Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 33, 2009, 205–215

CLASSIFICATION OF DIFFEOMORPHISMS OF $\mathbb{S}^4$ INDUCED BY QUATERNIONIC RICCATI EQUATIONS WITH PERIODIC COEFFICIENTS

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Abstract. The monodromy maps for the quaternionic Riccati equations with periodic coefficients $\dot{z} = zp(t)z + q(t)z + zr(t) + s(t)$ in $\mathbb{HP}^1$ are quternionic Möbius transformations. We prove that, like in the case of automorphisms of $\mathbb{CP}^1$, the quaternionic homografies are divided into three classes: hyperbolic, elliptic and parabolic.

1. Introduction

There exist many results about periodic solutions to differential equations of the form $\dot{z} = P(z,t)$ where P is a polynomial in $z \in \mathbb{C}$ with coefficients periodic in t, of period T (see [2], [4]–[6], [8], [9] for example).

In the case $\deg_z P \leq 2$, i.e. the Riccati equation, we can associate with it a linear second order equation $\ddot{y} + K(t)\dot{y} + L(t) = 0$ such that $z = M(t) \cdot \dot{y}/y$ and the periodic functions are determined by the coefficient functions in P. It implies that the 2-parameter family $\{g_s^t\}$ of non-autonomous flow maps are Möbius maps. In particular, the monodromy map $G = g_0^{2\pi}$ takes the form $G(z) = (az+b)/(cz+d)$, where we can assume that $ad - c = 1$, i.e. $G \in \mathrm{PSL}(2,\mathbb{C})$.

2000 *Mathematics Subject Classification*. Primary 34C25; Secondary 37C27, 37C60.
Key words and phrases. Quaternionic Riccati equation, Möbius map, periodic solution.
Supported by Polish MNiSzW Grant No 1/P03A/015/29.

The dynamics of any complex Möbius map is well known. It is equivalent, by an internal automorphism in the group $\mathrm{PSL}(2,\mathbb{C})$, to one of the three maps:

- the hyperbolic (or loxodromic) map $z \to \lambda z$, $0 \neq \lambda \in \mathbb{C} \setminus \mathbb{S}^1$;
- the elliptic map $z \to e^{i\alpha} z$, $\alpha \in \mathbb{R}$;
- the parabolic map $z \to z + 1$.

In the parabolic case the corresponding differential equation in the Riemann sphere $\mathbb{CP}^1 \simeq \mathbb{S}^2$ has only one periodic solution, of period T, which corresponds to the fixed point $z = \infty$ of G. In the hyperbolic case there are two periodic solutions corresponding to $z = 0$ and $z = \infty$. Also there are only two periodic solutions in the elliptic case when $\alpha/(2\pi)$ is irrational; if $\alpha/(2\pi) = p/q$, reduced rational ratio with $q > 0$, then all other solutions are periodic with the period qT.

When one studies periodic solutions only in $\mathbb{C} = \mathbb{CP}^1 \setminus \infty$ then one must exclude the periodic solutions in $\mathbb{CP}^1$ which pass through the point $z = \infty$. In this way examples of Riccati equations without periodic solutions were constructed in [2], [4]–[6], [8], [9].

Recently an interest arised in the study of periodic solutions of quaternionic Riccati equations of the type

$$\dot{z} = zp(t)z + q(t)z + zr(t) + s(t), \quad z \in \mathbb{H}, \tag{1.1}$$

where p, q, r, s are periodic functions in $t \in \mathbb{R}$ with values in the quaternionic line $\mathbb{H} \simeq \mathbb{R}^4$. In particular, J. Campos and J. Mawhin [1] proved existence of at least one T-periodic solution in $\mathbb{H}$ for the equation $\dot{z} = z^2 + s(t)$ with not too large $M = \sup|s(t)|$. Recently P. Wilczyński [7] has given some conditions for existence of at least two periodic solutions. Also quaternionic Riccati equations appear in the Euler fluid dynamics (see [3]).

The aim of his paper is to generalize the classification of complex monodromy maps associated with complex periodic Riccati equations to the quaternionic case. We shall prove the following statements:

1. With equation (1.1) one can associate a linear system

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad y_{1,2} \in \mathbb{H}, \tag{1.2}$$

with periodic quaternion-valued coefficients, such that any solution $z(t)$ to equation (1.1) is of the form $y_1(t) y_2^{-2}(t)$ for a solution $(y_1(t), y_2(t))^\top$ to system (1.2).

2. The 2-parameter group of non-autonomous flow maps associated with system (1.2) are of the form

$$\begin{pmatrix} a(s,t) & b(s,t) \\ c(s,t) & d(s,t) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

and the *monodromy map* $G = g_0^T$ for equation (1.1) takes the form of a quaternionic Möbius map

$$G(z) = (az + b)(cz + d)^{-1}, \tag{1.3}$$

where $a = a(0,T)$, $b = b(0,T)$, $c = c(0,T)$, $d = d(0,T)$.

3. Any Möbius automorphism of $\mathbb{HP}^1$ of the form (1.3) is equivalent, via a conjugation in the group $PGL(2, \mathbb{H})$, to one of the three (with $a, d \in \mathbb{C}$):

$z \to azd^{-1}$, $1 = |a| < |d|$ (hyperbolic);
$z \to azd^{-1}$, $|a| = |d| = 1$ (elliptic);
$z \to (az + 1)a^{-1}$, $|a| = 1$ (parabolic).

4. In the hyperbolic case G has the fixed points $z = 0$ (attractor) and $z = \infty$ (repeller) as the only periodic points.

5. Any elliptic map has two fixed points $z = 0$ and $z = \infty$ and the tori $\{z = z_1 + jz_2 \in \mathbb{C} + j\mathbb{C}: |z_1| =\text{const}, |z_2| = \text{const}\}$ are invariant for G.

If $\operatorname{Re} a^n \neq \operatorname{Re} d^n$ for any positive integer n then then G does not have other periodic points. If $\operatorname{Re} a^n = \operatorname{Re} d^n$ for some n then it has also a surface diffeomorphic to $\mathbb{CP}^1 \setminus \{0, \infty\}$ of periodic points of period n. If $a^n = d^m = 1$ for some m, n then the whole map is periodic with period $p = \operatorname{LCM}(m, n)$; if, additionally, $\operatorname{Re} a^q = \operatorname{Re} d^q$ for some smallest $q < p$ then there is a surface $\simeq \mathbb{CP}^1 \setminus \{0, \infty\}$ of periodic points of period q.

6. In the parabolic case the map G has only one periodic point corresponding to $z = \infty$.

We must underline that these results cannot be generalized to more general autonomous quaternionic differential equations with quadratic right-hand side, like $\dot{z} = azbz + zczd + ezf + gz + zh$.

The paper is organized as follows. In Sections 2 and 3 we present some preliminary general properties of Riccati equations and Möbius maps. The main results are in Section 4. In Appendix we present some properties of quaternions and quaternionic equations of degree ≤ 2.

2. Linear systems, Riccati equations and Möbius maps

The subject of this section is standard in the complex case, but in the quaternionic case the non-commutativity requires some care. Therefore we present all details in the below proofs.

Consider the linear system (1.2), i.e.

$$\dot{y} = Z(t)y, \quad y \in \mathbb{H}^2. \tag{2.1}$$

The solutions $y = \varphi(t; s, y_0) = \varphi(t)$ which satisfy the initial condition $\varphi(s; s, y_0) = y_0$ define the evolution maps $y_0 \to f_s^t(y_0) = \varphi(t; s, y_0)$. Treated as maps $f_s^t: \mathbb{R}^8 \to \mathbb{R}^8$ they are linear, i.e. $\mathbb{R}$-linear. But we can say more.

Lemma 2.1. *The maps f_s^t are left $\mathbb{H}$-linear, i.e.*

$$f_s^t(y) = \begin{pmatrix} a(s,t) & b(s,t) \\ c(s,t) & d(s,t) \end{pmatrix} y.$$

Proof. It follows from the Picard sequence

$$\varphi_0(t) = y_0, \qquad \varphi_n(t) = y_0 + \int_s^t Z(\tau)\varphi_{n-1}(\tau)\,d\tau. \qquad \square$$

The relation between systems (1.1) and (1.2) is given in the following

Lemma 2.2. *The change $z = y_1 y_2^{-1}$ leads to the following equation*

$$\dot{z} = A(t)z + B(t) - zC(t)z - zD(t).$$

Proof. Use

$$\frac{d}{dt}(y_2^{-1}) = -y_2^{-1}\dot{y}_2 y_2^{-1}$$

and (1.2). $\square$

Lemma 2.3. *The evolution maps for equation* (1.1) *are quaternionic fractional linear:*

$$g_s^t(z) = (a(s,t)z + b(s,t)) \cdot (c(s,t)z + d(s,t))^{-1}.$$

Proof. We have

$$(\alpha y_1 + \beta y_2)\cdot(\gamma y_1 + \delta y_2)^{-1} = [(\alpha z + \beta) y_2]\cdot[(\gamma z + \delta) y_2]^{-1} = (\alpha z + \beta)\cdot(\gamma z + \delta)^{-1}. \ \square$$

The map

$$F = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to G = (az+b)(cz+d)^{-1}$$

defines a homomorphism from the group $\mathrm{GL}(2,\mathbb{H})$, of invertible quaternionic matrices, to the group $\mathrm{PGL}(2,\mathbb{H}) = \mathrm{GL}(2,\mathbb{H})/\mathbb{R}I$ of automorphisms of the quaternionic projective space $\mathbb{HP}^2$.

Remark 2.1. Note the following non-standard formula

$$F^{-1} = \begin{pmatrix} (a - bd^{-1}c)^{-1} & -(a - bd^{-1}c)^{-1}bd^{-1} \\ -(d - ca^{-1}b)^{-1}ca^{-1} & (d - ca^{-1}b)^{-1} \end{pmatrix}, \quad ad \neq 0.$$

At this moment we have proved properties 1 and 2 from Introduction.

3. Fixed points

The fixed points of a Möbius map, i.e. such that $G(z) = z$, play important role in simplification of the map and further analysis of its dynamics.

Firstly note that when $c = 0$ the map is affine

$$G(z) = azd^{-1} + bd^{-1}. \tag{3.1}$$

Here the point $z = \infty$ is fixed. Indeed, in the chart $\zeta = z^{-1}$ we have

$$\zeta \to (G(\zeta^{-1}))^{-1} = d\zeta(a + c\zeta)^{-1}.$$

In the case $c \neq 0$ we consider finite fixed points (when they exist).

LEMMA 3.1. *If z_1 is a finite fixed point of G then in the chart $\zeta = (z - z_1)^{-1}$ the map G takes the affine form* (3.1). *Moreover, if the affine map* (3.1) *has a finite fixed point z_2 then, in the chart $\zeta = z - z_2$ it takes the linear form*

$$\zeta \to a\zeta d^{-1}. \tag{3.2}$$

PROOF. This proof is the same as in the complex case, e.g. with care in multiplication of quaternions. □

We consider now the problem of existence of fixed points of G. We assume that $c \neq 0$. Then we arrive to the following equation

$$zcz - az + zd - b = 0. \tag{3.3}$$

The change $z = c^{-1}w$ and the left multiplication by c gives the equation

$$w^2 - cac^{-1}w + wd - cb = 0. \tag{3.4}$$

LEMMA 3.2. *Equation* (3.4) *has at least one finite solution.*

PROOF. Let

$$X(w) = w^2 + \alpha w + w\beta + \gamma \tag{3.5}$$

be treated as a vector field on $\mathbb{R}^4$ (with $\alpha = -cac^{-1}$, $\beta = d$, $\gamma = -cb$).

We can define its index at infinity as the degree of the map

$$S^3_R \ni w \to \frac{X(w)}{|X(w)|} \in S^3_1,$$

where $S^3_r = \{|w| = r\}$ is a sphere of radius r and R is large. Of course, this degree equals 2.

If the vector field X has only isolated singular points (where $X(w) = 0$) then we apply the topological argument (like in the Fundamental Theorem of Algebra).

The example $w^2 + 1 = 0$ shows that the singular points may be non-isolated.

Anyway in the both cases there exists a singular point of X. □

Let us examine the situations when the equation $X(w)=0$ has a non-isolated solution, which we assume at $w=0$ (thus $\gamma=0$). Of course, then $|\alpha|+|\beta|\neq 0$ and $\det(\partial X/\partial w)(0)=0$.

By Lemma 5.5 in Appendix we have $\beta=-\mu\alpha\mu^{-1}$for some quaternion μ.

Note also that we can assume that $\operatorname{Re}\alpha=\operatorname{Re}\beta=0$, because the corresponding terms in (3.5) cancel themselves. Moreover, applying a conjugation we can assume that $\alpha=i\alpha_1\in i\mathbb{R}$ (see Lemma 5.3 in Appendix). In this case we find that

$$\ker\frac{\partial X}{\partial w}(0)=\mathbb{C}\mu^{-1}.$$

If $\alpha=\beta=-\mu\alpha\mu^{-1}$ then a translation of w leads to an equation of the form $(w+\alpha)^2=\delta$. So here the equation $X(w)=0$ with $\gamma=0$ defines a 2-dimensional surface diffeomorphic to $\mathbb{S}^2$.

The following lemma is proved in Appendix.

Lemma 3.3. *If $\alpha\neq\beta=-\mu\alpha\mu^{-1}$ and $\gamma=0$ then the point $w=0$ is the unique singular point of X. It follows that the only possibility of X to have non-isolated singular points is the case $X=(w+\alpha)^2+a$, $a>0$.*

Remark 3.4. In the complex case the fixed points of the Mobius map G correspond to the eigendirections of the corresponding linear operator F. In the quaternionic case this can be not true.

Indeed, if $\binom{z}{1}$ is an eigenvector of F, i.e. the equations $az+b=\lambda z$, $cz+d=\lambda$ are satisfied, then we get the equation

$$cz^2+(d-a)z-b=0$$

which is different than equation (3.3).

4. Classification

We can now present the promised classification of the quaternionic Möbius maps with respect to conjugation in the group $\mathrm{PGL}(2,\mathbb{H})$. We use the form (3.1).

Lemma 4.1. *After applying the conjugations $\lambda^{-1}G(\lambda z)$ and $G(z\mu)\mu^{-1}$ and rescaling $(a,d)\to(a\nu,d/\nu)$, $\nu\in\mathbb{R}$, in (3.1), we can assume the following*

$$\begin{aligned} a&=\widetilde{a}_0+i\widetilde{a}_1\in\mathbb{C},\quad |a|=1,\quad \widetilde{a}_0\geq 0,\quad \widetilde{a}_1\geq 0;\\ d&=\widetilde{d}_0+i\widetilde{d}_1\in\mathbb{C},\quad \widetilde{d}_1\geq 0.\end{aligned}\tag{4.1}$$

Moreover, this choice is unique and cannot be simplified.

Proof. The above conjugations result in the changes $a\to\lambda^{-1}a\lambda$ and $d\to\mu^{-1}d\mu$, respectively. Moreover, it is clear that they are the only possible changes in the class of affine transformations. Next we apply Lemmas 5.3, 5.4, 5.1 from Appendix; (the conjugation $a\to j^{-1}aj=\overline{a}$ allows to obtain $\operatorname{Im}a,\operatorname{Im}d\geq 0$). $\square$

Theorem 4.2. *Any Möbius quaternionic map is equivalent to a map of the affine form* (3.1) *with the restrictions* (4.1) *and satisfying one of the following conditions:*

(H) $|d| > 1$, $b = 0$ (*hyperbolic*);
(E) $|d| = 1$, $b = 0$ (*elliptic*);
(P) $a = d$, $b = 1$ (*parabolic*).

Any two different maps from the above list are non-equivalent.

Proof. The types (H) and (E) correspond to the form (3.2), i.e. with fixed points at $z = 0$ and $z = \infty$. The other types correspond to situations with only one fixed point at $z = \infty$.

Therefore we should consider the fixed point equation

$$Lz := az - zd = -b.$$

It does not have solutions when $b \notin \operatorname{Im} L$. Thus $\ker L \neq 0$.

By Lemma 5.5 from Appendix we find that $d = \nu a \nu^{-1}$, which is equivalent to the conditions $\operatorname{Re} a = \operatorname{Re} d$ and $|a| = |d|$ (see Lemmas 5.3 and Lemma 5.4 in Appendix), i.e. $d = a$.

If $d = a = 1$ then $L = 0$. Otherwise, $\dim_{\mathbb{R}} \ker L = 2$ in this case.

To find $\operatorname{Im} L$ we solve the equation $Lz = w$ with $z = z_1 + jz_2$, $w = w_1 + jw_2$, $z_{1,2}, w_{1,2} \in \mathbb{C}$. We arrive at the system

$$(a - d)z_1 = w_1, \quad (\overline{a} - d_1)z_2 = w_2.$$

If $d = a = 1$ then any $w \neq 0$ is outside $\operatorname{Im} L$. Then application of the conjugation $\lambda^{-1} G(\lambda z)$, $\lambda = b$, reduces b to $b = 1$ (type (P)).

If $d = a \neq 1$ then $\operatorname{Im} L = j\mathbb{C}$ and any $w = w_1 + j0 \in \mathbb{C} \setminus 0$ is outside $\operatorname{Im} L$. Moreover, any $v \notin \operatorname{Im} L$ is of the form $w_1 + j0 + u$, $u \in \ker L$. Therefore we can choose $b = b_1 + j0 \in \mathbb{C} \setminus 0$ (by a translation of z) and further application of the conjugation $\lambda^{-1} G(\lambda z)$ reduces it to $b = 1$ (type (P)). $\square$

Before analysis of the dynamical properties of the Möbius maps we make further simplifications and division of the set of elliptic and parabolic maps.

First, putting $a = e^{i\alpha}$ and $d = e^{i\beta}$ and $z = z_1 + jz_2$ (as before), the elliptic and parabolic maps take the following forms:

$$(z_1, z_2) \to (e^{i(\alpha-\beta)} z_1, e^{-i(\alpha+\beta)} z_2) \quad \text{(E)} \tag{4.2}$$

$$(z_1, z_2) \to (z_1 + 1, e^{-2i\alpha} z_2) \quad \text{(P)} \tag{4.3}$$

Next, we divide the class (E) into the following subclasses:

(EI) $\operatorname{Re} a^n \neq \operatorname{Re} d^n$ for any integer $n \geq 1$
(i.e. $(\alpha - \beta)/(2\pi)$ and $(\alpha + \beta)/(2\pi)$ are irrational);

(EII)$_n$ $\operatorname{Re} a^n = \operatorname{Re} d^n$ for some smallest integer $n \geq 1$ but a is not a root of 1;

(EIII)$_{m,n}$ $a^n = d^m = 1$ for some minimal positive integers m and n.

THEOREM 4.2. (a) *Any map of type* (H) *has $z = 0$ and $z = \infty$ as the only periodic points. The point $z = \infty$ is repelling and $z = 0$ is a global hyperbolic attractor for the affine part of $\mathbb{HP}^2$.*

(b) *Any elliptic map has two fixed points $z = 0$ and $z = \infty$ and the tori $\{|z_1| = r_1,\ |z_2| = r_2\}$ are invariant.*

If it is of type (EI) *then it does not have other periodic points. If it is of type* (EII)$_n$ *then it has also a surface diffeomorphic to $\mathbb{CP}^1 \setminus \{0, \infty\}$ (i.e. $\{z_1 = 0\}$ or $\{z_2 = 0\}$) of periodic points of period n.*

If it is of type (EIII)$_{m,n}$ *then the whole map is periodic with period $p = \mathrm{LCM}(m, n)$. If, additionally, $\operatorname{Re} a^q = \operatorname{Re} d^q$ for some smallest $q < p$ then there is a surface $\simeq \mathbb{CP}^1 \setminus \{0, \infty\}$ of periodic points of period q.*

(c) *Any parabolic map has $z = \infty$ as the only periodic point.*

PROOF. The point (a) is obvious. The points (b) and (c) easily follow from formulas (4.2) and (4.3). □

REMARK 4.4. Using he above classification of the monodromy maps of Riccati equations we can reprove the result of Campos and Mawhin [1]; moreover, with explicit bound. Namely we shall show that if

$$MT < \pi/4, \quad M^2 = \sup |s(t)|,$$

then the equation $\dot{z} = z^2 + s(t)$, $z \in H$ and $s(t)$ periodic with period T, has at least one periodic solution of period T.

Indeed, this equation has at least one T-periodic solution in $\mathbb{S}^4$. So, if there are no finite T-periodic solutions, then the point $z = \infty$ is passed by a periodic solution.

But in the chart $\zeta = z^{-1}$ we have $\dot{\zeta} = -1 - \zeta s(t) \zeta$.

If $\zeta = \varphi(t)$, $t_0 \leq t \leq t_0 + T$ is a solution such that $\varphi(t_0) = \infty$, then

$$\operatorname{Re} \dot{\varphi} < -1 + M^2 R^2, \tag{4.4}$$

where $R = \sup_{t_0 \leq t \leq t_0 + T} r(t)$ and $r(t) = |\varphi(t)|$.

On the other hand, the inequality $\dot{r} \leq 1 + M^2 r^2$ gives $MR \leq \tan(MT)$. If $\tan(MT) < 1$ then inequality (4.4) implies $\operatorname{Re} \dot{\varphi}(t) < 0$ for $t_0 \leq t \leq t_0 + T$. Of course, this solution cannot be T-periodic.

5. Appendix. Quaternions and quaternionic equations

Recall that the quaternion algebra $\mathbb{H}$ is the algebra with unity and generated by i, j, k with the relations $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$.

Any quaternion is of the form $q = q_0 + iq_1 + jq_2 + kq_3$, $q_{0,1,2,3} \in \mathbb{R}$ with the real part $\operatorname{Re} q = q_0$ and the imaginary part $\operatorname{Im} q = iq_1 + jq_2 + kq_3$ and with the conjugate $\overline{q} = q_0 - iq_1 - jq_2 - kq_3$. Its norm equals $|q| = (q_0^2 + q_1^2 + q_2^2 + q_3^2)^{1/2}$. In this paper we often represent quaternions in the form $z = z_1 + jz_2$, $z_{1,2} \in \mathbb{C}$.

Below we present some standard properties of quaternions.

Lemma 5.1. *If $z \in \mathbb{C}$ then $zj = \overline{z}j$.*

Lemma 5.2. *If $\zeta = i\zeta_1 + j\zeta_2 + k\zeta_3$ is imaginary then*

$$e^{\zeta} = \cos|\zeta| + \frac{\zeta}{|\zeta|}\sin|\zeta|$$

and $|e^{\zeta}| = 1$. Any quaternion with norm 1, i.e. unitary quaternion, is of this form.

Lemma 5.3. *The group of quaternions with the norm 1, i.e. $\mathbb{S}^3$, is is isomorphic to the group $SU(2)$ by the following correspondence*

$$i \to i\sigma_z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j \to i\sigma_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad k \to -i\sigma_x = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

This group acts on the space $\mathbb{R}^3 = i\mathbb{R} + j\mathbb{R} + k\mathbb{R}$ of imaginary quaternions by conjugations,

$$q \to Ad_z = zqz^{-1}.$$

The latter action is related with the action of the rotation group $\mathrm{SO}(3)$ *on* $\mathbb{R}^3$*, when one identifies the group* $\mathrm{SU}(2)$ *with the spin group* $\mathrm{Spin}(3)$ *equipped with the two-fold covering* $\mathrm{Spin}(3) \to \mathrm{SO}(3)$. (*The above Pauli matrices correspond to the generators of rotations around the three coordinate axes.*) *In particular, it follows that any imaginary quaternion is of the form*

$$\gamma e^{\zeta} i e^{-\zeta}, \quad \gamma > 0, \quad \operatorname{Re}\zeta = 0.$$

Lemma 5.4. *If $\operatorname{Im} q = 0$ and ζ is arbitrary then $e^{\zeta} q e^{-\zeta} = q$.*

Consider now quaternionic algebraic equations. We begin with the linear case.

Lemma 5.5. *Let $a \in \mathbb{C} \setminus \mathbb{R}$. Then the equation*

$$az + zb = 0, \quad z \in \mathbb{H},$$

has a nonzero solution if and only if $b = -\mu^{-1} a \mu$ *for some* $\mu \in \mathbb{H}$. *In such case any solution is of the form*

$$z = \zeta \mu, \quad \zeta \in \mathbb{C}.$$

Proof. Let $z_0 \neq 0$ be some solution. Then we have $b = -z_0^{-1} a z_0$. Putting $z = \zeta z_0$ the equation reads as $a\zeta = \zeta a$. It is clear that it must be $\zeta_2 = \zeta_3 = 0$.□

Now we can prove Lemma 3.3. It can be reduced to the following statement:

- *if* $b = -\mu^{-1} a \mu \neq a$ *then the equation*

$$z^2 + az + zb = 0$$

has unique solution $z = 0$.

Proof. Like in the proof of Lemma 3.2 we can assume that $\mathrm{Re}\, a = -\mathrm{Re}\, b = 0$. Next, using conjugation by means of unitary quaternions, i.e. rotations in the space of imaginary quaternions (see Lemma 5.3), and rescaling we can reduce a to i and b to $\alpha i + \beta j$, $\alpha^2 + \beta^2 = 1$. So we have the equation

$$z^2 + iz + z(\alpha i + \beta j) = 0, \quad \alpha \neq 1.$$

With $z = z_1 + j z_2$ this is equivalent to the following system

$$z_1^2 - |z_2|^2 + (1+\alpha) i z_1 - \beta \bar{z}_2 = 0, \tag{5.1}$$

$$2\mathrm{Re}\, z_1 \cdot z_2 - (1-\alpha) i z_2 + \beta \bar{z}_1 = 0. \tag{5.2}$$

Assume firstly that $\alpha = -1$. Then $\beta = \sqrt{1-\alpha^2} = 0$ and it is easy to see that we get $z_1 = z_2 = 0$.

Let $\beta \neq 0$. We put $z_1 = x + iy$. From equation (5.2) we get

$$z_2 = -\beta \frac{\bar{z}_1}{2x - (1-\alpha) i}.$$

Its substitution to equation (5.1) gives the following equation

$$x^2 - y^2 + 2ixy - (1-\alpha^2) \frac{x^2+y^2}{4x^2 + (1-\alpha)^2} + (1+\alpha)(ix - y) - (1-\alpha^2) \frac{x+iy}{2x - (1-\alpha) i} = 0.$$

The imaginary and real parts of the above equation give

$$y = -\frac{(1+\alpha)x^2}{2x^2 + 1 - \alpha}, \qquad x^2 - y^2 = 2y \frac{(1+\alpha)x^2}{2x^2 + 1 - \alpha}.$$

We see that $x^2 - y^2 = -2y^2$, i.e. $z_1 = z_2 = 0$. □

We finish this appendix with the following

PROPOSITION 5.6. *Any quadratic equation $zaz+bz+zc+d=0$, $a\neq 0$, has either one solution or two solutions or a 2-sphere of solutions in $\mathbb{H}$.*

PROOF. It is the same equation as the equation for fixed points of a Möbius map without fixed point at infinity (see equation (3.3)). From the classification Theorems 4.2 and 4.3 it follows that any Möbius map has either one fixed point or two fixed points or a sphere of fixed points. □

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Manuscript received October 24, 2007

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