r+\varepsilon_1)|x|, \quad \text{whenever } |x| > \rho. \tag{3.7}$$

Let

$$E_1 = \{t \in [0,T] : |x(t-\gamma(t))| > \rho\}, \quad E_2 = \{t \in [0,T] : |x(t-\gamma(t))| \le \rho\}.$$

By (3.7) we have

$$\begin{aligned}\int_0^T |g(x(t-\gamma(t)))|\,dt &= \bigg(\int_{E_1} |g(x(t-\gamma(t)))|\,dt + \int_{E_2} |g(x(t-\gamma(t)))|\,dt\bigg) \\ &\le T(r+\varepsilon_1)|x|_0 + Tg_\rho,\end{aligned} \tag{3.8}$$

where $g_\rho = \max_{|u|\le\rho} |g(u)|$. From (3.4) and (3.8) we have

$$\begin{aligned}\int_0^T |(Ax)'(t)|\,dt &\le \int_0^T |g(x(t-\gamma(t)))|\,dt + T|e|_0 \\ &\le T(r+\varepsilon_1)|x|_0 + Tg_\rho + T|e|_0.\end{aligned} \tag{3.9}$$

If $c_0 < 1/2$, from (3.3) and (3.9) we have

$$\int_0^T |x'(t)|\,dt \le \frac{T(r+\varepsilon_1)}{1-c_0-c_1T}|x|_0 + \frac{Tg_\rho + T|e|_0}{1-c_0-c_1T} + \frac{c_1Td}{1-c_0-c_1T}. \tag{3.10}$$

From (3.2) and (3.10) we have

$$\begin{aligned}|x|_0 &\le d + \int_0^T |x'(s)|\,ds \\ &\le d + \frac{T(r+\varepsilon_1)}{1-c_0-c_1T}|x|_0 + \frac{Tg_\rho + T|e|_0}{1-c_0-c_1T} + \frac{c_1Td}{1-c_0-c_1T}.\end{aligned} \tag{3.11}$$

By (3.5) and (3.11) there exists a constant $M_1 > 0$ such that $|x|_0 \le M_1$.

If $\sigma > 1$, from (3.6) and the condition $c_1T/(\sigma - 1) < 1$, similar to the above proof, we obtain that there exists a constant $M_2 > 0$ such that $|x|_0 \le M_2$. Then we have $|x|_0 < \max\{M_1, M_2\} + 1 := \overline{M}$.

Step 2. Let $\Omega_2 = \{x \in \operatorname{Ker} L : QNx = 0\}$, we shall prove that Ω_2 is a bounded set. For all $x \in \Omega_2$, when $x = a_0\varphi(t)$, $a_0 \in \mathbb{R}$, we have

$$\int_0^T g(a_0\varphi(t))\,dt = 0. \tag{3.12}$$

When $c_0 < 1/2$, we have

$$\begin{aligned}\varphi(t) &= A^{-1}(1) = 1 + \sum_{j=1}^{\infty}\prod_{i=1}^{j} c(t-(i-1)\tau)\\ &\ge 1 - \sum_{j=1}^{\infty}\prod_{i=1}^{j} c_0 = 1 - \frac{c_0}{1-c_0} = \frac{1-2c_0}{1-c_0} := \delta_1 > 0.\end{aligned}$$

Then we have $a_0 \le d/\delta_1$. Otherwise, for all $t \in [0,T]$, $a_0\varphi(t) > d$, from assumption (H1), we have

$$\int_0^T g(a_0\varphi(t))\,dt > 0$$

which is contradiction to (3.12). When $\sigma > 1$, we have

$$\begin{aligned}\varphi(t) &= A^{-1}(1) = -\frac{1}{c(t+\tau)} - \sum_{j=1}^{\infty}\prod_{i=1}^{j+1}\frac{1}{c(t+i\tau)}\\ &\le -\frac{1}{\sigma} - \sum_{j=1}^{\infty}\prod_{i=1}^{j+1}\frac{1}{\sigma} = -\frac{1}{\sigma-1} := \delta_2 < 0.\end{aligned}$$

Then we have $a_0 \le -d/\delta_2$. Otherwise, for all $t \in [0,T]$, $a_0\varphi(t) < -d$, from assumption (H1), we have

$$\int_0^T g(a_0\varphi(t))\,dt < 0$$

which is contradiction to (3.12). So Ω_2 is a bounded set.

Denote $|a_0\varphi(t)| \le \widehat{M}$ and $M = \max\{\overline{M}, \widehat{M}\} + 1$.

Step 3. Let $\Omega = \{x \in X : |x|_0 < M\}$, then $\Omega_1 \cup \Omega_2 \subset \Omega$. For all $(x,\lambda) \in \partial\Omega \times (0,1)$, from the above proof, $Lx \ne \lambda Nx$ is satisfied. Obviously, condition (b) of Lemma 2.2 is also satisfied. Now we prove that condition (c) of Lemma 2.2 is satisfied. Take the homotopy

$$H(x,\mu) = \mu x - (1-\mu)JQNx, \quad x \in \overline{\Omega} \cap \operatorname{Ker} L, \quad \mu \in [0,1],$$

where $J\colon \operatorname{Im} Q \to \operatorname{Ker} L$ is a homeomorphism with $Ja = a\varphi(t)$, $a \in \mathbb{R}$. For all $x \in \partial\Omega \cap \operatorname{Ker} L$, we have $x = a_1\varphi$, $a_1 \in \mathbb{R}$, $|a_1\varphi| = M > d$, then

$$H(x,\mu) = a_1\varphi\mu + (1-\mu)g(a_1\varphi).$$

By using assumption (H1), we have $H(x,\mu) \neq 0$. And then by the degree theory,

$$\begin{aligned}\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} &= \deg\{H(\,\cdot\,,0), \Omega \cap \operatorname{Ker} L, 0\} \\ &= \deg\{H(\,\cdot\,,1), \Omega \cap \operatorname{Ker} L, 0\} \\ &= \deg\{I, \Omega \cap \operatorname{Ker} L, 0\} \neq 0.\end{aligned}$$

Applying Lemma 2.2, we reach the conclusion. □

As applications, we consider an example:

Example 3.2.

$$\left(x(t) - \frac{1}{10}(2-\sin t)x(t-\tau)\right)' + g\left(x\left(t - \frac{1}{2}\sin t\right)\right) = \cos t, \tag{3.13}$$

where $\gamma(t) = (1/2)\sin t$, $e(t) = \cos t$, $c(t) = (1/10)(2-\sin t)$, $T = 2\pi$,

$$g(u) = \begin{cases} e^{\sin u} & \text{for } u \geq 0, \\ \dfrac{1}{101}u & \text{for } u < 0. \end{cases}$$

We have

$$\lim_{|x|\to\infty} \frac{|g(x)|}{|x|} < \frac{1}{100} := r,$$

$c_0 = 3/10$ and $c_1 = 1/10$. From simple calculation, we have

$$\frac{c_1T}{1-c_0} = \frac{2\pi}{7} < 1, \qquad \frac{Tr}{1-c_0-c_1T} = \frac{\pi}{35-10\pi} < 1.$$

Applying Theorem 3.1, (3.13) has at least one 2π-periodic solution.

References

[1] B. Du, L. Guo, W. Ge and S. Lu, *Periodic solutions for generalized Liénard neutral equation with variable parameter*, Nonlinear Anal. **70** (2009), 385–392.

[2] R. E. Gaines and J. L. Mawhin, *Coincidence Degree and Nonlinear Differential Equations*, Springer, Berlin, 1977.

[3] B. Liu and L. Huang, *Existence and uniqueness of periodic solutions for a kind of first order neutral functional differential equation*, J. Math. Anal. Appl. 3 **22** (2006), 121–132.

[4] S. Lu and W. Ge, *On the existence of periodic solutions for neutral functional differential equation*, Nonlinear Anal. **54** (2003), 1285–1306.

[5] S. Lu, W. Ge and Z. Zheng, *Periodic solutions to neutral differential equation with deviating arguments*, Appl. Math. Comput. **152** (2004), 17–27.

[6] S. Lu, J. Ren and W. Ge, *Problems of periodic solutions for a kind of second order neutral functional differential equation*, Appl. Anal. **82** (2003), 411–426.

[7] E. Serra, *Periodic solutions for some nonlinear differential equational equations of neutral type*, Nonlinear Anal. **17** (1991), 139–151.

Manuscript received March 2, 2008

Bo Du
Department of Mathematics
School of Science
Zhejiang Forestry College
Hangzhou 311300, P.R. China

E-mail address: dubo7307@163.com

Jianxin Zhao
Navy Submarine Academy
Qingdao, 266071, P.R. China

Weigao Ge
Department of Mathematics
Beijing Institute of Technology
Beijing, 100081, P.R. China