Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 33, 2009, 149–168

SHARKOVSKII'S THEOREM, DIFFERENTIAL INCLUSIONS, AND BEYOND

Jan Andres — Tomáš Fürst — Karel Pastor

Abstract. We explain why the Poincaré translation operators along the trajectories of upper-Carathéodory differential inclusions do not satisfy the exceptional cases, described in our earlier counter-examples, for upper semicontinuous maps. Such a discussion was stimulated by a recent paper of F. Obersnel and P. Omari, where they show that, for Carathéodory scalar differential equations, the existence of just one subharmonic solution (e.g. of order 2) implies the existence of subharmonics of all orders. We reprove this result alternatively just via a multivalued Poincaré translation operator approach. We also establish its randomized version on the basis of a universal randomization scheme developed recently by the first author.

1. Introduction

In order to obtain an applicable version to differential equations and inclusions of the Sharkovskiĭ cycle coexistence theorem (cf. [23]), we published a series of papers (see [4], [8]–[13]) related to appropriate classes of multivalued maps. Let us note that the standard Sharkovskiĭ theorem for single-valued maps does not apply respectively, more precisely, that it only leads to empty statements. The desired application then was the following.

2000 *Mathematics Subject Classification.* 34A60, 37E05, 37E15, 37H10, 47H04, 47H40.

Key words and phrases. Sharkovskiĭ-type theorems, multivalued maps with monotone margins, Poincaré translation operators, coexistence of infinitely many periodic solutions, no exceptions, deterministic and random differential inclusions, random periodic solutions.

Supported by the Council of Czech Government (MSM 6198959214).

Theorem 1.1 ([11]). *Consider the inclusion*

$$\dot{x} \in F(t,x), \quad F(t+1,x) \equiv F(t,x), \tag{1.1}$$

where $F\colon [0,1]\times\mathbb{R} \multimap \mathbb{R}$ *is an upper-Carathéodory mapping* (*see Section* 2 *below*), *and assume that all solutions of* (1.1) *extend to* $\mathbb{R}$. *If* (1.1) *has an* n*-periodic solution, then it also admits a* k*-periodic solution, for any* $k \lhd n$ (*i.e. for any* k *smaller than* n *in the Sharkovskiĭ ordering of positive integers*), *with at most two exceptions.*

By a *k-periodic solution* of (1.1), we mean here, as well as in the entire text, an absolutely continuous function $x\colon \mathbb{R} \to \mathbb{R}$ satisfying (1.1) almost everywhere (a.e.) such that $x(t) \equiv x(t+k)$, but $x(t) \not\equiv x(t+j)$, for $1 \le j < k$; $j,k \in \mathbb{N}$.

Remark 1.2. The two exceptional cases can be detected explicitly (see e.g. [12], [13]). For multivalued maps (but not for differential equations or inclusions, as we shall see below), the exceptions can be even witnessed by counter-examples (see e.g. [4], [9]). On the other hand, there are classes of multivalued maps for which a full analogy (i.e. with no exceptions) of the standard Sharkovskiĭ theorem holds (see e.g. [5], [6]).

Our prime interest in this paper is to eliminate the exceptional cases in Theorem 1.1. In fact, for differential equations and inclusions, even much more was already achieved by a completely different method in [21]. Hence, as concerns the improvement of Theorem 1.1, we would especially like to understand why the mentioned counter-examples for multivalued maps (cf. [4], [9]) do not occur in terms of differential equations and inclusions.

We shall generalize and reprove in a simpler way the result of Obersnel and Omari [21], for differential inclusions, in terms of multivalued maps with monotone margins. Furthermore, we shall randomize this result by means of a transformation (to the deterministic case) technique, developed recently in [1]. Finally, we shall supply some comments and formulate open problems.

2. Preliminaries

In the entire text, all topological spaces will be (separable) metric and all multivalued maps will have nonempty values, i.e. by $\varphi\colon X \multimap Y$, we mean $\varphi\colon X \to 2^Y \setminus \{\emptyset\}$. We collect definitions and important statements that will be needed in the sequel.

By a *fixed point* of φ, we mean $x \in X \cap Y \neq \emptyset$ such that $x \in \varphi(x)$. The set of fixed points of φ will be denoted by $\mathrm{Fix}(\varphi) := \{x \in X : x \in \varphi(x)\}$.

An upper semicontinuous (u.s.c.) map $\varphi\colon X \multimap Y$, where X and Y are metric spaces, with nonempty, compact and connected values is called an *M-map*. Let us recall that a multivalued mapping $\varphi\colon X \multimap Y$ is *upper semicontinuous* if $\varphi^{-1}(U) := \{x \in X : \varphi(x) \subset U\}$ is open in X, for every open subset U of Y.

A multivalued mapping $\varphi: X \multimap Y$ is called *lower semicontinuous at* $x \in \mathrm{Dom}(\varphi)$ if, for any sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x, there exists a sequence of elements $y_n \in \varphi(x_n)$ converging to y. It is said to be *lower semicontinuous* (l.s.c.) if it is lower semicontinuous at every point $x \in \mathrm{Dom}(\varphi)$. It is well-known that $\varphi: X \multimap Y$ is l.s.c. if and only if $\varphi^{-1}(U)$ is closed in X, for every closed subset U of Y. It is also well-known that $\varphi: X \multimap Y$ is lower semicontinuous if and only if the set $\varphi_+^{-1}(U) := \{x \in X : \varphi(x) \cap U \neq \emptyset\}$ is open in X, for any open subset U of Y.

We say that a multivalued mapping φ is *continuous* if it is both upper semicontinuous and lower semicontinuous.

By a *measurable space*, we shall mean, as usual, the triple $(\Omega, \mathcal{U}, \mu)$, where a set Ω is equipped with σ-algebra $\mathcal{U}$ of subsets and a countably additive measure μ on $\mathcal{U}$.

Denoting, for $\varphi: X \multimap Y$, $\varphi^{-1}(B)$ and $\varphi_+^{-1}(B)$ as above, i.e. $\varphi^{-1}(B) := \{x \in X : \varphi(x) \subset U\}$ and $\varphi_+^{-1}(B) := \{x \in X : \varphi(x) \cap B \neq \emptyset\}$, namely the small and large counter-images of $B \subset Y$, we can define (weakly) measurable multivalued maps as follows.

Definition 2.1. Let $(\Omega, \mathcal{U}, \mu)$ be a measurable space and Y be a separable metric space. A map $\varphi: \Omega \multimap Y$ with closed values is called *measurable* if $\varphi^{-1}(B) \in \mathcal{U}$, for each open $B \subset Y$, or equivalently, if $\varphi_+^{-1}(B) \in \mathcal{U}$, for each closed $B \subset Y$. It is called *weakly measurable* if $\varphi_+^{-1}(B) \in \mathcal{U}$, for each open $B \subset Y$, or equivalently, if $\varphi^{-1}(B) \in \mathcal{U}$, for each closed $B \subset Y$.

It is well-known that, for compact-valued maps $\varphi: \Omega \multimap Y$, the notions of measurability and weak measurability coincide. Moreover, if φ and ψ are measurable, then so is their Cartesian product $\varphi \times \psi$. For more details see e.g. [7, Chapter 1, Proposition 3.45].

A multivalued mapping $F: [0,1] \times \mathbb{R} \multimap \mathbb{R}$ is called *upper-Carathéodory* (shortly, *u-Carathéodory*) if it satisfies the following conditions:

(a) $F(\,\cdot\,, x): [0,1] \multimap \mathbb{R}$ is measurable, for every $x \in \mathbb{R}$,
(b) $F(t, \cdot\,): \mathbb{R} \multimap \mathbb{R}$ is u.s.c., for almost all $t \in [0,1]$,
(c) there exist $a, b > 0$ such that $\sup\{|y| : y \in F(t,x)\} \leq a + b|x|$, for almost all $t \in [0,1]$ and every $x \in \mathbb{R}$.

For more details concerning semicontinuous and semi-Carathéodory multivalued maps, we recommend the monographs [2], [7], [14], [17], [18].

Definition 2.2. Let X and Y be metric spaces. A lower semicontinuous mapping $\varphi: X \multimap Y$ with nonempty, compact and connected values will be called an *N-map*. If φ is both lower semicontinuous and upper semicontinuous (i.e. continuous), we shall call it an *S-map*.

Definition 2.3. By a *k-orbit* of a mapping φ, we mean a sequence $\{x_0, \dots, x_{k-1}\}$ such that

(a) $x_{i+1} \in \varphi(x_i)$, for all $i = 0, \dots, k-2$, $x_0 \in \varphi(x_{k-1})$, and
(b) the orbit is not a product orbit formed by going p-times around a shorter m-orbit, where $mp = k$.

If still

(c) $x_i \neq x_j$, for $i \neq j$; $i, j = 0, \dots, k-1$, then we speak about a *primary k-orbit*.

3. Structure of periodic solutions to differential inclusions

It is well-known (cf. [2, Corollary 1 on p. 121]) that if the right-hand side $F\colon [0,1] \times \mathbb{R} \multimap \mathbb{R}$ of inclusion (1.1) is e.g. continuous with nonempty, convex, compact values and such that

$$\mathrm{d}_H(F(t,x), F(t,y)) \leq L(t)|x-y|, \quad \text{for all } t \in [0,1] \text{ and } x, y \in \mathbb{R}, \tag{3.1}$$

with a nonnegative integrable function $L\colon [0,1] \to \mathbb{R}$, where d_H stands for the Hausdorff distance (for the definition and more details, see e.g. [7], [14], [17], [18]), then the solution map $\Phi\colon \mathbb{R} \multimap C([0,1],\mathbb{R})$ that associates to an initial point $x_0 \in \mathbb{R}$ the set of solutions $x(\,\cdot\,)$ of (1.1) satisfying $x(0) = x_0$, is Lipschitz with constant $\exp \int_0^1 L(t)\,dt$. The associated *Poincaré translation operator* $T_1\colon \mathbb{R} \multimap \mathbb{R}$, defined by

$$T_1(x_0) := \{x(1) : x(\,\cdot\,) \text{ is a solution of (1.1) with } x(0) = x_0\},$$

is an M-map (cf. [7, Chapter III.6]). In particular, it is a composition of the Lipschitz continuous map $\Phi\colon \mathbb{R} \multimap C([0,1],\mathbb{R})$ with a continuous evaluation map $e\colon C([0,1],\mathbb{R}) \to \mathbb{R}$ defined by $e(x) := x(1)$, for any $x \in C([0,1],\mathbb{R})$. Consequently, $T_1 = e \circ \Phi$ is a continuous M-map, i.e. an S-map.

Hence, because of the correspondence between periodic solutions of (1.1) and periodic orbits of T_1 (for more details, see Section 4 below), Theorem 5 in [6] applies here as follows.

Corllary 3.1. *Consider the inclusion* (1.1)*, where $F\colon [0,1] \times \mathbb{R} \multimap \mathbb{R}$ is a continuous mapping with nonempty, convex, compact values satisfying* (3.1)*. If* (1.1) *has an n-periodic solution, then it also admits a k-periodic solution, for every $k \lhd n$.*

Although the exceptional cases which were present in Theorem 1.1 are eliminated in this particular case, one can say much more. Using the upper and lower solutions technique, F. Obersnel and P. Omari recently obtained the following theorem (cf. [21], [22]).

Theorem 3.2. *Let $F(t,x) \equiv F(t+1,x)$, and let F be an upper-Carathéodory multivalued mapping* (*cf. Section* 2) *with nonempty convex and compact values. If there exists an n-periodic solution of* (1.1) *with $n > 1$, then for any $k \in \mathbb{N}$, there also exists a k-periodic solution of* (1.1). *In addition, the set χ_k of all k-periodic solutions of* (1.1) *has dimension at least k, as a subset of $L^\infty(\mathbb{R})$.*

Remark 3.3. As pointed out in [21], condition (c) in the definition of an upper-Carathéodory map (cf. Section 2) can be replaced by a more general condition

(c') for every $\rho > 0$, there exists $\gamma \in L^1(0,1)$ such that $\sup\{|y| : y \in F(t,x)\} \le \gamma(t)$, for almost all $t \in [0,1]$ and every $x \in [-\rho,\rho]$.

In fact, the authors of [21] formulated Theorem 3.2 only for Carathéodory differential equations, i.e. only for single-valued $F\colon [0,1] \times \mathbb{R} \to \mathbb{R}$, but in [21, Remark 3] and [22] they pointed out that they were able to extend the validity of the theorem to differential inclusions, as in Theorem 3.2. At the same time, they decided to omit the proof of such an extension, because it follows from their arguments for differential equations with only minor changes. However, for the sake of completeness, and since we would like to understand why our counter-examples for M-maps (cf. [4], [9]) do not occur in terms of differential equations and inclusions, we decided to present their proof here.

Proof. Let $x(\,\cdot\,)$ be an n-periodic solution of (1.1), where $n > 1$. Let $t_0 \in \mathbb{R}$ be such that $x(t_0) < x(t_0+1)$. Let $j \in \{1,\dots,n-1\}$ be such that

$$x(t_0) < x(t_0+1) \le \dots \le x(t_0+j) \quad \text{and} \quad x(t_0+j) > x(t_0+j+1).$$

Let also $l \in \{0,\dots,j-1\}$ be such that

$$x(t_0+l) < x(t_0+l+1) = x(t_0+j).$$

Then we have

$$\max\{x(t_0+l), x(t_0+j+1)\} < x(t_0+j).$$

Set

$$I := (\max\{x(t_0+l), x(t_0+j+1)\}, x(t_0+j)).$$

Define $\alpha, \beta\colon [t_0, t_0+1] \to \mathbb{R}$ by

$$\alpha(t) := x(t+l) \quad \text{and} \quad \beta(t) := x(t+j).$$

Then α and β are solutions of (1.1) such that

$$\alpha(t_0) < \beta(t_0), \quad \beta(t_0+1) < \alpha(t_0+1).$$

Set

$$s_1 := \sup\{s \in (t_0, t_0+1) : \alpha(t) < \beta(t), \text{ on } [t_0, s)\},$$
$$s_2 := \inf\{s \in (t_0, t_0+1) : \beta(t) < \alpha(t), \text{ on } (s, t_0+1]\}.$$

Pick any $p \in I$. By [16, Theorem 6, Chapter 2.7], there exists a solution $v(\,\cdot\,)$ of (1.1) with $v(t_0) = p$, which can be continued to the right up to a point $r_1 \leq s_1$, where either $v(r_1) = \alpha(r_1)$ or $v(r_1) = \beta(r_1)$. In both cases, we can extend $v(\,\cdot\,)$ onto $[t_0, s_2]$ so that

$$\min\{\alpha(t), \beta(t)\} \leq v(t) \leq \max\{\alpha(t), \beta(t)\},$$

for all $t \in [t_0, s_2]$, and $v(s_2) = \alpha(s_2) = \beta(s_2)$.

Similarly, there exists a solution $w(\,\cdot\,)$ of (1.1) with $w(t_0 + 1) = p$, which can be constructed to the left up to a point $r_2 \geq s_2$, where either $w(r_2) = \alpha(r_2)$ or $w(r_2) = \beta(r_2)$. In both cases, we can extend $w(\,\cdot\,)$ onto $[s_2, t_0 + 1]$ so that

$$\min\{\alpha(t), \beta(t)\} \leq w(t) \leq \max\{\alpha(t), \beta(t)\},$$

for all $t \in [s_2, t_0 + 1]$, and $w(s_2) = \alpha(s_2) = \beta(s_2)$. Set

$$u_p(t) := \begin{cases} v(t) & \text{for } t \in [t_0, s_2], \\ w(t) & \text{for } t \in [s_2, t_0 + 1]. \end{cases}$$

Then $u_p(\,\cdot\,)$ gives raise to a 1-periodic solution of (1.1) satisfying $u_p(t_0) = p$.

This already means that, for each $p \in I$, there exists a 1-periodic solution $u_p(\,\cdot\,)$ of (1.1) such that $u_p(t_0) = p$ and $u_p(s_2) = \alpha(s_2) = \beta(s_2)$. By the lattice structure of the solution set of (1.1), one can easily find and increasing sequence $\{u_m(\,\cdot\,)\}_{m\in\mathbb{N}}$ of 1-periodic solutions of (1.1) such that $u_m(t_0) < u_{m+1}(t_0)$ and $u_m(s_2) = u_0(s_2)$, for every $m \in \mathbb{N}$.

A k-periodic solution $v(\,\cdot\,)$ of (1.1), where $k > 1$, can be constructed by setting

$$v(t) := u_m(t), \quad \text{on } [s_2 + m + ik, s_2 + m + 1 + ik],$$

for every $m \in \{0, \ldots, k - 1\}$ and $i \in \mathbb{Z}$. Let us prove that $v(\,\cdot\,)$ has a minimal period of k. Since $v(\,\cdot\,)$ is continuous and nonconstant, $v(\,\cdot\,)$ has a minimal period $\tau > 0$. Suppose, by contradiction, that $\tau < k$. Notice that, by the definition of $v(\,\cdot\,)$, $\tau \neq 1$ and because k is a multiple of τ, one has $\tau \leq k/2$. This particularly implies that $\tau < k - 1$. If $\tau > 1$ (and so $k > 2$) we get

$$\max v = \max v \restriction_{[s_2, s_2+\tau]} \leq \max_{m=0,\ldots,k-2} u_m = \max u_{k-2} < \max u_{k-1} = \max v,$$

whereas if $\tau < 1$, we get

$$\max v = \max u_0 \restriction_{[s_2, s_2+\tau]} \leq \max u_0 < \max u_{k-1} = \max v.$$

In both cases, a contradiction is achieved. Hence, we conclude that $\tau = k$.

Finally, to prove that the dimension of χ_k is at least k, we show that $[0, 1]^k$ is embedded in χ_k. Let $\mathcal{K}_m$ be the set of all solutions $v(\,\cdot\,)$ of (1.1), on $[s_2, s_2 + 1]$, such that $u_m \leq v \leq u_{m+1}$. By [16, Theorem 6, Chapter 2], $\mathcal{K}_m$ is a continuum in $C([s_2, s_2 + 1], \mathbb{R})$. Let $\mathcal{T}_m$ be a totally ordered subset of $\mathcal{K}_m$. By [20, Lemma 3.6],

$\mathcal{T}_m$ is homeomorphic to a compact interval in $\mathbb{R}$. Extend all functions $v \in \mathcal{T}_m$ by 1-periodicity onto $\mathbb{R}$, so that each $v(\,\cdot\,)$ is a 1-periodic solution of (1.1). Define

$$\Phi_k\colon \prod_{m=0}^{k-1} \mathcal{T}_m \to L^\infty(\mathbb{R})$$

by setting

$$\Phi_k(v_0, \dots, v_{k-1})(t) := v_m(t), \quad \text{on } [s+m+ik, s+m+1+ik],$$

for every $m \in \{0, \dots, k-1\}$ and $i \in \mathbb{Z}$. Clearly, Φ_k is one-to-one and continuous, and so it is a homeomorphism between

$$\prod_{m=0}^{k-1} \mathcal{T}_m \quad \text{and} \quad \Phi\Big(\prod_{m=0}^{k-1} \mathcal{T}_m\Big) \subset \chi_k.$$

This completes the proof. □

Theorem 3.2 is rather surprising, because it is in no way related to Sharkovskiĭ's ordering. Any subharmonic solution, e.g. a 2-periodic solution (whence the title of [21]), implies, for any $k \in \mathbb{N}$, the existence of an infinite set of k-periodic solutions of (1.1). Nontrivial examples of equations and inclusions satisfying the assumptions of Theorem 3.2 were given in [20] and [9]. More precisely, in [20, Example 3.3, p. 355], the equation

$$\dot{x} = \sqrt{|x|} - \frac{1}{8\pi}|\arcsin(\sin(\pi t))| \tag{3.2}$$

was shown to possess an infinite set of k-periodic solutions, for each $k \in \mathbb{N}$, coexistning with complicated dynamics, and demonstrating the complexity of asymptotic behaviour.

In [9], we presented two such examples of differential inclusions. The linear inclusion

$$\dot{x} + cx \in P(t), \tag{3.3}$$

where $P(t) = [0, |\sin(\pi t)|]$, for $t \in (-\infty, \infty)$, admits again, for $c \neq 0$, k-periodic solutions, for every $k \in \mathbb{N}$. This is because P possesses a k-periodic selection $p_k \subset P$, for every $k \in \mathbb{N}$, in the form

$$p_k(t) := \begin{cases} 0 & \text{for } t \in [0, k-1], \\ |\sin(\pi t)| & \text{for } t \in [k-1, k]. \end{cases}$$

Similarly, consider the inclusion

$$\dot{x} \in F(x) + p(t), \tag{3.4}$$

where

$$F(x) := \begin{cases} 0 & \text{for } x \neq 0, \\ [-1, 1] & \text{for } x = 0, \end{cases}$$

and $p(t) = \cos(\pi t)$, for $t \in [0,1)$ and $p(t+1) \equiv p(t)$. According to [9, Example 1], inclusion (3.4) admits, for every $k \in \mathbb{N}$, a k-periodic solution $x_k(\,\cdot\,)$ in the form

$$x_k(t) := \begin{cases} 0 & \text{for } t \in [0, k-1], \\ \dfrac{1}{\pi}\sin(\pi t) & \text{for } t \in [k-1, k]. \end{cases}$$

Inclusion (3.3) satisfies the assumptions of Corollary 3.1, by which the associated Poincaré translation operator T_1 is an S-map, and so our counter-examples in [4], [9] (constructed for M-maps which are not S-maps) cannot occur for T_1 (for more details, see [6]).

On the other hand, for equation (3.2) and inclusion (3.4), the associated Poincaré operators are no longer S-maps. More precisely, for (3.4), T_1 can be easily calculated as

$$T_1(x) := \begin{cases} x & \text{for } x < -\dfrac{1}{\pi}, \\ \left[-\dfrac{1}{\pi}, 0\right] & \text{for } x\left[-\dfrac{1}{\pi}, 0\right], \\ x & \text{for } x > 0, \end{cases}$$

i.e. T_1 is "only" an M-map (see Figure 1).

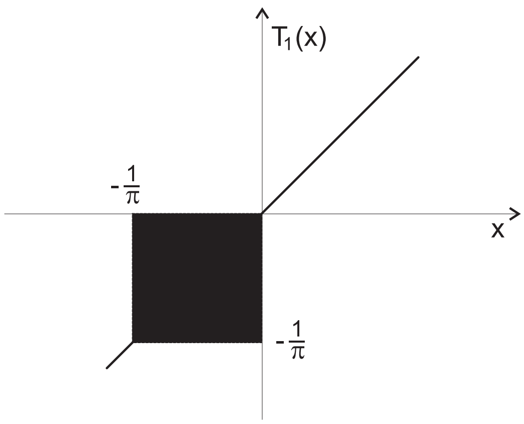


Figure 1. Poincaré operator T_1 for inclusion (3.4)

Numerical computations demonstrate that the M-map T_1 for equation (3.2) can be detected as in Figure 2.

Both Figures 1 and 2 already indicate the difference when compared to the counter-examples in [4], [9] which show the absence of the exceptional orbits. We would like to describe this difference in a more systematic way.

For the differential equation

$$\dot{x} = f(t, x) \tag{3.5}$$

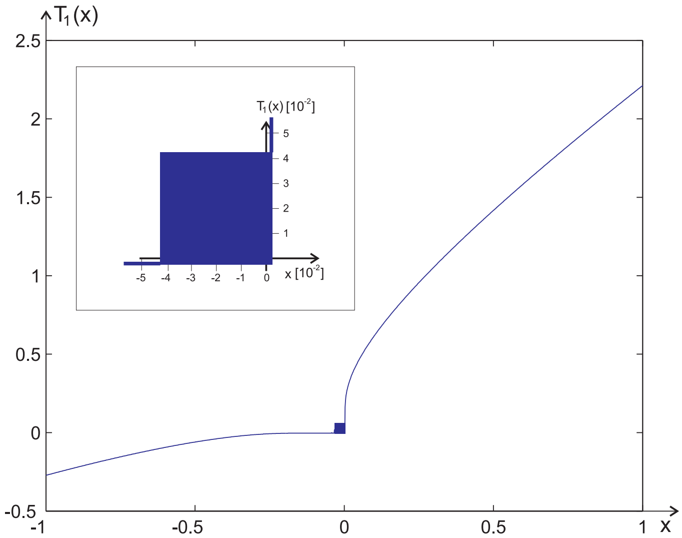


FIGURE 2. Poincaré operator T_1 for equation (3.2). The figure was generated using MatLab.

which satisfies a uniqueness condition, the related (single-valued) Poincaré operator can be easily shown to be strictly increasing. Define

$$T_t(x) := \{y(t) : y(\,\cdot\,) \text{ is a solution of (3.5) with } y(0) = x\}, \tag{3.6}$$

consider $x_1 < x_2$, denote $y_i := T_t(x_i)$, for $i = 1, 2$, and assume, on the contrary, that $y_1 \geq y_2$. It is obvious that the solution starting at x_1 has to cross the trajectory of the solution starting at x_2, which is a contradiction to the uniqueness assumption. The strict monotonicity of T_t therefore demonstrates that, in the case of uniqueness, the standard Sharkovskiĭ theorem does not apply, because no periodic points can occur.

Hence, let us now consider equation (3.5) without a uniqueness assumption or, more generally, the differential inclusion (1.1). The related *Poincaré operator* $T_t\colon \mathbb{R} \multimap \mathbb{R}$, defined again by

$$T_t(x) := \{y(t) : y(\,\cdot\,) \text{ is a solution of (1.1) with } y(0) = x\},$$

becomes obviously multivalued. A natural question arises, whether T_t preserves any type of monotonicity.

A multivalued map $\varphi\colon \mathbb{R} \multimap \mathbb{R}$ is monotone if

$$(x - y)(f_x - f_y) \geq 0, \quad \text{for all } f_x \in \varphi(x) \text{ and } f_y \in \varphi(y).$$

However, the assumption of monotonicity is rather severe. It can be shown that a monotone multivalued map is necessarily single-valued on a G_δ-set which is dense in the interior of $\mathrm{Dom}(\varphi)$ (see [14, Proposition 4.2]). This indicates that, in this context, the notion of monotonicity is not suitable for multivalued Poincaré operators.

On the other hand, *marginal maps* (cf. [2, pp. 51–54]) called also *value functions* (cf. [18, Chapter 1.3.0]), namely point-supremum and point-infimum value functions will be shown to preserve monotonicity.

Let us define

$$T_t^*(x) := \sup\{y : y \in T_t(x)\}, \tag{3.7}$$

$$T_{t*}(x) := \inf\{y : y \in T_t(x)\}. \tag{3.8}$$

Note that in the case of upper-Carathéodory right-hand side F (for the definition, see Section 2), the associated Poincaré operator T_t is known to be an M-map and, in particular, to have compact values (cf. [7, Chapter III.9]), and so the supremum and infimum in (3.7) and (3.8) can be replaced by maximum and minimum, respectively. The functions T_t^* (resp. T_{t*}) are known to be upper semicontinuous (resp. lower semicontinuous) in the sense of single-valued maps (see [18, Proposition 3.3 in Chapter 1.3.0], [2, Corollary 2 in Chapter 1.2]).

We will show that both T_t^* and T_{t*} are nondecreasing. Consider $x_1 < x_2$, denote $y_i := T_t^*(x_i)$, for $i = 1, 2$, and suppose, on the contrary, that $y_1 > y_2$. There exist solutions $z_i\colon [0, t] \to \mathbb{R}$ with $z_i(0) = x_i$ and $z_i(t) = y_i$, for $i = 1, 2$. The solutions z_1 and z_2 necessarily cross each other at some point $t = t_0$. But then the function $\widetilde{z}\colon [0, t] \to \mathbb{R}$ defined by

$$\widetilde{z}(s) := \begin{cases} z_2(s) & \text{for } s \le t_0, \\ z_1(s) & \text{for } s \ge t_0, \end{cases}$$

is also a solution, and consequently $y_1 \in T_t(x_2)$ which is a contradiction to y_2 being the maximal element of $T_t(x_2)$. We can show analogously that T_{t*} is also nondecreasing.

Proposition 3.4. *Let the right-hand side F of inclusion* (1.1) *be an upper-Carathéodory map. Then the marginal maps T_t^* (resp. T_{t*}) defined in* (3.7) (*resp.* (3.8)) *are upper semicontinuous* (*resp. lower semicontinuous*) *in the sense of single-valued maps and nondecreasing.*

Since the marginal maps of the M-maps in the mentioned counter-examples in [4], [9] are not monotone, such counter-examples cannot apply to differential equations or inclusions. In other words, Proposition 3.4 demonstrates that the class of M-maps is considerably wider than the class of Poincaré maps.

4. Simpler proof of Theorem 3.2 in terms of maps

Let us now show an extremely simple proof of Theorem 3.2 in terms of multivalued maps. In fact, we shall formulate its significant generalization.

We say that a multivalued mapping $\varphi\colon \mathbb{R} \multimap \mathbb{R}$ has *monotone margins* on a dense set D in $\mathbb{R}$ if the single-valued mappings $\varphi^*\colon D \to \mathbb{R}\cup\{\infty\}$ and $\varphi_*\colon D \to \mathbb{R}\cup\{-\infty\}$ defined by

$$\varphi^*(x) := \sup\{y : y \in \varphi(x)\}, \quad \varphi_*(x) := \inf\{y : y \in \varphi(x)\},$$

are either both nonincreasing or both nondecreasing functions.

Theorem 4.1. *Let D be a dense set in $\mathbb{R}$ and let $\varphi\colon \mathbb{R} \multimap \mathbb{R}$ be a multivalued mapping with nonempty connected values whose margins φ^*, φ_* are either both nondecreasing or both nonincreasing on D. If φ has an n-orbit $\{x_0, \dots, x_{n-1}\} \subset D$ with $n > 1$, $n \in \mathbb{N}$, then φ has also a primary k-orbit, for every $k \in \mathbb{N}$. Moreover, the set of all k-orbits (as a subset of $\mathbb{R}^k$) has dimension k.*

Proof. We proceed by two steps and restrict ourselves to the case of nondecreasing margins on D. The case of nonincreasing margins can be treated analogously.

Step 1. Let $\{x_0, \dots, x_{n-1}\} \subset D$ be the given n-orbit of φ. Denote $a := \min\{x_0, \dots, x_{n-1}\}$ and let $b, c \in \{x_0 \dots, x_{n-1}\}$, $b \neq a$, $c \neq a$, be such that $b \in \varphi(a)$ and $a \in \varphi(c)$. There are two possibilities:

- $b \geq c$. Since φ_* is nondecreasing on D, $c \in \varphi(a)$ and $\varphi(a)$ is connected, it holds that $a \in \varphi(a)$. Further, since φ^* is nondecreasing on D as well, $b \in \varphi(a)$ and the connectedness of $\varphi(c)$ imply that $c \in \varphi(c)$.
- $b < c$. Again $a \in \varphi(a)$, because φ_* is nondecreasing on D, $b \in \varphi(a)$ and $\varphi(a)$ is connected. Using $a \in \varphi(c)$, $b \in \varphi(a)$ and the property of the margins again, it follows from the connectedness of $\varphi(b)$ that $a \in \varphi(b)$ and $b \in \varphi(b)$.

Summing up the previous results, φ has a fixed point a, and a 2-orbit, say $\{a, b\}$, satisfying $a \in \varphi(a)$, $b \in \varphi(a)$, $b \in \varphi(b)$ and $a \in \varphi(b)$. Since φ has connected values and nondecreasing margins on D, we have that $([a, b] \cap D) \times [a, b]$ is a subset of the graph Γ_φ of φ.

Step 2. We will show that φ has a primary k-orbit, for every $k \in \mathbb{N}$. We can choose the points $\{z_0, \dots, z_{k-1}\}$ from the set $[a, b] \cap D$ e.g. in the following way:

$$a = z_0 < z_1 < \dots < z_{k-2} < z_{k-1} = b.$$

Due to Step 1, $\{z_0, \dots, z_{k-1}\}$ is the primary k-orbit of φ.

Notice that, in fact, any k-tuple $\{z_0, \dots z_{k-1}\}$ with $a \leq z_i \leq b$, $z_i \in D$, for $i = 0, \dots k-1$, which satisfies condition (b) in Definition 2.3 is a k-orbit of φ. Therefore, the set of all such k-tuples has dimension k. $\square$

Corollary 4.2. *The conclusion of Theorem* 4.1 *holds for M-maps and N-maps* (*and so S-maps*) *with monotone margins and so, in particular, for Poincaré* *translation operators* $T_t\colon \mathbb{R} \multimap \mathbb{R}$ *along the trajectories of* (1.1). *Thus, Theorem* 4.1 *can be also interpreted, via the Poincaré operators* T_1*, in terms of subharmonic solutions of* (1.1).

Proof. We restrict ourselves only to the special case of Poincaré operators T_1. Since T_1 is an M-map (see [7, Chapter III.9]), it has in particular nonempty connected values. According to Proposition 3.4, the related marginal maps T_1^* and T_{1*} are nondecreasing. So, in order to apply Theorem 4.1 via Poincaré operators T_1 along the trajectories of (1.1), it is sufficient to realize that if $\{x_0, \ldots, x_{k-1}\}$ is a k-orbit of T_1, then any solution of (1.1) with $x(0) = x_0, \ldots, x(k-1) = x_{k-1}$ and $x(k) = x_0$ becomes, after a k-periodic prolongation, k-periodic. Obviously, if two k-orbits of T_1 differ e.g. in x_i, then the corresponding k-periodic solutions differ at $t = i$. Let us note that any k-periodic orbit of T_1 can determine, in general, many k-periodic solutions of (1.1). So, there need not be a one-to-one correspondence between k-periodic orbits of T_1 and k-periodic solutions of (1.1).

As concerns the least dimension k of the set of all k-periodic solutions of (1.1) (as a subset of L^∞; cf. Theorem 3.2), notice (in view of the proof of Theorem 4.1) that $[a,b] \subset T_1([a,b])$. Thus, there exist solutions $\alpha, \beta\colon [0,1] \to \mathbb{R}$ such that

$$\alpha(0) = a, \quad \alpha(1) = b \quad \text{and} \quad \beta(0) = b, \quad \beta(1) = a.$$

Hence, we can proceed exactly in the same way as in the proof of Theorem 3.2, to make the conclusion about the dimension of the set of k-periodic solutions. □

5. Randomization of Theorem 3.2

Our final goal is to randomize Theorem 3.2 via random Poincaré translation operators. For this, we need the following definitions of random operators and random periodic orbits given in [1].

Definition 5.1 (random operator). Let $\varphi\colon \Omega \times X \multimap X$ be a multivalued map with nonempty closed values, where $\Omega = (\Omega, \mathcal{U}, \mu)$ is a complete measurable space ([1]), $\mathcal{U}$ is a σ-algebra of subsets of Ω, μ is a countably additive measure, and X is a separable metric space. We say that φ is a *random operator* if it is product-measurable (measurable in the whole), i.e. measurable w.r.t. the minimal σ-algebra $\mathcal{U} \otimes \mathcal{B}(X)$, generated by $\mathcal{U} \times \mathcal{B}(X)$, where $\mathcal{B}(X)$ denotes the Borel sets of X.

([1]) i.e. $\mathcal{U} = \widehat{\mathcal{U}}$, where $\widehat{\mathcal{U}} = \bigcap_\nu \mathcal{U}_\nu$, with a positive bounded measure ν on $(\Omega, \mathcal{U})$, and $\mathcal{U}_\nu$ is the ν-completion of $\mathcal{U}$.

Definition 5.2 (random k-orbit). Let $\varphi\colon \Omega\times X \multimap X$ be a random operator. A sequence of measurable maps $\{\xi_i\}_{i=0}^{k-1}$, where $\xi_i\colon \Omega \to X$, $i = 0,\dots,k-1$, is called a *random k-orbit*, associated to φ, if

(a) $\xi_{i+1}(\omega) \in \varphi(\omega,\xi_i(\omega))$, $i = 0,\dots,k-2$, and $\xi_0(\omega) \in \varphi(\omega,\xi_{k-1}(\omega))$, for almost all $\omega \in \Omega$,

(b) the random k-orbit is not a random product orbit formed by going p-times around a shorter m-orbit, where $mp = k$.

The following crucial proposition will allow us to transform the study of random periodic orbits to the deterministic case.

Proposition 5.3 (transformation to deterministic case). *Let $\varphi\colon \Omega\times X \multimap X$ be a random operator. Then φ possesses, for some integer $k > 1$, a random k-orbit if and only if $\varphi(\omega,\,\cdot\,)\colon X \multimap X$ admits, for almost every $\omega \in \Omega$, a k_ω-orbit with some $k_\omega \in \mathbb{N}$ such that $k_\omega|k$, and there exists a measurable subset $\Omega_0 \subset \Omega$ with $\mu(\Omega_0) > 0$ such that $\varphi(\omega,\,\cdot\,)$ admits, for almost all $\omega \in \Omega_0$, an m-orbit, for some $m > 1$ with $m|k$.*

Remark 5.4. Proposition 5.3 is only a particular case (sufficient for our needs here) of a more general statement in [1], where a complete characterization of random k-orbits, for a given $k \in \mathbb{N}$, is given in a deterministic way.

Now, consider the random system

$$\dot{x}(\omega,t) \in F(\omega,t,x(\omega,t)), \tag{5.1}$$

where $F(\omega,t,x) \equiv F(\omega,t+1,x)$, and assume that $F\colon \Omega\times[0,1]\times\mathbb{R}^n \multimap \mathbb{R}^n$ is a *random u-Carathéodory map*, i.e.

(a) $F(\,\cdot\,,\,\cdot\,,x)\colon \Omega\times[0,1] \multimap \mathbb{R}^n$ is product-measurable, for all $x \in \mathbb{R}^n$,

(b) $F(\omega,t,\,\cdot\,)\colon \mathbb{R}^n \multimap \mathbb{R}^n$ is u.s.c. for almost all $(\omega,t) \in \Omega\times[0,1]$,

(c) there exists $a,b > 0$ such that $\sup\{|y| : y \in F(\omega,t,x)\} \le a + b|x|$, for almost all $(\omega,t) \in \Omega\times[0,1]$ and all $x \in \mathbb{R}^n$,

with nonempty, convex and compact values.

Definition 5.5 (random solution). By a (*random*) *solution* of (5.1), we mean a function $x\colon \Omega\times\mathbb{R} \to \mathbb{R}^n$ such that $x(\,\cdot\,,t)$ is measurable, $x(\omega,\,\cdot\,)$ is (locally) absolutely continuous and x satisfies (5.1), for almost all $(\omega,t) \in \Omega\times[0,1]$; the derivative $\dot{x}$ is considered w.r.t. t. By a *random k-periodic solution* of (5.1), we mean a random solution x of (5.1) such that $x(\omega,t) \equiv x(\omega,t+k)$.

Associate with (5.1), for some $t_0 \in [0,1]$, the *random Poincaré translation operator* $T_k\colon \Omega\times\mathbb{R}^n \multimap \mathbb{R}^n$ along the trajectories of (5.1) defined (in a deterministic way) as follows:

$$T_k(\omega,x_0) := \{x(t_0+k) : x(\,\cdot\,) \text{ is a solution of } \dot{x} \in F_\omega(t,x), \text{ with } x(t_0) = x_0\}, \tag{5.2}$$

where

$$\dot{x} \in F_\omega(t,x) \quad (:= F(\omega,t,x)), \tag{5.3}$$

is a one parameter family of deterministic inclusions, obtained from (5.1) by fixing $\omega \in \Omega$.

It is known that (cf. [7], [15]) $T_k = T_1^k$ and that T_k is a random M-map, i.e. a random operator with compact connected values and $T_k(\omega, \cdot\,)$ is an M-map.

On the other hand, in order to determine random periodic solutions of (5.1) by means of random periodic orbits of the associated random Poincaré operator (5.2), we must still prove the following important proposition.

Proposition 5.6. *System* (5.1) *possesses a random k-periodic solution if and only if T_1 admits, with a suitable $t_0 \in [0,1]$, a random k-orbit* (*random periodic orbit of order k*) *in the sense of Definition* 5.2.

Proof. Let $\{\xi_i\}_{i=0}^{k-1}$ be a given (fixed) random k-orbit of the random Poincaré operator T_1, associated with (5.1), which is defined by means of (5.2). Let

$$\widetilde{P}\colon \Omega \multimap \mathrm{AC}([t_0,t_0+1]) \times \ldots \times \mathrm{AC}([t_0+k-1,t_0+k])$$

prescribe to $\omega \in \Omega$, in a component-like way, solutions $x(\omega,t)$ of (5.1), where $x_i(\omega,t_0+i) = \xi_i(\omega)$, $x_i = x|_{[t_0+i,t_0+i+1]}$, $i = 0,\ldots,k-1$, and $x_{k-1}(\omega,t_0+k) = \xi_0(\omega)$, namely $\widetilde{P}\colon \omega \multimap \{x_i(\omega,t)\}_{i=0}^{k-1}$. Our claim is to show that $\widetilde{P}$ is a measurable operator having a (single-valued) measurable selection $\widetilde{x} \subset \widetilde{P}$ which represents a random k-periodic solution $\widetilde{x}(\omega,t)$ of (5.1), where $\widetilde{x}(\omega,t_i) = \xi_i(\omega)$, $i = 0,1,\ldots,k-1$.

Hence, define

$$\widehat{T}\colon \Omega \times (\mathbb{R}^n)^k \overset{P}{\multimap} \mathrm{AC}_\omega([t_0,t_0+1]) \times \ldots \times \mathrm{AC}_\omega([t_0+k-1,t_0+k]) \xrightarrow{\widehat{e}} (\mathbb{R}^n)^k,$$

where

$$\mathrm{AC}_\omega([t_0,t_0+j]) := \{x : x(\omega,\cdot\,) \in \mathrm{AC}([t_0,t_0+j])\}, \quad j = 1,\ldots,k,$$

$$P := \begin{cases} P_1\colon (\omega,x_0) \multimap \{x_0(\omega,\cdot\,)\}, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ P_k\colon (\omega,x_{k-1}) \multimap \{x_{k-1}(\omega,\cdot\,)\}, \end{cases}$$

$$\widehat{e} := \begin{cases} x_0(\omega,\cdot\,) \to x_0(\omega,1) - \xi_1(\omega), \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ x_{k-1}(\omega,\cdot\,) \to x_{k-1}(\omega,k) - \xi_k(\omega)(= \xi_0(\omega)), \end{cases}$$

where $x(\omega,\cdot\,)$ is a solution of (5.3) such that $x_i(\omega,t_i) = x_i$, $i = 0,\ldots,k-1$.

Then $P = P(\omega,x)$, and subsequently the superposition

$$\widehat{P} = P(\omega,\{\xi_i(\omega)\}_{i=1}^{k-1}),$$

are measurable operators with compact values. This follows from the randomness of $P_1, \dots, P_k$ (proved in [15]), and the facts that a Cartesian product of measurable operators is also measurable (cf. [7, Chapter I.3]) and that a product-measurability implies the superpositional measurability (cf. [3]).

Because of

$$\widetilde{P}^{-1}(A) = \widehat{P}^{-1}(A \cap \widehat{e}^{-1}(0)),$$

for every closed A, and compact values of $\widehat{P}$, $\widetilde{P}$ is indeed a measurable operator (see Preliminaries), as claimed. Applying the Kuratowski–Ryll-Nardzewski theorem (cf. e.g. [7, pp. 48–50]), there exists a single-valued measurable selection $\widetilde{x} \subset \widetilde{P}$,

$$\widetilde{x}\colon \Omega \to \mathrm{AC}([t_0, t_0+1]) \times \dots \times \mathrm{AC}([t_0+k-1, t_0+k]), \quad i = 0, \dots, k-1,$$

representing the desired random k-periodic solution $\widetilde{x}(\omega, t)$ of (5.1), where $\widetilde{x}(\omega, t_0+i) = \xi_i(\omega)$, $i = 0, \dots, k-1$. $\square$

An alternative proof of Proposition 5.6 can be done as follows.

Proof (Alternative proof of Proposition 5.6). Consider system (5.3) and define the solution operators $S_r\colon \Omega \times \mathbb{R}^n \multimap (\mathrm{AC}([t_0, t_0+k]))^n$ by the formula:

$$S_r(\omega, x_r) := \{x \in (\mathrm{AC}([t_0, t_0+k]))^n : x \text{ is a solution of (5.3) with } x(t_0+r) = x_r\},$$

where $r = 0, \dots, k$.

Since $S_r(\omega, x_r) = S_r^+(\omega, x_r) \cup S_r^-(\omega, x_r)$, where

$$S_r^+(\omega, x_r) := \begin{cases} \emptyset & \text{for } t \in [t_0, t_0+r], \\ S_r(\omega, x_r) & \text{for } t \in [t_0+r, t_0+k], \end{cases}$$

$$S_r^-(\omega, x_r) := \begin{cases} S_r(\omega, x_r) & \text{for } t \in [t_0, t_0+r], \\ \emptyset & \text{for } t \in [t_0+r, t_0+k], \end{cases}$$

and

$$S_r^-(\omega, x_r) = \widetilde{S}_r^+(\omega, x_r),$$

where

$$\widetilde{S}_r^+(\omega, x_r) := \{x \in (\mathrm{AC}([-t_0-r, -t_0]))^n : x \text{ is a solution of } \dot{x} \in -F_\omega(-t, x) \text{ with } x(-t_0-r) = x_r\},$$

S_r must be a product-measurable operator, for every $r = 0, \dots, k$. This follows from the product-measurability of S_r^+ and $\widetilde{S}_r^+$, $r = 0, \dots, k$, proved in [15], and the fact that the union of two measurable operators is also measurable (cf. [7, Chapter I.3]).

Now, for a given random k-orbit $\{\xi_i\}_{i=0}^{k-1}$ define the intersection $S\colon\Omega \multimap (\mathrm{AC}([t_0, t_0 + k]))^n$ of compositions $S_r(\omega, \xi_r(\omega))$, $r = 0, \dots, k$, i.e.

$$S(\omega) := \bigcap_{r=0}^{k} S_r(\omega, \xi_r(\omega)), \quad \text{where } \xi_0(\omega) \equiv \xi_k(\omega).$$

The definition of a k-orbit (see Definition 5.2) guarantees that S has nonempty values. Moreover, since S_r^+, resp. S_r^- are, according to [15], random M-maps, the set of values must be compact.

Since the product-measurability implies a superpositional measurability (cf. [3]) and the intersection of product-measurable operators is also product-measurable (cf. [7, Chapter I.3]), S is a measurable operator.

Thus, applying the Kuratowski–Ryll-Nardzewski selection theorem (cf. e.g. [7, pp. 48–50]), there exists a single-valued measurable selection $x \subset S$,

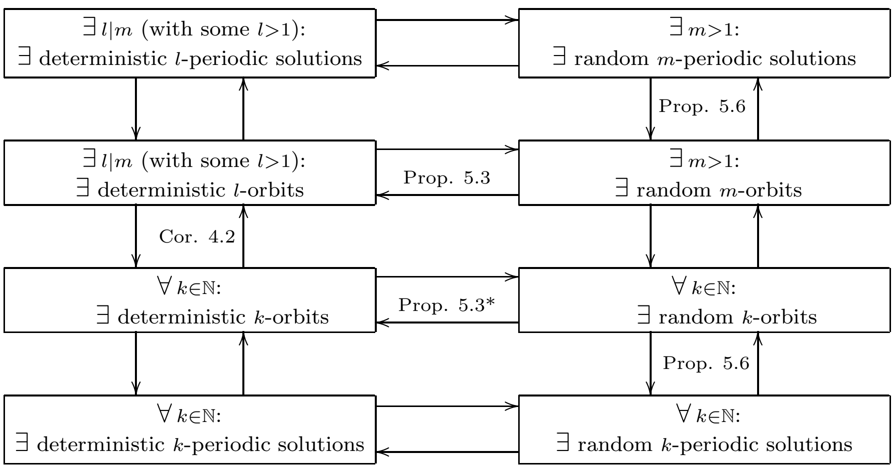
$$x\colon \Omega \to (\mathrm{AC}([t_0, t_0 + k]))^n$$

which is the desired random k-periodic solution $x(\omega, t)$ of (5.1), where $x(\omega, t_0 + i) = \xi_i(\omega)$, $i = 0, \dots, k - 1$. □

We are ready to give the main result of this section.

Theorem 5.7. *Consider the scalar* ($n = 1$) *inclusion* (5.1) *whose right-hand side F is a random u-Carathéodory map. If* (5.1) *possesses a random m-periodic solution, for some $m > 1$, then it also admits a random k-periodic solution, for every $k \in \mathbb{N}$.*

Proof. We can use the following randomization scheme for periodic solutions:



- *deterministic orbits* are related to the deterministic Poincaré operators $T_1(\omega, \cdot)$, with a suitable $t_0 \in [0, 1]$, associated with the inclusions (5.3)

(l-orbits, with some $l > 1$, are considered for fixed $\omega \in \Omega_0$, where $\Omega_0 \subset \Omega$ is a measurable subset such that $\mu(\Omega_0) > 0$),

- *random orbits* are related to the random Poincaré operator $T_1(\,\cdot\,,\,\cdot\,)$, with a suitable $t_0 \in [0,1]$, associated with the inclusion (5.1),
- *deterministic periodic solutions* are related to the deterministic inclusions (5.3) (l-periodic solutions, with some $l > 1$, are considered for fixed $\omega \in \Omega_0$, where $\Omega_0 \subset \Omega$ is a measurable subset such that $\mu(\Omega_0) > 0$),
- *random periodic solutions* are related to the random inclusion (5.1).

The deterministic (left-hand) part of the scheme is based on the investigations in the foregoing section. The inner (vertical) equivalence about the coexistence of periodic orbits of the Poincaré operators is due to Corollary 4.2. The outer (vertical) equivalences reflect the obvious fact that every k-orbit determines a k-periodic solution and, vice versa, every k-periodic solution determines, in a suitable point $t_0 \in [0,1]$, a k-orbit.

Because of Proposition 5.3 we can write the horizontal equivalences in the inner part related to Poincaré operators. The asterisk in the symbol Proposition 5.3* indicates that the implication from the random to the deterministic directions was obtained by means of a joint application of Corollary 4.2. As a consequence, we get the right-hand vertical equivalence in the inner part.

The upper and lower horizontal equivalences in the random (right-hand) part are due to Proposition 5.6. As a consequence, we obtain the upper and lower horizontal equivalences for periodic solutions in outer part of the scheme. □

REMARK 5.8. Observe that although (thanks to the inner part) Theorem 5.7 solves only the implication concerning the coexistence of random periodic solutions of all orders in the lower right-hand corner implied by the existence of random subharmonics of order higher than 1 in the upper right-hand corner (the reverse implication is trivial), we have to our disposal the scheme with ten equivalences. Theorem 5.7 can be, therefore, directly improved in this way. Unfortunately, because of the application of the Aumann-type and Kuratowski–Ryll-Nardzewski selection theorems in the proofs of Proposition 3.4 (cf. [1]) and Proposition 5.6, we lost the information about the topological dimension of the solution sets, as given in Theorem 3.2. On the other hand, for each $k \in \mathbb{N}$, the cardinality of the set of random k-periodic solutions must be bigger or at least equal to the cardinality of the set of random k-orbits of the associated random Poincaré operators.

EXAMPLE 5.9. Consider the random linear inclusion

$$\dot{x}(\omega,t) + cx(\omega,t) \in P(t) + r(\omega), \quad x\colon \Omega\times\mathbb{R}\to\mathbb{R},\ \omega\in\Omega, \tag{5.4}$$

where c is a real constant, $P(t) \equiv P(t+1)$, $P(t) := [0, |\sin(\pi t)|]$, for $t \in [0,1]$, and $r\colon \Omega \to \mathbb{R}$ is a random perturbation.

Since, for each $m \in \mathbb{N}$, there exists an m-periodic selection $p_m \subset P$ of P, namely $p_m(t) \equiv p_m(t+m)$, where

$$p_m(t) := \begin{cases} 0 & \text{for } t \in [0, m-1], \\ |\sin(\pi t)| & \text{for } t \in [m-1, m], \end{cases}$$

every random m-periodic solution of the random equation

$$\dot{x}(\omega, t) + cx(\omega, t) = p_m(t) + r(\omega), \quad x\colon \Omega \times \mathbb{R} \to \mathbb{R},\ \omega \in \Omega, \tag{5.5}$$

satisfies the inclusion (5.4).

Consider still the family of deterministic equations

$$\dot{x} + cx = p_m(t) + r_\omega, \quad x \in \mathbb{R}, \tag{5.6}$$

where $r_\omega = r(\omega)$, for each fixed $\omega \in \Omega$.

Since $p_m(t) + r_\omega \equiv p_m(t+m) + r_\omega$, equation (5.6) has, for any $c \neq 0$, exactly one m-periodic solution. In view of the randomization scheme, inclusion (5.5) therefore possesses a random m-periodic solution which also satisfies inclusion (5.4), for all $m \in \mathbb{N}$.

Example 5.10. Consider the random nonlinear equation

$$\dot{x}(\omega, t) = \sqrt{|x(\omega, t)|} - \frac{1}{16\pi}|1 + r(\omega)|\,|\arcsin(\sin(\pi t))|,$$

where $r\colon \Omega \to \mathbb{R}$ is a random perturbation such that $|r(\omega)| \leq 1/2$.

It can be proved exactly in the same way as in [20, Example 3.3 on p. 235] that, for each fixed $\omega \in \Omega$, the deterministic equation

$$\dot{x} = \sqrt{|x|} - \frac{1}{16\pi}|1 + r_\omega|\,|\arcsin(\sin(\pi t))|,$$

where $r_\omega = r(\omega)$, admits, for every $m \in \mathbb{N}$, an m-periodic solution passing through the origin $(t, x) = (0, 0)$. In view of the randomization scheme, equation (3.6) therefore possesses, for every $m \in \mathbb{N}$, a random m-periodic solution.

Observe that the conclusions in both illustrating examples were possible even without an explicit application of Theorem 5.7.

6. Concluding remarks and open problems

A deeper insight into the theory of scalar ordinary differential inclusions allows us to claim that the class of the associated Poincaré operators is too narrow, for the exceptional absent orbits of M-maps, illustrated by counter-examples in our earlier papers [4], [9]. More precisely, because of the monotone margins of Poincaré's operators, the existence of large sets of periodic orbits has, rather surprisingly, nothing to do with the Sharkovskii ordering. Despite this fact, not only period three, but each nontrivial (i.e. of order greater than 1) period implies all periods, and subsequently each subharmonic of order greater than 1 implies the existence of large sets of subharmonics of all orders.

In these lines, we have to understand what monotone margins mean for multivalued mappings $\varphi\colon\mathbb{R}^n \multimap \mathbb{R}^n$. The standard definition of monotonicity (see e.g. [14]) says that a single-valued mapping $s\colon\mathbb{R}^n \to \mathbb{R}^n$ is monotone if

$$\langle s(y) - s(x), y - x\rangle \geq 0, \quad \text{for every } x, y \in \mathbb{R}^n.$$

Being stimulated by vector optimization (see e.g. [19]), one can equip $\mathbb{R}^n$ with a cone C and define

$$\begin{aligned}\overline{\varphi}(x) := \{y \in \mathbb{R}^n \mid \exists h \in S_{\mathbb{R}^n} \cap C : \langle y, h\rangle = \sup\{\langle z, h\rangle : z \in \varphi(x),\ h \in S_{\mathbb{R}^n} \cap C\},\\ y \text{ is a cluster point of } \varphi(x)\},\\ \underline{\varphi}(x) := \{y \in \mathbb{R}^n \mid \exists h \in S_{\mathbb{R}^n} \cap C : \langle y, h\rangle = \inf\{\langle z, h\rangle : z \in \varphi(x), h \in S_{\mathbb{R}^n} \cap C\},\\ y \text{ is a cluster point of } \varphi(x)\},\end{aligned}$$

for every $x \in \mathbb{R}^n$, where $S_{\mathbb{R}^n}$ denotes the unit sphere in $\mathbb{R}^n$. The marginal maps φ^*, φ_* can then be defined as arbitrary single-valued selections from $\overline{\varphi}$ and $\underline{\varphi}$, respectively. On this basis, we would like to establish elsewhere the following triangular generalization of Theorem 4.1.

CONJECTURE 6.1. *Assume that $\varphi\colon\mathbb{R}^n \multimap \mathbb{R}^n$ is a multivalued map with nonempty connected values such that φ has the triangular structure. Assume, furthermore, that there exist monotone marginal maps φ^*, φ_* in the sense indicated above. If φ has an n-orbit with $n > 1$, $n \in \mathbb{N}$, then φ has also a primary k-orbit, for any $k \in \mathbb{N}$.*

If Conjecture 6.1 can be affirmatively solved, it could be randomized in the same way as in Theorem 3.2. On the other hand, it would be also nice to deduce the topological dimension of sets of random periodic solutions.

Acknowledgements. We are grateful to Professors Franco Obersnel and Pierpaolo Omari (University of Trieste, Italy) for sending us the manuscript of the paper [20] and for fruitful discussions about the proof of Theorem 3.2.

REFERENCES

[1] J. ANDRES, *Randomization of Sharkovskiĭ-type theorems*, Proc. Amer. Math. Soc. **136** (2008), 1385–1395; *Erratum*, Proc. Amer. Math. Soc. **136** (2008), 3733–3734.

[2] J.-P. AUBIN AND A. CELLINA, *Differential Inclusions*, Springer, Berlin, 1984.

[3] J. APPELL, E. DEPASCALE, N. H. THÁI AND P. P. ZABREĬKO, *Multi-valued superpositions*, Dissertationes Math. **345** (1995), 1–97.

[4] J. ANDRES, J. FIŠER AND L. JÜTTNER, *On a multivalued version of the Sharkovskiĭ theorem and its application to differential inclusions*, Set-Valued Anal. **10** (2002), 1–14.

[5] J. Andres, T. Fürst and K. Pastor, *Period two implies all periods for a class of* ODE*s: a multivalued map approach*, Proc. Amer. Math. Soc. **135** (2007), 3187–3191.

[6] ———, *Full analogy of Sharkovskiĭ's theorem for lower semicontinuous maps*, J. Math. Anal. Appl. **340** (2008), 1132–1144.

[7] J. Andres and L. Górniewicz, *Topological Fixed Point Principles for Boundary Value Problems*, Kluwer, Dordrecht, 2003.

[8] J. Andres and L. Jüttner, *Period three plays a negative role in a multivalued version of Sharkovskiĭ's theorem*, Nonlinear Anal. **51** (2002), 1101–1104.

[9] J. Andres, L. Jüttner and K. Pastor, *On a multivalued version of the Sharkovskiĭ theorem and its application to differential inclusions* II, Set-Valued Anal. **13** (2005), 47–68.

[10] J. Andres and K. Pastor, *On a multivalued version of the Sharkovskiĭ theorem and its application to differential inclusions,* III, Topol. Methods Nonlinear Anal. **22** (2003), 369–386.

[11] ———, *A version of Sharkovskiĭ's theorem for differential equations*, Proc. Amer. Math. Soc. **133** (2005), 449–453.

[12] J. Andres, K. Pastor and P. Šnyrychová, *A multivalued version of Sharkovskiĭ's theorem holds with at most two exceptions*, J. Fixed Point Theory Appl. **2** (2007), 153–170.

[13] J. Andres, P. Šnyrychová and P. Szuca, *Sharkovskiĭ's theorem for connectivity G_δ-relations*, Internat. J. Bifur. Chaos **16** (2006), 2377–2393.

[14] K. Deimling, *Multivalued Differential Equations*, W. de Gruyter, Berlin, 1992.

[15] F. S. De Blasi, L. Górniewicz and G. Pianigiani, *Random topological degree and random differential inclusions*, unpublished manuscript (1998).

[16] A. F. Filippov, *Differential Equations with Discontinuous Right-hand Sides*, Kluwer, Dordrecht, 1988.

[17] L. Górniewicz, *Topological Fixed Point Theory of Multivalued Mappings*, Springer, Berlin, 2006, 2nd edition.

[18] S. Hu and N. S. Papageorgiou, *Handbook of Multivalued Analysis*, vol. 1, Kluwer, Dordrecht, 1997.

[19] J. Jahn, *Vector Optimization*, Springer, Berlin, 2004.

[20] F. Obersnel and P. Omari, *Old and new results for first order periodic* ODE*s without uniqueness: a comprehensive study by lower and upper solutions*, Adv. Nonlin. Stud. **4** (2004), 323–376.

[21] ———, *Period two implies chaos for a class of* ODE*s*, Proc. Amer. Math. Soc. **135** (2007), 2055–2058.

[22] ———, private communication.

[23] A. N. Sharkovskiĭ, *Coexistence of cycles of a continuous map of a line into itself*, Ukrainian Math. J. **16** (1964), 61–71 (in Russian); English transl. in Internat. J. Bifur. Chaos **5** (1995), 1263–1273.

Manuscript received December 13, 2007

Jan Andres, Tomáš Fürst and Karel Pastor
Department of Mathematical Analysis
and Applications of Mathematics
Faculty of Science
Palacký University
Tomkova 40
779 00 Olomouc-Hejčín, CZECH REPUBLIC

E-mail address: andres@inf.upol.cz, tomas.furst@seznam.cz, pastor@inf.upol.cz