ON NONSYMMETRIC THEOREMS FOR \((H, G)\)-COINCIDENCES

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Abstract. Let \(X\) be a compact Hausdorff space, \(\varphi : X \to S^n\) a continuous map into the \(n\)-sphere \(S^n\) that induces a nonzero homomorphism \(\varphi^* : H^n(S^n; \mathbb{Z}_p) \to H^n(X; \mathbb{Z}_p)\), \(Y\) a \(k\)-dimensional CW-complex and \(f : X \to Y\) a continuous map. Let \(G\) a finite group which acts freely on \(S^n\). Suppose that \(H \subset G\) is a normal cyclic subgroup of a prime order. In this paper, we define and we estimate the cohomological dimension of the set \(A_\varphi(f, H, G)\) of \((H, G)\)-coincidence points of \(f\) relative to \(\varphi\).

1. Introduction

K. D. Joshi [10] has proved a nonsymmetric generalization of the Borsuk–Ulam theorem [1], in which the \(n\)-sphere \(S^n\) is replaced by a certain compact subset \(X\) of the \((n + 1)\)-dimensional Euclidean space \(\mathbb{R}^{n+1}\). In this context, a pair of points \(x, y \in X\) are said to be antipodal if \(y = -\lambda x\), for some \(\lambda > 0\). The Joshi’s theorem shows that for every continuous map \(f : X \to \mathbb{R}^n\) there exist antipodal points \(x, y \in X\) such that \(f(x) = f(y)\).

K. Borsuk has suggested to define antipodal points in an arbitrary space in the following way: \(x_1, x_2 \in X\) are said to be antipodal points relative to an

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essential map \( \varphi: X \to S^n \) if \( \varphi(x_1) = -\varphi(x_2) \). Using the Borsuk’s suggestion, Spieć \([11]\) has proved that if \( X \) is a compact Hausdorff space and if \( \varphi: X \to S^n \) is an essential map, then for every continuous map \( f: X \to \mathbb{R}^k \), the covering dimension of the set

\[
A_{\varphi}(f) = \{ x \in X : \text{there exists } y \in X, \text{ such that } \varphi(x) = -\varphi(y) \text{ and } f(x) = f(y) \}
\]

is not less than \( n - k \), obtaining thus a generalization of the Joshi’s theorem.

M. Izydorek \([7]\) has extended the proposition of Borsuk for a cyclic group \( G \) of order prime which acts freely on a \( n \)-dimensional sphere \( S^n \) and has proved the following generalization of the Spieć’s theorem: if \( X \) is a compact Hausdorff space and if \( \varphi: X \to S^n \) is an essential map, then for every continuous map \( f: X \to \mathbb{R}^k \), the covering dimension of the set

\[
A_{\varphi}(f) = \{ x_1, \ldots, x_p \} \text{ such that } \varphi(x_1) = g^{-1}\varphi(x_2) = \ldots = g^{-1}p \varphi(x_p) \text{ and } f(x_1) = \ldots = f(x_p) \}
\]

is not less than \( n - (p - 1)k \), where \( g \) is a fixed generator of \( G \). Moreover, if \( \mathbb{R}^k \) is replace by a generalized \( k \)-dimensional manifold \( M^k \) over \( \mathbb{Z}_p \), then an analogous theorem has been proved (see \([7, \text{Theorem 4}]\)).

Gonçalves, Jaworowski and Pergher \([3]\) have defined \((H,G)\)-coincidence for a continuous map \( f \) from a \( n \)-sphere \( S^n \) into a \( k \)-dimensional CW-complex \( Y \), where \( G \) is a finite group which acts freely on \( S^n \) and have proved that if \( H \) is a nontrivial normal cyclic subgroup of a prime order, then

\[
\text{cohom.dim } A(f,H,G) \geq n - |G|k,
\]

where \( A(f,H,G) \) is the set of \((H,G)\)-coincidence points of \( f \) and cohom.dim denotes the cohomological dimension. The other papers closely related to \([3]\) are \([4]–[6], [8], [9] \) and \([13]\).

The purpose of this paper is to define the set \( A_{\varphi}(f,H,G) \) of \((H,G)\)-coincidence points of a continuous map \( f: X \to Y \) relative to an essential map \( \varphi: X \to S^n \), where \( X \) is a compact Hausdorff space, \( Y \) is a topological space, \( G \) is a finite group which acts freely on the \( n \)-dimensional sphere \( S^n \) and \( H \) is a subgroup of \( G \). Using this definition, under certain conditions, we estimate the cohomological dimension of the set \( A_{\varphi}(f,H,G) \). Specifically, we will prove the following nonsymmetric version of the main theorem of \([3]\):

**Theorem 1.1.** Let \( X \) be a compact Hausdorff space, \( Y \) a \( k \)-dimensional CW-complex and \( \varphi: X \to S^n \) an essential map. Given a finite group \( G \) which acts freely on \( S^n \) and \( H \) a normal cyclic subgroup of prime order, then for every
continuous map \( f: X \to Y \) such that \( f^*: H^n(Y; \mathbb{Z}_p) \to H^n(X; \mathbb{Z}_p) \) is trivial for \( i \geq 1 \), cohom.dim \( A_\varphi(f, H, G) \geq n - |G|k \).

For the proof of Theorem 1.1, it was fundamental to prove the following version of the main Theorem of [3],

**Theorem 1.2.** Let \( X \) be a paracompact Hausdorff space, \( Y \) a \( k \)-dimensional CW-complex, \( G \) a finite group which acts freely on \( X \) and \( H \subset G \) a normal cyclic subgroup of prime order. Let \( f: X \to Y \) be a continuous map such that \( f^*: H^n(Y; \mathbb{Z}_p) \to H^n(X; \mathbb{Z}_p) \) is trivial for \( i \geq 1 \). Suppose that the \( \mathbb{Z}_p \)-index of \( X \) is greater than or equal to \( n \), then the \( \mathbb{Z}_p \)-index of the set \( A(f, H, G) \) is greater than or equal to \( n - |G|k \). Consequently, cohom.dim \( A(f, H, G) \geq n - |G|k \).

2. Preliminaries

Throughout this paper the symbols \( H \) and \( H^* \) will denote Čech homology and cohomology groups with coefficients in \( \mathbb{Z}_p \), unless otherwise indicated. If \( G \) is a group which acts on a topological space \( X \), we will denote by \( X^G \) the orbit space \( X/G \).

We start by introducing some basic notions and definitions as follows.

**2.1. \((H, G)\)-coincidence.** Suppose that \( X, Y \) are topological spaces, \( G \) is a group acting freely on \( X \) and \( f: X \to Y \) is a map. If \( H \) is a subgroup of \( G \), then \( H \) acts on the right on each orbit \( Gx \) of \( G \) as follows: if \( y \in Gx \) and \( y = gx \), \( g \in G \), then \( hy = ghx \) (such action may depend on the choice of the reference point \( x \)). Following [4], [6], [9] the concept of \( G \)-coincidence is generalized as follows: a point \( x \in X \) is said to be a \((H, G)\)-coincidence point of \( f \) if \( f \) sends every orbit of the action of \( H \) on the \( G \)-orbit of \( x \) to a single point (see [5]). We will denote by \( A(f, H, G) \) the set of all \((H, G)\)-coincidence points of \( f \). If \( H \) is the trivial subgroup, then every point of \( X \) is a \((H, G)\)-coincidence. If \( H = G \), this is the usual definition of coincidence. If \( G = \mathbb{Z}_p \) with \( p \) prime, then a nontrivial \((H, G)\)-coincidence point is a \( G \)-coincidence point.

**2.2. The space \( X_\varphi \) and the set \( A_\varphi(f, H, G) \).** Let us consider \( X \) a compact Hausdorff space and an essential map \( \varphi: X \to S^n \). Suppose \( G \) be a finite group of order \( r \) which acts freely on \( S^n \) and \( H \) be a subgroup of order \( p \) of \( G \). Let \( G = \{g_1, g_2, \ldots , g_r\} \) be a fixed enumeration of elements of \( G \), where \( g_1 \) is the identity of \( G \). A nonempty space \( X_\varphi \) can be associated with the essential map \( \varphi: X \to S^n \) as follows:

\[
X_\varphi = \{(x_1, \ldots , x_r) \in X^r : \varphi(x_1) = (g_2)^{-1}\varphi(x_2) = \ldots = (g_r)^{-1}\varphi(x_r)\} = \{(x_1, \ldots , x_r) \in X^r : g_1\varphi(x_1) = \varphi(x_i), \ i = 1, \ldots , r\},
\]

where \( X^r \) denotes the \( r \)-fold cartesian product of \( X \). The set \( X_\varphi \) is a closed subset of \( X^r \) and so it is compact. We define a \( G \)-action on \( X_\varphi \) as follows: for
each \( g_i \in G \) and for each \( (x_1, \ldots, x_r) \in X_\varphi \),

\[
    g_i(x_1, \ldots, x_r) = (x_{\sigma_i(1)}, \ldots, x_{\sigma_i(r)}),
\]

where the permutation \( \sigma_i \) is defined by \( \sigma_i(k) = j \), \( g_i g_j = g_j \). We observe that if \( x = (x_1, \ldots, x_r) \in X_\varphi \) then \( x_i \neq x_j \), for any \( i \neq j \) and therefore \( G \) acts freely on \( X_\varphi \).

Now, let us consider a continuous map \( f : X \to Y \), where \( Y \) is a topological space and \( \tilde{f} : X_\varphi \to Y \) given by \( \tilde{f}(x_1, \ldots, x_r) = f(x_1) \). Let \( y = (x_1, \ldots, x_r) \in A(\tilde{f}, H, G) \) and consider the orbit \( G y = \{g_1 y, g_2 y, \ldots, g_y y\} \). Note that

(i) From (2.1), we have that for each \( i \), the 1-th coordinate of \( g_i y \) is \( x_i \).

(ii) The action of \( H \) on \( G y \) determines a partition of the orbit \( G y \) in \( s = (r/p) \) disjoint suborbits, and we can be rewrite

\[
    G y = \{g_1 y, \ldots, g_{s_1} y; \ldots; g_{s_2} y, \ldots, g_{s_r} y\},
\]

where \( \{1, \ldots, r\} \leftrightarrow \{1, \ldots, l_p; \ldots; j_1, \ldots, j_{p_1}; \ldots; s_1, \ldots, s_p\} \) is a bijection.

Since \( y \) is a \((H,G)\)-coincidence point of \( \tilde{f} \), it follows from (i) and (ii) that,

\[
    f(x_{1_1}) = \ldots = f(x_{1_p}); \ldots; f(x_{s_1}) = \ldots = f(x_{s_p}); \ldots; f(x_{s_1}) = \ldots = f(x_{s_p}).
\]

In these conditions, we have the following

**Definition 2.1.** The set \( A_\varphi(f, H, G) \) of \((H,G)\)-coincidence points of \( f \) relative to \( \varphi \) is defined by

\[
    A_\varphi(f, H, G) = A(\tilde{f}, H, G) = \{(x_1, \ldots, x_r) \in X^r : g_i \varphi(x_1) = \varphi(x_i), \ i = 1, \ldots, r \text{ and } f(x_{j_1}) = \ldots = f(x_{j_s}), \ j = 1, \ldots, s\}.
\]

**Remark 2.2.** Let us observe that if \( G = H = Z_p \),

\[
    A_\varphi(f, H, G) = A_\varphi(f) = \{(x_1, \ldots, x_p) \in X^p : g_i \varphi(x_1) = \varphi(x_i), \ i = 1, \ldots, p \text{ and } f(x_1) = \ldots = f(x_p)\}.
\]

**2.3. The \( Z_p \)-index.** Suppose that the cyclic group \( G = Z_p \) of order prime \( p \) acts freely on a Hausdorff and paracompact space \( X \). Then \( X \to X^* \) is a principal \( Z_p \)-bundle and one can take \( h: X^* \to BZ_p \) a classifying map for the \( G \)-bundle \( X \to X^* \).

**Remark 2.3.** It is well known that if \( \hat{h} \) is another classifying map for the principal \( Z_p \)-bundle \( X \to X^* \), then there is a homotopy between \( h \) and \( \hat{h} \).

We will consider the following definition for the \( Z_p \)-index of \( X \) (see [7]).
DEFINITION 2.4. We say that the $\mathbb{Z}_p$-index of $X$ is greater than or equal to $k$ if the homomorphism $h^*: H^k(B\mathbb{Z}_p) \to H^k(X^*)$ is nontrivial. We say that the $\mathbb{Z}_p$-index of $X$ is equal to $k$ if it is greater than or equal to $k$ and moreover $h^*: H^i(B\mathbb{Z}_p) \to H^i(X^*)$ is zero, for any $i \geq k + 1$.

REMARK 2.5. A model for $B\mathbb{Z}_2$ the classifying space for $\mathbb{Z}_2$ is the infinite real projective space $P^\infty$. Then $H^*(B\mathbb{Z}_2) \cong H^*(P^\infty)$ is isomorphic to $\mathbb{Z}_2[a]$, where $a \in H^1(P^\infty)$ is the generator. The generator of $H^i(B\mathbb{Z}_2)$ is $a^i$ for any $i \geq 0$. If $p > 2$ a model for $B\mathbb{Z}_p$ the classifying space for $\mathbb{Z}_p$ is the infinite lens space $L^\infty_p = S^\infty/\mathbb{Z}_p$. Thus, $H^i(B\mathbb{Z}_p) = H^i(L^\infty_p) \cong \mathbb{Z}_p$ for any $i \geq 0$ and given any nonzero element $a \in H^1(L^\infty_p)$, one has that $b = \beta(a)$ is a nonzero element of $H^2(L^\infty_p)$, where $\beta: H^1(L^\infty_p) \to H^2(L^\infty_p)$ is the Bockstein homomorphism. More generally, a generator $\mu \in H^i(B\mathbb{Z}_p)$ is given by

\[ \mu = \begin{cases} a \cdot b^{(i-1)/2} & \text{if } i \text{ is odd} \\ b^{i/2} & \text{if } i \text{ is even} \end{cases} \]

2.4. The Smith special cohomology groups with coefficients in $\mathbb{Z}_p$.

In this work, we will be considering the definition of the Smith special cohomology groups with coefficients in $\mathbb{Z}_p$ in the sense of [2]. Smith homology and cohomology were originally defined in [12] and in a series of subsequent papers. A systematic exposition of the Smith theory can be found in [2]. Let $X$ be a topological space; given a finite group $G$ of prime order $p$ which acts freely on $X$, let $g$ be a fixed generator of $G$ and put

\[ \sigma = 1 + g + g^2 + \ldots + g^{p-1} \text{ and } \tau = 1 - g, \]

in the group ring $\mathbb{Z}_p(G)$. We have that $\sigma = \tau^{p-1}$. If $\rho = \tau^i$, we put $\overline{\rho} = \tau^{p-i}$, then $\tau = \overline{\sigma}$ and $\sigma = \overline{\tau}$. There exists an exact sequence with coefficients in $\mathbb{Z}_p$ [2, p.125],

\[ \xrightarrow{\delta} H^n_{\rho}(X) \xrightarrow{\partial^*} H^n(X) \xrightarrow{T} H^n(X^*) \xrightarrow{\delta} H^{n+1}_{\rho}(X^*) \xrightarrow{\partial^*} \]

called Smith exact sequence, where $H^n_{\rho}(X)$ denotes the Smith special cohomology groups and $T$ is the transfer homomorphism.

REMARK 2.6. The Smith cohomology groups are natural with respect to $\mathbb{Z}_p$-equivariant maps, that is, if $f: X \to Y$ is a $\mathbb{Z}_p$-equivariant map then $f$ induces homomorphism $f^*: H^n_{\rho}(Y) \to H^n_{\rho}(X)$ which commutes with the homomorphisms in the Smith sequence.

3. The $\mathbb{Z}_p$-index of $X_{\varphi}$

Let us consider the free $G$-space $X_{\varphi}$ as defined in Section 2.2. If $H \subset G$ is a cyclic subgroup of prime order $p$, then $X_{\varphi}$ is a free $H \cong \mathbb{Z}_p$-space. In these conditions, as in [7, Theorem 3], we have the following
Theorem 3.1. Let $X$ be a compact Hausdorff space and $\varphi: X \to S^n$ an essential map. Then, the $\mathbb{Z}_p$-index of $X_\varphi$ is equal to $n$.

Proof. For $i = 1, \ldots, r$, let us consider the maps $\varphi_i: X \to S^n$ given by $\varphi_i(x) = (g_i)^{-1}\varphi(x)$, where $g_i \in G$. Then, we can define the map

$$\psi = \varphi_1 \times \ldots \times \varphi_r: X^r \to [S^n]^r,$$

where $[S^n]^r$ denotes the $r$-fold cartesian product of $S^n$, such that $X_\varphi = \psi^{-1}\Delta[S^n]^r$, where $\Delta[S^n]^r$ is the diagonal in $[S^n]^r$. In these conditions, we prove the following

Lemma 3.2. The homomorphism $\psi^*: H^*([S^n]^r) \to H^*(X^r)$ induced by $\psi: X^r \to [S^n]^r$ is a monomorphism in each dimension.

Proof. Let $m$ be an integer and for each $t = 1, \ldots, r$ consider $H^m([S^n]^r)$. If $m$ is not divisible by $n$ then $H^m([S^n]^r) = 0$ and the result follows. Suppose that $m$ is divisible by $n$; one then has that there exists $\alpha = 0, 1, \ldots, n$ such that $m = \alpha n$. Let us consider the following commutative diagram

$$
\begin{array}{ccc}
H^\alpha([S^n]^r) \otimes H^0(S^n) \oplus H^{(\alpha-1)n}([S^n]^r) & \to & H^\alpha([S^n]^r) \\
\downarrow{(\varphi_1 \times \ldots \times \varphi_t)}^* \otimes \varphi_{t+1}^* & & \downarrow{(\varphi_1 \times \ldots \times \varphi_{t+1})}^* \\
H^\alpha(X^t) \otimes H^0(X) \oplus H^{(\alpha-1)n}(X^t) & \to & H^\alpha(X^t) 
\end{array}
$$

(3.1)

Applying the Künneth’s formula, we have that the upper row of the above diagram is an isomorphism and the lower row is a monomorphism, for each $\alpha = 0, 1, \ldots$ and $t = 1, \ldots, r$.

The proof will be done by induction on $t$. We assume inductively that for some $t = 1, \ldots, r - 1$ and for each $\alpha = 0, 1, \ldots$ the homomorphism $(\varphi_1 \times \ldots \times \varphi_t)^*: H^\alpha([S^n]^r) \to H^\alpha(X^t)$ is a monomorphism and we will show that $(\varphi_1 \times \ldots \times \varphi_{t+1})^* \otimes \varphi_{t+1}^*$ is a monomorphism.

The result will follow from the commutativity of the diagram (3.1). By induction hypothesis it suffices to show that $\varphi_{t+1}^*$ is a monomorphism. For this, observe that $\varphi_i^* = \varphi_j^*$, for any $1 \leq i, j \leq r$. If $\varphi_{t+1}^*$ is not a monomorphism, then $\varphi_i^*$ is not a monomorphism, for any $1 \leq i \leq t$, which completes the proof. \hfill $\square$

The next step is to use Lemma 3.2 to show that the homomorphism induced by $\psi|_{X_\varphi}: X_\varphi \to \Delta[S^n]^r$ is a monomorphism. Let us consider the following commutative diagram

$$
\begin{array}{ccc}
H^n(X^r, X_\varphi) & \xrightarrow{j^*} & H^n(X^r) \\
\downarrow{\psi^*} & & \downarrow{\psi^*} \\
H^n([S^n]^r, \Delta[S^n]^r) & \xrightarrow{\psi^*} & H^n([S^n]^r) \\
\end{array}
$$

(3.2)

Theorem 3.1 follows from this diagram, where $j^*$ is the inclusion map.
whose rows are exact. If $\gamma$ is a generator of $H^n(S^n) \cong \mathbb{Z}_p$ let us denote by $\alpha_i$ the element $q_i^*(\gamma) \in H^n([S^n]^r)$, where $q_i: [S^n]^r \to S^n$ is the natural projection on the $i$-th coordinate for $i = 1, 2, \ldots, r$. Let us observe that $q_i \circ q_j = q_{ij}$ for any $1 \leq i, j \leq r$, where $i: [S^n]^r \hookrightarrow [S^n]^r$ is the natural inclusion. In this way, one has that

$$i^*(\alpha_i) = i^* \circ q_i^*(\gamma) = (q_i \circ i)^*(\gamma) = (q_j \circ i)^*(\gamma) = i^*(\alpha_j).$$

**Lemma 3.3.** $(\psi|_{X_{\varphi}})^*: H^n(\Delta[S^n]^r) \to H^n(X_{\varphi})$ is a monomorphism.

**Proof.** Since $\Delta[S^n]^r$ is homeomorphic to $S^n$, it follows from (3.3) that $i^*(\alpha_i)$ is a nonzero element in $H^n(\Delta[S^n]^r)$ for any $i = 1, \ldots, r$. Thus, it suffices to show that $(\psi|_{X_{\varphi}})^*(i^*(\alpha_1)) \neq 0$. Let us assume that this does not happen. From the diagram (3.2) we have that

$$k^* \circ \psi^*(\alpha_1) = (\psi|_{X_{\varphi}})^* \circ i^*(\alpha_1) = 0,$$

which implies that $\psi^*(\alpha_1) \in \ker(k^*) = \text{Im}(j^*)$ and there exists an element $U \in H^n(X^r, X_{\varphi})$ such that

$$j^*(U) = \psi^*(\alpha_1) \neq 0,$$

since by Lemma 3.2 $\psi^*$ is a monomorphism and $\alpha_1 = q_1^*(\gamma) \in H^n([S^n]^r)$ is a nonzero element. Let us consider the following commutative diagram,

$$\begin{array}{ccc}
H^{(r-1)n}(X^r, X^r - X_{\varphi}) & \xrightarrow{j^*} & H^{(r-1)n}(X^r) \\
\xrightarrow{\psi^*} & D^{-1} & \xrightarrow{k^*} H^{(r-1)n}(X^r - X_{\varphi}) \\
\bigg\downarrow & \bigg\downarrow & \bigg\downarrow \\
H_n(\Delta[S^n]^r) & \xrightarrow{i^*} & H_n([S^n]^r)
\end{array}$$

where the first and the second rows are exact, $D$ is the Alexander–Spanier Duality which is an isomorphism and all others maps are induced by appropriate inclusions.

Let us denote by $a_1, \ldots, a_r$ the elements of $H_n([S^n]^r)$ which are conjugated to $a_1, \ldots, a_r$ in $H^n([S^n]^r)$. More precisely,

$$\langle \alpha_j, \alpha_i \rangle = \begin{cases} 
1 & \text{if } i = j, \\
0 & \text{if } i \neq j,
\end{cases}$$

where the map $\langle \cdot, \cdot \rangle: H^n([S^n]^r) \times H_n([S^n]^r) \to \mathbb{Z}_p$ denotes the Kronecker product. Let us denote by $c \in H_n(\Delta[S^n]^r)$ the conjugated element to $i^*(\alpha_1) = \ldots = i^*(\alpha_r)$ in $H^n(\Delta[S^n]^r)$. Then for any $j = 1, \ldots, r$ one has $\langle i^*(\alpha_j), c \rangle = 1.$
Furthermore, it follows from properties of the Kronecker product that for any $j = 1, \ldots, r$

$$\langle j^*(\alpha_j), c \rangle = \langle \alpha_j, i_*(c) \rangle = 1,$$

which implies that

\begin{equation}
(3.6) \quad i_*(c) = \sum_{i=1}^{r} a_i.
\end{equation}

Let us consider for each $i = 1, \ldots, r$ the elements

$$\beta_i = \alpha_1 \wedge \ldots \wedge \alpha_{i-1} \wedge \hat{\alpha}_i \wedge \alpha_{i+1} \wedge \ldots \wedge \alpha_r \in H^{(n-1)r}([S^n]^r),$$

where the symbol $\hat{\alpha}_i$ means that the element $\alpha_i$ is omitted.

To simplify notation, here we will also denote by $[S^n]^r$ the generator of $H_{nr}([S^n]^r)$, which is called the fundamental class of $[S^n]^r$. We will show that

$$D^{-1}(a_i) = (-1)^{i+1}\beta_i, \quad \text{that is,} \quad (-1)^{i+1}\beta_i \wedge [S^n]^r = a_i.$$

In fact, it follows from properties of the cup and cap products with respect to the Kronecker product and by definition of $\beta_i$ that

\begin{equation}
(3.7) \quad \langle \alpha_i, (-1)^{i+1}\beta_i \wedge [S^n]^r \rangle = \langle \alpha_i \wedge (-1)^{i+1}\beta_i, [S^n]^r \rangle
\end{equation}

$$= \langle (-1)^{i+1}(-1)^{n(i-1)}\alpha_1 \wedge \ldots \wedge \alpha_r, [S^n]^r \rangle
\end{equation}

$$= \langle \alpha_1 \wedge \ldots \wedge \alpha_r, [S^n]^r \rangle = 1$$

observing that $\alpha_1 \wedge \ldots \wedge \alpha_r$ is the generator of $H_{nr}([S^n]^r)$. Thus,

$$D^{-1}\left(\sum_{i=1}^{r} a_i\right) = \sum_{i=1}^{r} D^{-1}(a_i) = \sum_{i=1}^{r} (-1)^{i+1}\beta_i.$$

It follows from (3.6), (3.7) and from commutativity of diagram (3.5) that

$$j_1^* \circ D^{-1}(c) = D^{-1} \circ i_*(c) = \sum_{i=1}^{r} (-1)^{i+1}\beta_i,$$

that is,

$$\sum_{i=1}^{r} (-1)^{i+1}\beta_i \in \text{Im}(j_1^*).$$

Since the second row of diagram (3.5) is exact, one has that

$$k_1^*\left(\sum_{i=1}^{r} (-1)^{i+1}\beta_i\right) = 0.$$

By using again the commutativity of diagram (3.5)

$$k^* \circ \psi^*\left(\sum_{i=1}^{r} (-1)^{i+1}\beta_i\right) = \psi^* \circ k_1^*\left(\sum_{i=1}^{r} (-1)^{i+1}\beta_i\right) = 0,$$
which implies that 
\[ \psi^* \left( \sum_{i=1}^r (-1)^{i+1} \beta_i \right) \in \text{Ker}(k^*) = \text{Im}(j^*). \]

Thus, there exists an element \( V \in H^{(r-1)n}(X^r, X^r - X_\varphi) \) such that

\[ j^*(V) = \psi^* \left( \sum_{i=1}^p (-1)^{i+1} \beta_i \right) \neq 0, \]

since \( \psi^* \) is a monomorphism. Using the naturality of the cup product in the following diagram

\[
\begin{array}{ccc}
H^n(X^r) \otimes H^{(r-1)n}(X^r) & \xymatrix{ \ar[r]^-{\sim} & } & H^{rn}(X^r) \\
\psi^* \ar@{|->}[u] & & \psi^* \ar@{|->}[u] \\
H^n([S^n]^r) \otimes H^{(r-1)n}([S^n]^r) & \xymatrix{ \ar[r]^-{\sim} & } & H^{rn}([S^n]^r)
\end{array}
\]

and observing that
\[
\sum_{i=2}^r \alpha_i \sim (-1)^{i+1} \beta_i = 0
\]

one has that

\[ \psi^* (\alpha_1 \sim \ldots \sim \alpha_r) = \psi^*(\alpha_1) \sim \psi^* \left( \sum_{i=1}^r (-1)^{i+1} \beta_i \right) \]

\[ = \psi^*(\alpha_1 \sim \beta_1 + \sum_{i=2}^r \alpha_1 \sim (-1)^{i+1} \beta_i) \]

\[ = \psi^*(\alpha_1 \sim \beta_1) = \psi^*(\alpha_1 \sim \ldots \sim \alpha_r) \neq 0, \]

since \( \alpha_1 \sim \ldots \sim \alpha_r \) is the generator of \( H^{rn}([S^n]^r) \) and \( \psi^* \) is a monomorphism.

On the other hand, from naturality of the cup product in the diagram

\[
\begin{array}{ccc}
H^n(X^r, X_\varphi) \otimes H^{(r-1)n}(X^r, X^r - X_\varphi) & \xymatrix{ \ar[r]^-{\sim} & } & H^{rn}(X^r, X^r) \\
\psi^* \ar@{|->}[u] & & \psi^* \ar@{|->}[u] \\
H^n(X^r) \otimes H^{(r-1)n}(X^r) & \xymatrix{ \ar[r]^-{\sim} & } & H^{rn}(X^r)
\end{array}
\]

and from equations (3.4) and (3.8) we conclude that

\[ \psi^*(\alpha_1 \sim \ldots \sim \alpha_r) = \psi^*(\alpha_1) \sim \psi^* \left( \sum_{i=1}^p (-1)^{i+1} \beta_i \right) \]

\[ = j^*(U) \sim j^*(V) = j^*(U \sim V) = j^*(0) = 0, \]

which contradicts (3.9). This completes the proof. \( \square \)
Now, let us consider the map \( \theta = q_1 \circ \psi|_{X_\varphi}: X_\varphi \to S^n \), where \( q_1: \Delta[S^n]^r \to S^n \) is the natural projection on the 1-th coordinate, which is an homeomorphism. Since by Lemma 3.3, \((\psi|_{X_\varphi})^* \) is a monomorphism, one then has that \( \theta^*: H^n(S^n) \to H^n(X_\varphi) \) is a monomorphism. Note that, if \( (x_1, \ldots, x_r) \in X_\varphi \), we have that for each \( i, g_i \theta(x_1, \ldots, x_r) = g_i \varphi(x_1) = \varphi(x_i) = \theta g_i(x_1, \ldots, x_r) \), thus \( \theta \) is a \( G \)-equivariant map, and consequently, \( \theta \) is an \( H \)-equivariant map, where \( H \subset G \) is a cyclic subgroup of prime order. Thus, in particular for \( \rho = \sigma \), we can consider the homomorphism induced by \( \theta \), \( \theta^\sigma_\sigma^\rho: H^n(S^n) \to H^n(X_\varphi) \), where \( H^n_\rho \) denotes the \( n \)-dimensional Smith special cohomology group with coefficients in \( \mathbb{Z}_p \) in the sense of Section 2.4.

By remarks in [2, Results following 5.2] whose dual holds in cohomology, \( i^*: H^n(S^n) \to H^n_\rho(S^n) \) is an isomorphism, and since \( \theta^*: H^n(S^n) \to H^n(X_\varphi) \) is a monomorphism it follows that \( \theta^\sigma_\sigma^\rho \) is a monomorphism. To conclude that the \( \mathbb{Z}_p \)-index of \( X_\varphi \) is equal to \( n \) it suffices to verify that the map between the orbit spaces \( \bar{\theta}: X_\varphi/H \to S^n/H \) induces a monomorphism in cohomology. From results in [2, (3.10), p. 125], we have that \( H^n_\rho(S^n) \cong H^n(S^n/H) \) and \( H^n_\rho(X_\varphi) \cong H^n(X_\varphi/H) \), and considering the commutative diagram

\[
\begin{array}{ccc}
H^n_\rho(S^n) & \xrightarrow{\theta^\sigma_\sigma^\rho} & H^n_\rho(X_\varphi) \\
\cong \downarrow & \hspace{1cm} & \cong \\
H^n(S^n/H) & \xrightarrow{\bar{\theta}^\sigma} & H^n(X_\varphi/H)
\end{array}
\]

it follows that \( \bar{\theta}^\sigma: H^n(S^n/H) \to H^n(X_\varphi/H) \) is a monomorphism. Therefore, the \( \mathbb{Z}_p \)-index of \( X_\varphi \) is equal to \( n \).

\[ \square \]

4. Proof of Theorems 1.1 and 1.2

PROOF OF THEOREM 1.2. By following the similar steps of [3], we first prove Theorem 1.2 in the case that \( G = H = \mathbb{Z}_p \), where \( p \geq 2 \). We need to show that the \( \mathbb{Z}_p \)-index of the set \( A_f = \{ x \in X : f(x) = f(gx) = \ldots = f(g^{p-1}x) \} \) is greater than or equal to \( n - pk \). For this, let us consider \( F: X \to Y^p \) given by \( F(x) = (f(x), f(gx), \ldots, f(g^{p-1}x)) \), where \( Y^p = Y \times \ldots \times Y \) denotes the \( p \)-fold cartesian product of \( Y \) and \( g \) is a fixed generator of \( \mathbb{Z}_p \). In these conditions, we prove the following

**LEMMA 4.1.** The homomorphism \( F^*: H^q(Y^p) \to H^q(X) \) induced by the map \( F: X \to Y^p \) is zero for any \( q \geq 1 \).

**Proof.** We have that \( F = (f_0 \times \ldots \times f_{p-1}) \circ d \), where \( d: X \to X^p \) is the diagonal map and \( f_i(x) = f(g^ix), \) for any \( x \in X \) and \( i = 0, \ldots, p-1 \). In this way, it suffices to show that \((f_0 \times f_1 \times \ldots \times f_{p-1})^*: H^q(Y^p) \to H^q(X^p) \) is trivial
for any \( q \geq 1 \). Let us consider the following commutative diagram

\[
\begin{array}{ccc}
\bigoplus_{i+j=q} H^i(Y^t) \otimes H^j(Y) & \xrightarrow{=} & H^q(Y^{t+1}) \\
(f_0 \times \ldots \times f_t)^* \otimes f_{t+1}^* & & (f_0 \times \ldots \times f_{t+1})^* \\
\bigoplus_{i+j=q} H^i(X^t) \otimes H^j(X) & \xrightarrow{=} & H^q(X^{t+1})
\end{array}
\]

(4.1)

Since \( Y \) is a CW-complex, applying the Künneth’s formula we have that the upper row of diagram (4.1) is an isomorphism for any \( q = 1, 2, \ldots \) and \( t = 1, \ldots, p - 1 \).

The proof will be done by induction on \( t \). Suppose inductively that \((f_0 \times \ldots \times f_t)^*: H^t(Y^t) \rightarrow H^t(X^t)\) is zero for some \( t = 1, \ldots, p - 1 \) and for each \( i = 1, 2, \ldots \). By hypothesis, \( f \) induces the zero homomorphism in each dimension; in particular \( f_{t+1}^* \) is zero and thus \((f_0 \times \ldots \times f_t)^* \otimes f_{t+1}^*\) is trivial. It follows from commutativity of the diagram (4.1) that \((f_0 \times \ldots \times f_{t+1})^*\) is zero, which completes the proof.

We can define a \( \mathbb{Z}_p \)-action on \( Y^p \) as follows: for each \((y_1, \ldots, y_p) \in Y^p \)

\[ g(y_1, \ldots, y_{p-1}, y_p) = (y_p, y_1, \ldots, y_{p-1}). \]

Since \( p \) is a prime, this action is free on \( Y^p - \Delta \), where \( \Delta \) is the diagonal in \( Y^p \). Let us observe that \( A_f = F^{-1}(\Delta) \), thus \( F \) determines a \( \mathbb{Z}_p \)-equivariant map \( F_0: X - A_f \rightarrow Y^p - \Delta \), which induces a map between the orbit spaces \( F_0: [X - A_f]^* \rightarrow [Y^p - \Delta]^* \). In these conditions, we prove that

**Lemma 4.2.** The map \( F_0^*: H^{pk}([Y^p - \Delta]^*) \rightarrow H^{pk}([X - A_f]^*) \) is zero.

**Proof.** Let us consider the map of pairs \((F, F_0): (X, X - A_f) \rightarrow (Y, Y - \Delta)\). One then has the following commutative diagram

\[
\begin{array}{cccccc}
H^{pk}(X) & \xrightarrow{i^*} & H^{pk}(X - A_f) & \xrightarrow{H^{pk+1}(X, X - A_f)} & H^{pk+1}(Y, Y - \Delta) \\
\downarrow F_0^* & & \downarrow F_0^* & & \downarrow (F, F_0)^* \\
H^{pk}(Y^p) & \xrightarrow{j^*} & H^{pk}(Y^p - \Delta) & \xrightarrow{H^{pk+1}(Y^p, Y^p - \Delta)} & H^{pk+1}(Y^p, Y^p - \Delta)
\end{array}
\]

where the homomorphisms \( i^* \) and \( j^* \) are induced by appropriate inclusions. Since \( \dim(Y^p) \) is less than or equal to \( pk \) we have that \( H^{pk+1}(Y^p, Y^p - \Delta) \) is trivial and thus \( j^* \) is surjective. On the other hand \( F_0: (X - A_f) \rightarrow (Y^p - \Delta) \) is
a $\mathbb{Z}_p$-equivariant map and it follows from Remark 2.6 that the diagram
\[
\begin{array}{ccc}
H^{pk}(X - A_f) & \xrightarrow{T} & H^{pk}([X - A_f]^*) \\
F_0^* & \downarrow & \uparrow F_0^*
\end{array}
\quad
\begin{array}{ccc}
H^{pk}(Y^p - \Delta) & \xrightarrow{T} & H^{pk}([Y^p - \Delta]*) \\
F_0^* & \downarrow & \uparrow F_0^*
\end{array}
\]

between the Smith sequences of $X - A_f$ and $Y^p - \Delta$ is commutative and since
$H^{pk+1}_p(Y^p - \Delta)$ is zero, $T$ is surjective.

Putting together these diagrams, one obtains a new commutative diagram
\[
\begin{array}{ccc}
H^{pk}(X) & \xrightarrow{T} & H^{pk}([X - A_f]^*) \\
F^* & \downarrow & \uparrow F_0^*
\end{array}
\quad
\begin{array}{ccc}
H^{pk}(Y^p) & \xrightarrow{T} & H^{pk}([Y^p - \Delta]*) \\
F_0^* & \downarrow & \uparrow F_0^*
\end{array}
\]

where the horizontal sequences are not necessarily exacts, but the composition $T \circ j^*$ is surjective. Therefore, as $F^*$ is zero by Lemma 4.1, it follows from commutativity of the diagram (4.2) that $F_0^*$ is zero. $\square$

Let $h: X^* \to B\mathbb{Z}_p$ be a classifying map for the principal $\mathbb{Z}_p$-bundle $X \to X^*$. Then the compositions $h \circ i_1: A_f^* \to B\mathbb{Z}_p$ and $h \circ i_2: [X - A_f]^* \to B\mathbb{Z}_p$ are classifying maps for the following principal $\mathbb{Z}_p$-bundles $A_f \to A_f^*$ and $X - A_f \to [X - A_f]^*$ respectively, where the maps $i_1: A_f^* \to X^*$ and $i_2: [X - A_f]^* \to X^*$ are induced by the inclusions between the orbit spaces.

Let us consider $G: Y^p - \Delta \to B\mathbb{Z}_p$ a classifying map for the principal $\mathbb{Z}_p$-bundle $Y^p - \Delta \to [Y^p - \Delta]^*$. Since $F_0: X - A_f \to Y^p - \Delta$ is a $\mathbb{Z}_p$-equivariant map, one has that
\[
G \circ F_0: [X - A_f]^* \to B\mathbb{Z}_p
\]
also classifies the principal $\mathbb{Z}_p$-bundle $X - A_f \to [X - A_f]^*$. In this way,
\[
i_2^* \circ h^* = F_0^* \circ G^*: H^*(B\mathbb{Z}_p) \to H^*([X - A_f]^*).
\]

To conclude that the $\mathbb{Z}_p$-index of $A_f$ is greater than or equal to $n - pk$, it suffices to show that $i_1^* \circ h^*(\mu) \neq 0$, where $\mu$ is the generator of $H^{n-pk}(B\mathbb{Z}_p)$.

We first consider the case when $k$ is odd. Let us observe that $n$ must be necessarily odd, since $p > 2$ is a prime. Then, $n - pk$ is even and it follows from Remark 2.5 that $\mu = b^{(n-pk)/2} \in H^{n-pk}(B\mathbb{Z}_p)$. Suppose that $i_1^* \circ h^*(\mu) = 0$. From continuity of the cohomology, there exists a neighbourhood $V$ of $A_f$ in $X$ which is invariant by the action of $\mathbb{Z}_p$ and such that $i_1^* \circ h^*(\mu) = 0$ in $H^{n-pk}(V^*)$. From the exact cohomology sequence of the pair $(X^*, V^*)$ one has
\[
h^*(\mu) \in \text{Im}[H^{n-pk}(X^*, V^*) \to H^{n-pk}(X^*)].
\]
Since \( pk \) is odd from Remark 2.5 \( \eta = a \sim b^{(pk-1)/2} \) is a generator of \( H^{pk}(B\mathbb{Z}_p) \). It follows from Lemma 4.2 and (4.3) that

\[
i_2^* \circ h^*(\eta) = \mathcal{T}_0^* \circ G^*(\eta) = 0 \in H^{pk}([X - A_f]^*)
\]

and from the exact cohomology sequence of the pair \( (X^*, [X - A_f]^*) \) one has

\[
(4.5) \quad h^*(\eta) \in \text{Im} [H^{pk}(X^*, [X - A_f]^*) \rightarrow H^{pk}(X^*)].
\]

Thus from (4.4), (4.5) and by the naturality of the cup product we have

\[
h^*(\eta \sim \mu) = h^*(\eta) \sim h^*(\mu) \in \text{Im} [H^n(X^*, [X - A_f]^* \cup V^*) \rightarrow H^n(X^*)].
\]

Let us note that the element

\[
\eta \sim \mu = a \sim b^{(pk-1)/2} \sim b^{(n-pk)/2} = a \sim b^{(n-1)/2}
\]

is a generator of \( H^n(B\mathbb{Z}_p) \). Furthermore,

\[
H^n(X^*, [X - A_f]^* \cup V^*) = H^n(X^*, X^*) = 0
\]

and then \( h^*(\eta \sim \mu) = 0 \in H^n(X^*) \), that is, \( h^*: H^n(B\mathbb{Z}_p) \rightarrow H^n(X^*) \) is trivial which contradicts the hypothesis that the \( \mathbb{Z}_p \)-index of \( X \) is greater than or equal to \( n \).

If \( k \) is even, then \( n - pk \) is odd and \( pk \) is even. In this case, the proof is analogous to the previous case, considering now the generators

\[
\mu = a \sim b^{(n-pk-1)/2} \in H^{n-pk}(B\mathbb{Z}_p) \quad \text{and} \quad \eta = b^{(pk)/2} \in H^{pk}(B\mathbb{Z}_p).
\]

Let us examine the case where \( G = \mathbb{Z}_2 \). Here, \( n \) can be any positive integer and the generator of \( H^{n-2k}(B\mathbb{Z}_2) \) is \( \mu = a^{n-2k} \). To show that the \( \mathbb{Z}_2 \)-index of \( A_f \) is greater than or equal to \( n - 2k \), it suffices to prove that \( i_2^* \circ h^*(\mu) \neq 0 \). Let us assume that \( i_2^* \circ h^*(\mu) = 0 \). Then there exists a neighbourhood \( V \) of \( A_f \) in \( X \) which is invariant with respect to the action and such that \( i_2^* \circ h^*(\mu) = 0 \) in \( H^{n-2k}(V^*) \). From exact cohomology sequence of the pair \( (X^*, V^*) \) one has that

\[
(4.6) \quad h^*(\mu) \in \text{Im} [H^{n-2k}(X^*, V^*) \rightarrow H^{n-2k}(X^*)].
\]

On the other hand, \( \eta = a^{2k} \) is the generator of \( H^{2k}(B\mathbb{Z}_2) \) and it follows from Lemma 4.2 and (4.3) that

\[
i_2^* \circ h^*(\eta) = \mathcal{T}_0^* \circ G^*(\eta) = 0 \in H^{2k}([X - A_f]^*).
\]

Moreover, from exact cohomology sequence of \( (X^*, [X - A_f]^*) \) one has that

\[
(4.7) \quad h^*(\eta) \in \text{Im} [H^{2k}(X^*, [X - A_f]^*) \rightarrow H^{2k}(X^*)].
\]

Thus, from (4.6), (4.7) and by the naturality of the cup product we have

\[
h^*(\eta \sim \mu) = h^*(\eta) \sim h^*(\mu) \in \text{Im} [H^n(X^*, [X - A_f]^* \cup V^*) \rightarrow H^n(X^*)].
\]
Let us observe that \( \eta \sim \mu = a^{2k} \sim a^{n-2k} = a^n \) is the generator of \( H^n(B\mathbb{Z}_2) \). Furthermore, \( H^n(X^*, [X - A_1]^* \cup V^*) = H^n(X^*, X^*) \) is trivial and then \( h^* \eta \sim \mu = 0 \in H^n(X^*) \) which contradicts the hypothesis that the \( \mathbb{Z}_2 \)-index of \( X \) is greater than or equal to \( n \). This concludes the proof of Theorem 1.2 in the case \( G = H = \mathbb{Z}_p \).

For the general case, suppose that \( G \) is a finite group which acts freely on \( X \) and let \( H \subset G \) be a normal cyclic subgroup of prime order \( p \). We denote by \( s = |G|/p \), the number of the left cosets of \( G/H \) and let \( a_1, \ldots, a_s \) be a set of representatives of the cosets. We define the map \( F : X \to Y^s \) by

\[
(4.8) \quad F(x) = (f(a_1x), \ldots, f(a_sx)).
\]

We need to show that

\[
A(f, H, G) = A_F = \{ x \in X : F(x) = F(hx), \text{ for all } h \in H \}.
\]

Let \( x \) be a point in the set \( A(f, H, G) \), then \( f \) collapses each orbit determined by the action of \( H \) on \( a_ix \) to a single point, for each \( i = 1, \ldots, s \). If \( h \in H \)

\[
F(hx)_i = f(ha_ix),
\]

for each \( i = 1, \ldots, s \) which implies that \( F(x) = F(hx) \). Therefore \( x \in A_F \). The proof of the another inclusion is entirely analogous.

To conclude, let us observe that \( H \cong \mathbb{Z}_p \) acts freely on \( X \) by restriction and by hypothesis the \( \mathbb{Z}_p \)-index of \( X \) is greater than or equal to \( n \). By using Lemma 4.1 for the map \( F : X \to Y^s \) defined in (4.8) one has that \( F^*: H^q(Y^s) \to H^q(X) \) is trivial for any \( q \geq 1 \). Since dimension of \( Y^s \) is \( ks \) and Theorem 1.2 is true for \( G = \mathbb{Z}_p \) we can conclude that the \( \mathbb{Z}_p \)-index of \( A_F = A(f, H, G) \) is greater than or equal \( n - pk(|G|/p) = n - |G|k \) and this completes the proof. \( \square \)

**Proof of Theorem 1.1.** Let \( \tilde{f} : X_\varphi \to Y \) given by \( \tilde{f}(x_1, \ldots, x_r) = f(x_1) \), that is \( \tilde{f} = f \circ \pi_1 \) where \( \pi_1 \) is the natural projection on the 1-th coordinate. By hypothesis \( f \) induces the zero homomorphism in each dimension, then we have that \( \tilde{f}^*: H^i(Y) \to H^i(X_\varphi) \) is trivial for any \( i \geq 1 \). Moreover, the \( \mathbb{Z}_p \)-index of \( X_\varphi \) is equal to \( n \) by Theorem 3.1. In this way, \( X_\varphi \) and \( \tilde{f} \) satisfy the hypotheses of Theorem 1.2 which implies that the \( \mathbb{Z}_p \)-index of the set \( A(\tilde{f}, H, G) \) is greater than or equal to \( n - |G|k \). By Definition 2.1 \( A_\varphi(f, H, G) = A(\tilde{f}, H, G) \), and then cohom.dim \( A_\varphi(f, H, G) \geq n - |G|k \). \( \square \)

**Remark 4.3.** In the particular case that \( G = H = \mathbb{Z}_p \) with \( p \) prime, Volovikov in [13, Theorem 3.2] proved a version of Theorem 1.1.
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