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ON NONSYMMETRIC THEOREMS FOR (H,G)-COINCIDENCES

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Abstract. Let X be a compact Hausdorff space, $\varphi\colon X \to S^n$ a continuous map into the n-sphere S^n that induces a nonzero homomorphism $\varphi^*\colon H^n(S^n;\mathbb{Z}_p) \to H^n(X;\mathbb{Z}_p)$, Y a k-dimensional CW-complex and $f\colon X \to Y$ a continuous map. Let G a finite group which acts freely on S^n. Suppose that $H \subset G$ is a normal cyclic subgroup of a prime order. In this paper, we define and we estimate the cohomological dimension of the set $A_\varphi(f,H,G)$ of (H,G)-coincidence points of f relative to φ.

1. Introduction

K. D. Joshi [10] has proved a nonsymmetric generalization of the Borsuk–Ulam theorem [1], in which the n-sphere S^n is replaced by a certain compact subset X of the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. In this context, a pair of points $x, y \in X$ are said to be *antipodal* if $y = -\lambda x$, for some $\lambda > 0$. The Joshi's theorem shows that for every continuous map $f\colon X \to \mathbb{R}^n$ there exist antipodal points $x, y \in X$ such that $f(x) = f(y)$.

K. Borsuk has suggested to define antipodal points in an arbitrary space in the following way: $x_1, x_2 \in X$ are said to be *antipodal points relative to an*

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essential map[1] $\varphi: X \to S^n$ if $\varphi(x_1) = -\varphi(x_2)$. Using the Borsuk's suggestion, Spież [11] has proved that if X is a compact Hausdorff space and if $\varphi: X \to S^n$ is an essential map, then for every continuous map $f: X \to \mathbb{R}^k$, the covering dimension of the set

$$A_\varphi(f) = \{x \in X : \text{there exists } y \in X, \text{ such that } \varphi(x) = -\varphi(y) \text{ and } f(x) = f(y)\}$$

is not less than $n-k$, obtaining thus a generalization of the Joshi's theorem.

M. Izydorek [7] has extended the proposition of Borsuk for a cyclic group G of order prime which acts freely on a n-dimensional sphere S^n and has proved the following generalization of the SpiePlr ż's theorem: if X is a compact Hausdorff space and if $\varphi: X \to S^n$ is an essential map, then for every continuous map $f: X \to \mathbb{R}^k$, the covering dimension of the set

$$\begin{aligned} A_\varphi(f) = \{x \in X : {}& \text{there exists } x_2, \ldots, x_p \text{ such that} \\ & \varphi(x) = g^{-1}\varphi(x_2) = \ldots = g^{1-p}\varphi(x_p) \text{ and } f(x) = f(x_2) = \ldots = f(x_p)\} \end{aligned}$$

is not less than $n-(p-1)k$, where g is a fixed generator of G. Moreover, if $\mathbb{R}^k$ is replace by a generalized k-dimensional manifold M^k over $\mathbb{Z}_p$, then an analogous theorem has been proved (see [7, Theorem 4]).

Gonçalves, Jaworowski and Pergher [3] have defined (H,G)-concidence for a continuous map f from a n-sphere S^n into a k-dimensional CW-complex Y, where G is a finite group which acts freely on S^n and have proved that if H is a nontrivial normal cyclic subgroup of a prime order, then

$$\operatorname{cohom.dim} A(f,H,G) \geq n - |G|k,$$

where $A(f,H,G)$ is the set of (H,G)-coincidence points of f and cohom.dim denotes the cohomological dimension. The other papers closely related to [3] are [4]–[6], [8], [9] and [13].

The purpose of this paper is to define the set $A_\varphi(f,H,G)$ of (H,G)-coincidence points of a continuous map $f: X \to Y$ relative to an essential map $\varphi: X \to S^n$, where X is a compact Hausdorff space, Y is a topological space, G is a finite group which acts freely on the n-dimensional sphere S^n and H is a subgroup of G. Using this definition, under certain conditions, we estimate the cohomological dimension of the set $A_\varphi(f,H,G)$. Specifically, we will prove the following nonsymmetric version of the main theorem of [3]:

Theorem 1.1. *Let X be a compact Hausdorff space, Y a k-dimensional* CW*-complex and $\varphi: X \to S^n$ an essential map. Given a finite group G which acts freely on S^n and H a normal cyclic subgroup of prime order, then for every*

(1) A map $\varphi: X \to S^n$ is said to be an *essential map* if φ induces nonzero homomorphism $\varphi^*: H^n(S^n;\mathbb{Z}_p) \to H^n(X;\mathbb{Z}_p)$.

continuous map $f\colon X \to Y$ *such that* $f^*\colon H^i(Y;\mathbb{Z}_p) \to H^i(X;\mathbb{Z}_p)$ *is trivial for* $i \geq 1$, cohom.dim $A_\varphi(f,H,G) \geq n - |G|k$.

For the proof of Theorem 1.1, it was fundamental to prove the following version of the main Theorem of [3],

Theorem 1.2. *Let* X *be a paracompact Hausdorff space,* Y *a* k*-dimensional* CW*-complex,* G *a finite group which acts freely on* X *and* $H \subset G$ *a normal cyclic subgroup of prime order. Let* $f\colon X \to Y$ *be a continuous map such that* $f^*\colon H^i(Y;\mathbb{Z}_p) \to H^i(X;\mathbb{Z}_p)$ *is trivial, for* $i \geq 1$*. Suppose that the* $\mathbb{Z}_p$*-index of* X *is greater than or equal to* n*, then the* $\mathbb{Z}_p$*-index of the set* $A(f,H,G)$ *is greater than or equal to* $n - |G|k$*. Consequently,* cohom.dim $A(f,H,G) \geq n - |G|k$.

2. Preliminaries

Throughout this paper the symbols H_* and H^* will denote Čech homology and cohomology groups with coefficients in $\mathbb{Z}_p$, unless otherwise indicated. If G is a group which acts on a topological space X, we will denote by X^* the orbit space X/G.

We start by introducing some basic notions and definitions as follows.

2.1. (H,G)**-coincidence.** Suppose that X,Y are topological spaces, G is a group acting freely on X and $f\colon X \to Y$ is a map. If H is a subgroup of G, then H acts on the right on each orbit Gx of G as follows: if $y \in Gx$ and $y = gx$, $g \in G$, then $hy = ghx$ (such action may depend on the choice of the reference point x). Following [4], [6], [9] the concept of G-coincidence is generalized as follows: a point $x \in X$ is said to be a (H,G)*-coincidence point of* f if f sends every orbit of the action of H on the G-orbit of x to a single point (see [5]). We will denote by $A(f,H,G)$ the set of all (H,G)-coincidence points of f. If H is the trivial subgroup, then every point of X is a (H,G)-coincidence. If $H = G$, this is the usual definition of coincidence. If $G = Z_p$ with p prime, then a nontrivial (H,G)-coincidence point is a G-coincidence point.

2.2. The space X_φ **and the set** $A_\varphi(f,H,G)$**.** Let us consider X a compact Hausdorff space and an essential map $\varphi\colon X \longrightarrow S^n$. Suppose G be a finite group of order r which acts freely on S^n and H be a subgroup of order p of G. Let $G = \{g_1, g_2, \dots, g_r\}$ be a fixed enumeration of elements of G, where g_1 is the identity of G. A nonempty space X_φ can be associated with the essential map $\varphi\colon X \to S^n$ as follows:

$$\begin{aligned} X_\varphi &= \{(x_1,\dots,x_r) \in X^r : \varphi(x_1) = (g_2)^{-1}\varphi(x_2) = \dots = (g_r)^{-1}\varphi(x_r)\} \\ &= \{(x_1,\dots,x_r) \in X^r : g_i\varphi(x_1) = \varphi(x_i),\ i = 1,\dots,r\}, \end{aligned}$$

where X^r denotes the r-fold cartesian product of X. The set X_φ is a closed subset of X^r and so it is compact. We define a G-action on X_φ as follows: for each $g_i \in G$ and for each $(x_1, \dots, x_r) \in X_\varphi$,

$$g_i(x_1, \dots, x_r) = (x_{\sigma_{g_i}(1)}, \dots, x_{\sigma_{g_i}(r)}), \tag{2.1}$$

where the permutation σ_{g_i} is defined by $\sigma_{g_i}(k) = j$, $g_k g_i = g_j$. We observe that if $x = (x_1, \dots, x_r) \in X_\varphi$ then $x_i \neq x_j$, for any $i \neq j$ and therefore G acts freely on X_φ.

Now, let us consider a continuous map $f: X \to Y$, where Y is a topological space and $\widetilde{f}: X_\varphi \to Y$ given by $\widetilde{f}(x_1, \dots, x_r) = f(x_1)$. Let $y = (x_1, \dots, x_r) \in A(\widetilde{f}, H, G)$ and consider the orbit $Gy = \{g_1 y, g_2 y, \dots, g_r y\}$. Note that

(i) From (2.1), we have that for each i, the 1-th coordinate of $g_i y$ is x_i.

(ii) The action of H on Gy determines a partition of the orbit Gy in $s = (r/p)$ disjoint suborbits, and we can be rewrite

$$Gy = \{g_{1_1} y, \dots, g_{1_p} y; \dots; g_{j_1} y, \dots, g_{j_p} y; \dots; g_{s_1} y, \dots, g_{s_p} y\},$$

where $\{1, \dots, r\} \leftrightarrow \{1_1, \dots, 1_p; \dots; j_1, \dots, j_p; \dots; s_1, \dots, s_p\}$ is a bijection.

Since y is a (H, G)-coincidence point of $\widetilde{f}$, it follows from (i) and (ii) that,

$$f(x_{1_1}) = \dots = f(x_{1_p}); \dots; f(x_{j_1}) = \dots = f(x_{j_p}); \dots; f(x_{s_1}) = \dots = f(x_{s_p}).$$

In these conditions, we have the following

Definition 2.1. The set $A_\varphi(f, H, G)$ of (H, G)-coincidence points of f relative to φ is defined by

$$\begin{aligned} A_\varphi(f, H, G) = A(\widetilde{f}, H, G) = \{&(x_1, \dots, x_r) \in X^r : \\ &g_i \varphi(x_1) = \varphi(x_i),\ i = 1, \dots, r \text{ and } f(x_{j_1}) = \dots = f(x_{j_p}),\ j = 1, \dots, s\}. \end{aligned}$$

Remark 2.2. Let us observe that if $G = H = \mathbb{Z}_p$,

$$\begin{aligned} A_\varphi(f, H, G) = A_\varphi(f) = \{&(x_1, \dots, x_p) \in X^p : \\ &g_i \varphi(x_1) = \varphi(x_i),\ i = 1, \dots, p \text{ and } f(x_1) = \dots = f(x_p)\}. \end{aligned}$$

2.3. The $\mathbb{Z}_p$-index. Suppose that the cyclic group $G = \mathbb{Z}_p$ of order prime p acts freely on a Hausdorff and paracompact space X. Then $X \to X^*$ is a principal $\mathbb{Z}_p$-bundle and one can take $h: X^* \to B\mathbb{Z}_p$ a classifying map for the G-bundle $X \to X^*$.

Remark 2.3. It is well known that if $\widehat{h}$ is another classifying map for the principal $\mathbb{Z}_p$-bundle $X \to X^*$, then there is a homotopy between h and $\widehat{h}$.

We will consider the following definition for the $\mathbb{Z}_p$-index of X (see [7]).

DEFINITION 2.4. We say that the $\mathbb{Z}_p$-index of X is greater than or equal to k if the homomorphism $h^*: H^k(B\mathbb{Z}_p) \to H^k(X^*)$ is nontrivial. We say that the $\mathbb{Z}_p$-index of X is equal to k if it is greater than or equal to k and moreover $h^*: H^i(B\mathbb{Z}_p) \to H^i(X^*)$ is zero, for any $i \geq k+1$.

REMARK 2.5. A model for $B\mathbb{Z}_2$ the classifying space for $\mathbb{Z}_2$ is the infinite real projective space P^∞. Then $H^*(B\mathbb{Z}_2) \cong H^*(P^\infty)$ is isomorphic to $\mathbb{Z}_2[a]$, where $a \in H^1(P^\infty)$ is the generator. The generator of $H^i(B\mathbb{Z}_2)$ is a^i for any $i \geq 0$. If $p > 2$ a model for $B\mathbb{Z}_p$ the classifying space for $\mathbb{Z}_p$ is the infinite lens space $L_p^\infty = S^\infty/\mathbb{Z}_p$. Thus, $H^i(B\mathbb{Z}_p) = H^i(L_p^\infty) \cong \mathbb{Z}_p$ for any $i \geq 0$ and given any nonzero element $a \in H^1(L_p^\infty)$, one has that $b = \beta(a)$ is a nonzero element of $H^2(L_p^\infty)$, where $\beta: H^1(L_p^\infty) \to H^2(L_p^\infty)$ is the Bockstein homomorphism. More generally, a generator $\mu \in H^i(B\mathbb{Z}_p)$ is given by

$$\mu = \begin{cases} a \smile b^{(i-1)/2} & \text{if } i \text{ is odd} \\ b^{i/2} & \text{if } i \text{ is even.} \end{cases}$$

2.4. The Smith special cohomology groups with coefficients in $\mathbb{Z}_p$. In this work, we will be considering the definition of the Smith special cohomology groups with coefficients in $\mathbb{Z}_p$ in the sense of [2]. Smith homology and cohomology were originally defined in [12] and in a series of subsequent papers. A systematic exposition of the Smith theory can be found in [2]. Let X be a topological space; given a finite group G of prime order p which acts freely on X, let g be a fixed generator of G and put

$$\sigma = 1 + g + g^2 + \ldots + g^{p-1} \quad \text{and} \quad \tau = 1 - g,$$

in the group ring $\mathbb{Z}_p(G)$. We have that $\sigma = \tau^{p-1}$. If $\rho = \tau^i$, we put $\overline{\rho} = \tau^{p-i}$, then $\tau = \overline{\sigma}$ and $\sigma = \overline{\tau}$. There exists an exact sequence with coefficients in $\mathbb{Z}_p$ [2, p.125],

$$\longrightarrow H_\rho^n(X) \xrightarrow{\rho^*} H^n(X) \xrightarrow{\mathcal{T}} H^n(X^*) \xrightarrow{\delta} H_\rho^{n+1}(X) \xrightarrow{\rho^*}$$

called Smith exact sequence, where $H_\rho^*(X)$ denotes the Smith special cohomology groups and $\mathcal{T}$ is the transfer homomorphism.

REMARK 2.6. The Smith cohomology groups are natural with respect to $\mathbb{Z}_p$-equivariant maps, that is, if $f: X \to Y$ is a $\mathbb{Z}_p$-equivariant map then f induces homomorphism ${f_\rho}^*: H_\rho^*(Y) \to H_\rho^*(X)$ which commutes with the homomorphisms in the Smith sequence.

3. The $\mathbb{Z}_p$-index of X_φ

Let us consider the free G-space X_φ as defined in Section 2.2. If $H \subset G$ is a cyclic subgroup of prime order p, then X_φ is a free $H \cong \mathbb{Z}_p$-space. In these conditions, as in [7, Theorem 3], we have the following

Theorem 3.1. *Let X be a compact Hausdorff space and $\varphi\colon X \longrightarrow S^n$ an essential map. Then, the $\mathbb{Z}_p$-index of X_φ is equal to n.*

Proof. For $i = 1,\dots,r$, let us consider the maps $\varphi_i\colon X \to S^n$ given by $\varphi_i(x) = (g_i)^{-1}\varphi(x)$, where $g_i \in G$. Then, we can define the map

$$\psi = \varphi_1 \times \dots \times \varphi_r\colon X^r \to [S^n]^r,$$

where $[S^n]^r$ denotes the r-fold cartesian product of r copies of S^n, such that $X_\varphi = \psi^{-1}\Delta[S^n]^r$, where $\Delta[S^n]^r$ is the diagonal in $[S^n]^r$. In these conditions, we prove the following

Lemma 3.2. *The homomorphism $\psi^*\colon H^*([S^n]^r) \to H^*(X^r)$ induced by $\psi\colon X^r \to [S^n]^r$ is a monomorphism in each dimension.*

Proof. Let m be an integer and for each $t = 1,\dots,r$ consider $H^m([S^n]^t)$. If m is not divisible by n then $H^m([S^n]^t) = 0$ and the result follows. Suppose that m is divisible by n; one then has that there exists $\alpha = 0,1,\dots$, such that $m = \alpha n$. Let us consider the following commutative diagram

$$\begin{tikzcd} H^{\alpha n}([S^n]^t) \otimes H^0(S^n) \oplus H^{(\alpha-1)n}([S^n]^t) \otimes H^n(S^n) \arrow[r, "\times"] \arrow[d, "(\varphi_1\times\dots\times\varphi_t)^*\otimes\varphi_{t+1}^*"'] & H^{\alpha n}([S^n]^{t+1}) \arrow[d, "(\varphi_1\times\dots\times\varphi_{t+1})^*"] \\ H^{\alpha n}(X^t) \otimes H^0(X) \oplus H^{(\alpha-1)n}(X^t) \otimes H^n(X) \arrow[r, "\times"'] & H^{\alpha n}(X^{t+1}) \end{tikzcd} \tag{3.1}$$

Applying the Künneth's formula, we have that the upper row of the above diagram is an isomorphism and the lower row is a monomorphism, for each $\alpha = 0,1,\dots$ and $t = 1,\dots,r$.

The proof will be done by induction on t. We assume inductively that for some $t = 1,\dots,r-1$ and for each $\alpha = 0,1,\dots$ the homomorphism $(\varphi_1 \times \dots \times \varphi_t)^*\colon H^{\alpha n}([S^n]^t) \to H^{\alpha n}(X^t)$ is a monomorphism and we will show that $(\varphi_1 \times \dots \times \varphi_t)^* \otimes \varphi_{t+1}^*$ is a monomorphism.

The result will follow from the commutativity of the diagram (3.1). By induction hypothesis it suffices to show that φ_{t+1}^* is a monomorphism. For this, observe that $\varphi_i^* = \varphi_j^*$, for any $1 \le i,j \le r$. If φ_{t+1}^* is not a monomorphism, then φ_i^* is not a monomorphism, for any $1 \le i \le t$, which completes the proof. $\square$

The next step is to use Lemma 3.2 to show that the homomorphism induced by $\psi|_{X_\varphi}\colon X_\varphi \to \Delta[S^n]^r$ is a monomorphism. Let us consider the following commutative diagram

$$\begin{tikzcd} H^n(X^r, X_\varphi) \arrow[r, "j^*"] & H^n(X^r) \arrow[r, "k^*"] & H^n(X_\varphi) \\ H^n([S^n]^r, \Delta[S^n]^r) \arrow[r] \arrow[u, "\psi^*"] & H^n([S^n]^r) \arrow[r, "i^*"'] \arrow[u, "\psi^*"] & H^n(\Delta[S^n]^r) \arrow[u, "(\psi|_{X_\varphi})^*"'] \end{tikzcd} \tag{3.2}$$

whose rows are exact. If γ is a generator of $H^n(S^n) \cong \mathbb{Z}_p$ let us denote by α_i the element $q_i^*(\gamma) \in H^n([S^n]^r)$, where $q_i\colon [S^n]^r \to S^n$ is the natural projection on the i-th coordinate for $i = 1, 2, \dots, r$. Let us observe that $q_i \circ i = q_j \circ i$ for any $1 \le i, j \le r$, where $i\colon \Delta[S^n]^r \hookrightarrow [S^n]^r$ is the natural inclusion. In this way, one has that

$$i^*(\alpha_i) = i^* \circ q_i^*(\gamma) = (q_i \circ i)^*(\gamma) = (q_j \circ i)^*(\gamma) = i^*(\alpha_j). \tag{3.3}$$

Lemma 3.3. *$(\psi|_{X_\varphi})^*\colon H^n(\Delta[S^n]^r) \to H^n(X_\varphi)$ is a monomorphism.*

Proof. Since $\Delta[S^n]^r$ is homeomorphic to S^n, it follows from (3.3) that $i^*(\alpha_i)$ is a nonzero element in $H^n(\Delta[S^n]^r)$ for any $i = 1, \dots, r$. Thus, it suffices to show that $(\psi|_{X_\varphi})^*(i^*(\alpha_1)) \ne 0$. Let us assume that this does not happen. From the diagram (3.2) we have that

$$k^* \circ \psi^*(\alpha_1) = (\psi|_{X_\varphi})^* \circ i^*(\alpha_1) = 0,$$

which implies that $\psi^*(\alpha_1) \in \operatorname{Ker}(k^*) = \operatorname{Im}(j^*)$ and there exists an element $U \in H^n(X^r, X_\varphi)$ such that

$$j^*(U) = \psi^*(\alpha_1) \ne 0, \tag{3.4}$$

since by Lemma 3.2 ψ^* is a monomorphism and $\alpha_1 = q_1^*(\gamma) \in H^n([S^n]^r)$ is a nonzero element. Let us consider the following commutative diagram,

$$\begin{tikzcd}
H^{(r-1)n}(X^r, X^r - X_\varphi) \arrow[r, "j^*"] & H^{(r-1)n}(X^r) \arrow[r, "k^*"] & H^{(r-1)n}(X^r - X_\varphi) \\
H^{(r-1)n}([S^n]^r, [S^n]^r - \Delta[S^n]^r) \arrow[u, "\psi^*"] \arrow[r, "j_1^*"] & H^{(r-1)n}([S^n]^r) \arrow[u, "\psi^*"'] \arrow[r, "k_1^*"] & H^{(r-1)n}([S^n]^r - \Delta[S^n]^r) \arrow[u, "\psi^*"'] \\
H_n(\Delta[S^n]^r) \arrow[u, "D^{-1}"] \arrow[r, "i_*"'] & H_n([S^n]^r) \arrow[u, "D^{-1}"'] &
\end{tikzcd} \tag{3.5}$$

where the first and the second rows are exact, D is the Alexander–Spanier Duality which is an isomorphism and all others maps are induced by appropriate inclusions.

Let us denote by $a_1, \dots, a_r$ the elements of $H_n([S^n]^r)$ which are conjugated to $\alpha_1, \dots, \alpha_r$ in $H^n([S^n]^r)$. More precisely,

$$\langle \alpha_j, a_i \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \ne j, \end{cases}$$

where the map $\langle \cdot, \cdot \rangle\colon H^n([S^n]^r) \times H_n([S^n]^r) \to \mathbb{Z}_p$ denotes the Kronecker product. Let us denote by $c \in H_n(\Delta[S^n]^r)$ the conjugated element to $i^*(\alpha_1) = \dots = i^*(\alpha_r)$ in $H^n(\Delta[S^n]^r)$. Then for any $j = 1, \dots, r$ one has $\langle i^*(\alpha_j), c \rangle = 1$.

Furthermore, it follows from properties of the Kronecker product that for any $j = 1, \dots, r$

$$\langle i^*(\alpha_j), c\rangle = \langle \alpha_j, i_*(c)\rangle = 1,$$

which implies that

$$i_*(c) = \sum_{i=1}^{r} a_i. \tag{3.6}$$

Let us consider for each $i = 1, \dots, r$ the elements

$$\beta_i = \alpha_1 \smile \dots \smile \alpha_{i-1} \smile \widehat{\alpha}_i \smile \alpha_{i+1} \smile \dots \smile \alpha_r \in H^{(n-1)r}([S^n]^r),$$

where the symbol $\widehat{\alpha}_i$ means that the element α_i is omitted.

To simplify notation, here we will also denote by $[S^n]^r$ the generator of $H_{nr}([S^n]^r)$, which is called *the fundamental class of* $[S^n]^r$. We will show that

$$D^{-1}(a_i) = (-1)^{i+1}\beta_i, \quad \text{that is,} \quad (-1)^{i+1}\beta_i \frown [S^n]^r = a_i.$$

In fact, it follows from properties of the cup and cap products with respect to the Kronecker product and by definition of β_i that

$$\begin{aligned} \langle \alpha_i, (-1)^{i+1}\beta_i \frown [S^n]^r\rangle &= \langle \alpha_i \smile (-1)^{i+1}\beta_i, [S^n]^r\rangle \\ &= \langle (-1)^{i+1}(-1)^{n(n(i-1))}\alpha_1 \smile \dots \smile \alpha_r, [S^n]^r\rangle \\ &= \langle \alpha_1 \smile \dots \smile \alpha_r, [S^n]^r\rangle = 1 \end{aligned} \tag{3.7}$$

observing that $\alpha_1 \smile \dots \smile \alpha_r$ is the generator of $H^{nr}([S^n]^r)$. Thus,

$$D^{-1}\bigg(\sum_{i=1}^{r} a_i\bigg) = \sum_{i=1}^{r} D^{-1}(a_i) = \sum_{i=1}^{r}(-1)^{i+1}\beta_i.$$

It follows from (3.6), (3.7) and from commutativity of diagram (3.5) that

$$j_1^* \circ D^{-1}(c) = D^{-1} \circ i_*(c) = \sum_{i=1}^{r}(-1)^{i+1}\beta_i,$$

that is,

$$\sum_{i=1}^{r}(-1)^{i+1}\beta_i \in \operatorname{Im}(j_1^*).$$

Since the second row of diagram (3.5) is exact, one has that

$$k_1^*\bigg(\sum_{i=1}^{r}(-1)^{i+1}\beta_i\bigg) = 0.$$

By using again the commutativity of diagram (3.5)

$$k^* \circ \psi^*\bigg(\sum_{i=1}^{r}(-1)^{i+1}\beta_i\bigg) = \psi^* \circ k_1^*\bigg(\sum_{i=1}^{r}(-1)^{i+1}\beta_i\bigg) = 0,$$

which implies that

$$\psi^*\bigg(\sum_{i=1}^{r}(-1)^{i+1}\beta_i\bigg) \in \operatorname{Ker}(k^*) = \operatorname{Im}(j^*).$$

Thus, there exists an element $V \in H^{(r-1)n}(X^r, X^r - X_\varphi)$ such that

$$j^*(V) = \psi^*\bigg(\sum_{i=1}^{p}(-1)^{i+1}\beta_i\bigg) \neq 0, \tag{3.8}$$

since ψ^* is a monomorphism. Using the naturality of the cup product in the following diagram

$$\begin{tikzcd}
H^n(X^r) \otimes H^{(r-1)n}(X^r) \arrow[r, "\smile"] & H^{rn}(X^r) \\
H^n([S^n]^r) \otimes H^{(r-1)n}([S^n]^r) \arrow[u, "\psi^*"] \arrow[r, "\smile"'] & H^{rn}([S^n]^r) \arrow[u, "\psi^*"']
\end{tikzcd}$$

and observing that

$$\sum_{i=2}^{r} \alpha_1 \smile (-1)^{i+1}\beta_i = 0$$

one has that

$$\begin{aligned}
\psi^*(\alpha_1) \smile \psi^*\bigg(\sum_{i=1}^{r}(-1)^{i+1}\beta_i\bigg) &= \psi^*\bigg(\alpha_1 \smile \sum_{i=1}^{r}(-1)^{i+1}\beta_i\bigg) \\
&= \psi^*\bigg(\alpha_1 \smile \beta_1 + \sum_{i=2}^{r} \alpha_1 \smile (-1)^{i+1}\beta_i\bigg) \\
&= \psi^*(\alpha_1 \smile \beta_1) = \psi^*\Big(\alpha_1 \smile \ldots \smile \alpha_r\Big) \neq 0,
\end{aligned} \tag{3.9}$$

since $\alpha_1 \smile \ldots \smile \alpha_r$ is the generator of $H^{rn}([S^n]^r)$ and ψ^* is a monomorphism.

On the other hand, from naturality of the cup product in the diagram

$$\begin{tikzcd}
H^n(X^r, X_\varphi) \otimes H^{(r-1)n}(X^r, X^r - X_\varphi) \arrow[r, "\smile"] \arrow[d, "j^*"'] & H^{rn}(X^r, X^r) \arrow[d, "j^*"] \\
H^n(X^r) \otimes H^{(r-1)n}(X^r) \arrow[r, "\smile"'] & H^{rn}(X^r)
\end{tikzcd}$$

and from equations (3.4) and (3.8) we conclude that

$$\begin{aligned}
\psi^*(\alpha_1 \smile \ldots \smile \alpha_r) &= \psi^*(\alpha_1) \smile \psi^*\bigg(\sum_{i=1}^{p}(-1)^{i+1}\beta_i\bigg) \\
&= j^*(U) \smile j^*(V) = j^*(U \smile V) = j^*(0) = 0,
\end{aligned}$$

which contradicts (3.9). This completes the proof. □

Now, let us consider the map $\theta = q_1 \circ \psi|_{X_\varphi}: X_\varphi \to S^n$, where $q_1: \Delta[S^n]^r \to S^n$ is the natural projection on the 1-th coordinate, which is an homeomorphism. Since by Lemma 3.3, $(\psi|_{X_\varphi})^*$ is a monomorphism, one then has that $\theta^*: H^n(S^n) \to H^n(X_\varphi)$ is a monomorphism. Note that, if $(x_1, \ldots, x_r) \in X_\varphi$, we have that for each i, $g_i\theta(x_1, \ldots, x_r) = g_i\varphi(x_1) = \varphi(x_i) = \theta g_i(x_1, \ldots, x_r)$, thus θ is a G-equivariant map, and consequently, θ is a H-equivariant map, where $H \subset G$ is a cyclic subgroup of prime order. Thus, in particular for $\rho = \sigma$, we can consider the homomorphism induced by θ, $\theta^*_\sigma: H^n_\sigma(S^n) \to H^n_\sigma(X_\varphi)$, where H^n_σ denotes the n-dimensional Smith special cohomology group with coefficients in $\mathbb{Z}_p$ in the sense of Section 2.4.

By remarks in [2, Results following 5.2] whose dual holds in cohomology, $i^*: H^n(S^n) \to H^n_\sigma(S^n)$ is an isomorphism, and since $\theta^*: H^n(S^n) \to H^n(X_\varphi)$ is a monomorphism it follows that θ^*_σ is a monomorphism. To conclude that the $\mathbb{Z}_p$-index of X_φ is equal to n it suffices to verify that the map between the orbit spaces $\overline{\theta}: X_\varphi/H \to S^n/H$ induces a monomorphism in cohomology. From results in [2, (3.10), p. 125], we have that $H^n_\sigma(S^n) \cong H^n(S^n/H)$ and $H^n_\sigma(X_\varphi) \cong H^n(X_\varphi/H)$, and considering the commutative diagram

$$\begin{tikzcd}
H^n_\sigma(S^n) \arrow[r, "\theta^*_\sigma"] & H^n_\sigma(X_\varphi) \\
H^n(S^n/H) \arrow[u, "\cong"] \arrow[r, "\overline{\theta}^*"'] & H^n(X_\varphi/H) \arrow[u, "\cong"']
\end{tikzcd}$$

it follows that $\overline{\theta}^*: H^n(S^n/H) \to H^n(X_\varphi/H)$ is a monomorphism. Therefore, the $\mathbb{Z}_p$-index of X_φ is equal to n. □

4. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.2. By following the similar steps of [3], we first prove Theorem 1.2 in the case that $G = H = Z_p$, where $p \geq 2$. We need to show that the $\mathbb{Z}_p$-index of the set $A_f = \{x \in X : f(x) = f(gx) = \ldots = f(g^{p-1}x)\}$ is greater than or equal to $n - pk$. For this, let us consider $F: X \to Y^p$ given by $F(x) = (f(x), f(gx), \ldots, f(g^{p-1}x))$, where $Y^p = Y \times \ldots \times Y$ denotes the p-fold cartesian product of Y and g is a fixed generator of $\mathbb{Z}_p$. In these conditions, we prove the following

Lemma 4.1. *The homomorphism $F^*: H^q(Y^p) \to H^q(X)$ induced by the map $F: X \to Y^p$ is zero for any $q \geq 1$.*

Proof. We have that $F = (f_0 \times \ldots \times f_{p-1}) \circ d$, where $d: X \to X^p$ is the diagonal map and $f_i(x) = f(g^i x)$, for any $x \in X$ and $i = 0, \ldots, p-1$. In this way, it suffices to show that $(f_0 \times f_1 \times \ldots \times f_{p-1})^*: H^q(Y^p) \to H^q(X^p)$ is trivial for any $q \geq 1$. Let us consider the following commutative diagram

$$\begin{tikzcd} \bigoplus_{i+j=q} H^i(Y^t)\otimes H^j(Y) \arrow[r, "\cong"] \arrow[d, "(f_0\times\ldots\times f_t)^*\otimes f_{t+1}^*"'] & H^q(Y^{t+1}) \arrow[d, "(f_0\times\ldots\times f_{t+1})^*"] \\ \bigoplus_{i+j=q} H^i(X^t)\otimes H^j(X) \arrow[r, "\times"'] & H^q(X^{t+1}) \end{tikzcd} \tag{4.1}$$

Since Y is a CW-complex, applying the Künneth's formula we have that the upper row of diagram (4.1) is an isomorphism for any $q = 1,2,\ldots$ and $t = 1,\ldots,p-1$.

The proof will be done by induction on t. Suppose inductively that $(f_0 \times \ldots \times f_t)^*\colon H^i(Y^t) \to H^i(X^t)$ is zero for some $t = 1,\ldots,p-1$ and for each $i = 1,2,\ldots$. By hypothesis, f induces the zero homomorphism in each dimension; in particular f_{t+1}^* is zero and thus $(f_0 \times \ldots \times f_t)^* \otimes f_{t+1}^*$ is trivial. It follows from commutativity of the diagram (4.1) that $(f_0 \times \ldots \times f_{t+1})^*$ is zero, which completes the proof. $\square$

We can define a $\mathbb{Z}_p$-action on Y^p as follows: for each $(y_1,\ldots,y_p) \in Y^p$ $g(y_1,\ldots,y_{p-1},y_p) = (y_p,y_1,\ldots,y_{p-1})$. Since p is a prime, this action is free on $Y^p - \Delta$, where Δ is the diagonal in Y^p. Let us observe that $A_f = F^{-1}(\Delta)$, thus F determines a $\mathbb{Z}_p$-equivariant map $F_0\colon X - A_f \to Y^p - \Delta$, which induces a map between the orbit spaces $\overline{F}_0\colon [X - A_f]^* \to [Y^p - \Delta]^*$. In these conditions, we prove that

Lemma 4.2. *The map* $\overline{F}_0^*\colon H^{pk}([Y^p - \Delta]^*) \to H^{pk}([X - A_f]^*)$ *is zero.*

Proof. Let us consider the map of pairs $(F,F_0)\colon (X, X - A_f) \to (Y, Y - \Delta)$. One then has the following commutative diagram

$$\begin{tikzcd} \arrow[r] & H^{pk}(X) \arrow[r, "i^*"] & H^{pk}(X - A_f) \arrow[r] & H^{pk+1}(X, X - A_f) \arrow[r] & {} \\ \arrow[r] & H^{pk}(Y^p) \arrow[r, "j^*"'] \arrow[u, "F^*"] & H^{pk}(Y^p - \Delta) \arrow[r] \arrow[u, "F_0^*"'] & H^{pk+1}(Y^p, Y^p - \Delta) \arrow[r] \arrow[u, "(F,F_0)^*"'] & {} \end{tikzcd}$$

where the homomorphisms i^* and j^* are induced by appropriate inclusions. Since $\dim(Y^p)$ is less than or equal to pk we have that $H^{pk+1}(Y^p, Y^p - \Delta)$ is trivial and thus j^* is surjective. On the other hand $F_0\colon (X - A_f) \to (Y^p - \Delta)$ is a $\mathbb{Z}_p$-equivariant map and it follows from Remark 2.6 that the diagram

$$\begin{tikzcd}
H^{pk}(X - A_f) \arrow[r, "\mathcal{T}"] & H^{pk}([X - A_f]^*) \arrow[r] & H^{pk+1}_\rho(X - A_f) \\
H^{pk}(Y^p - \Delta) \arrow[r, "\mathcal{T}"'] \arrow[u, "{F_0}^*"] & H^{pk}([Y^p - \Delta]^*) \arrow[r] \arrow[u, "\overline{F}_0^*"'] & H^{pk+1}_\rho(Y^p - \Delta) \arrow[u]
\end{tikzcd}$$

between the Smith sequences of $X - A_f$ and $Y^p - \Delta$ is commutative and since $H^{pk+1}_\rho(Y^p - \Delta)$ is zero, $\mathcal{T}$ is surjective.

Putting together these diagrams, one obtains a new commutative diagram

$$\begin{tikzcd}
H^{pk}(X) \arrow[r, "i^*"] & H^{pk}(X - A_f) \arrow[r, "\mathcal{T}"] & H^{pk}([X - A_f]^*) \\
H^{pk}(Y^p) \arrow[r, "j^*"'] \arrow[u, "F^*"] & H^{pk}(Y^p - \Delta) \arrow[r, "\mathcal{T}"'] \arrow[u, "F_0^*"'] & H^{pk}([Y^p - \Delta]^*) \arrow[u, "\overline{F}_0^*"']
\end{tikzcd} \tag{4.2}$$

where the horizontal sequences are not necessarily exacts, but the composition $\mathcal{T} \circ j^*$ is surjective. Therefore, as F^* is zero by Lemma 4.1, it follows from commutativity of the diagram (4.2) that ${\overline{F}_0}^*$ is zero. $\square$

Let $h: X^* \to B\mathbb{Z}_p$ be a classifying map for the principal $\mathbb{Z}_p$-bundle $X \to X^*$. Then the compositions $h \circ i_1: A_f^* \to B\mathbb{Z}_p$ and $h \circ i_2: [X - A_f]^* \to B\mathbb{Z}_p$ are classifying maps for the following principal $\mathbb{Z}_p$-bundles $A_f \to A_f^*$ and $X - A_f \to [X - A_f]^*$ respectively, where the maps $i_1: A_f^* \to X^*$ and $i_2: [X - A_f]^* \to X^*$ are induced by the inclusions between the orbit spaces.

Let us consider $G: Y^p - \Delta \to B\mathbb{Z}_p$ a classifying map for the principal $\mathbb{Z}_p$-bundle $Y^p - \Delta \to [Y^p - \Delta]^*$. Since $F_0: X - A_f \to Y^p - \Delta$ is a $\mathbb{Z}_p$-equivariant map, one the has that

$$G \circ \overline{F}_0: [X - A_f]^* \to B\mathbb{Z}_p$$

also classifies the principal $\mathbb{Z}_p$-bundle $X - A_f \to [X - A_f]^*$. In this way,

$$i_2^* \circ h^* = \overline{F}_0^* \circ G^*: H^*(B\mathbb{Z}_p) \to H^*([X - A_f]^*). \tag{4.3}$$

To conclude that the $\mathbb{Z}_p$-index of A_f is greater than or equal to $n-pk$, it suffices to show that $i_1^* \circ h^*(\mu) \neq 0$, where μ is the generator of $H^{n-pk}(B\mathbb{Z}_p)$.

We first consider the case when k is odd. Let us observe that n must be necessarily odd, since $p > 2$ is a prime. Then, $n - pk$ is even and it follows from Remark 2.5 that $\mu = b^{(n-pk)/2} \in H^{n-pk}(B\mathbb{Z}_p)$. Suppose that $i_1^* \circ h^*(\mu) = 0$. From continuity of the cohomology, there exists a neighbourhood V of A_f in X which is invariant by the action of $\mathbb{Z}_p$ and such that $i_1^* \circ h^*(\mu) = 0$ in $H^{n-pk}(V^*)$. From the exact cohomology sequence of the pair (X^*, V^*) one has

$$h^*(\mu) \in \mathrm{Im}\,[H^{n-pk}(X^*, V^*) \to H^{n-pk}(X^*)]. \tag{4.4}$$

Since pk is odd from Remark 2.5 $\eta = a \smile b^{(pk-1)/2}$ is a generator of $H^{pk}(B\mathbb{Z}_p)$. It follows from Lemma 4.2 and (4.3) that

$$i_2^* \circ h^*(\eta) = \overline{F}_0^* \circ G^*(\eta) = 0 \in H^{pk}([X - A_f]^*)$$

and from the exact cohomology sequence of the pair $(X^*, [X - A_f]^*)$ one has

$$h^*(\eta) \in \mathrm{Im}\,[H^{pk}(X^*, [X - A_f]^*) \to H^{pk}(X^*)]. \tag{4.5}$$

Thus from (4.4), (4.5) and by the naturality of the cup product we have

$$h^*(\eta \smile \mu) = h^*(\eta) \smile h^*(\mu) \in \mathrm{Im}\,[H^n(X^*, [X - A_f]^* \cup V^*) \to H^n(X^*)].$$

Let us note that the element

$$\eta \smile \mu = a \smile b^{(pk-1)/2} \smile b^{(n-pk)/2} = a \smile b^{(n-1)/2}$$

is a generator of $H^n(B\mathbb{Z}_p)$. Furthermore,

$$H^n(X^*, [X - A_f]^* \cup V^*) = H^n(X^*, X^*) = 0$$

and then $h^*(\eta \smile \mu) = 0 \in H^n(X^*)$, that is, $h^*\colon H^n(B\mathbb{Z}_p) \to H^n(X^*)$ is trivial which contradicts the hypothesis that the $\mathbb{Z}_p$-index of X is greater than or equal to n.

If k is even, then $n - pk$ is odd and pk is even. In this case, the proof is analogous to the previous case, considering now the generators

$$\mu = a \smile b^{(n-pk-1)/2} \in H^{n-pk}(B\mathbb{Z}_p) \quad \text{and} \quad \eta = b^{(pk)/2} \in H^{pk}(B\mathbb{Z}_p).$$

Let us examine the case where $G = \mathbb{Z}_2$. Here, n can be any positive integer and the generator of $H^{n-2k}(B\mathbb{Z}_2)$ is $\mu = a^{n-2k}$. To show that the $\mathbb{Z}_2$-index of A_f is greater than or equal to $n-2k$, it suffices to prove that $i_1^* \circ h^*(\mu) \neq 0$. Let us assume that $i_1^* \circ h^*(\mu) = 0$. Then there exists a neighbourhood V of A_f in X which is invariant with respect to the action and such that $i_1^* \circ h^*(\mu) = 0$ in $H^{n-2k}(V^*)$. From exact cohomology sequence of the pair (X^*, V^*) one has that

$$h^*(\mu) \in \mathrm{Im}\,[H^{n-2k}(X^*, V^*) \to H^{n-2k}(X^*)]. \tag{4.6}$$

On the other hand, $\eta = a^{2k}$ is the generator of $H^{2k}(B\mathbb{Z}_2)$ and it follows from Lemma 4.2 and (4.3) that

$$i_2^* \circ h^*(\eta) = \overline{F}_0^* \circ G^*(\eta) = 0 \in H^{2k}([X - A_f]^*).$$

Moreover, from exact cohomology sequence of $(X^*, [X - A_f]^*)$ one has that

$$h^*(\eta) \in \mathrm{Im}\,[H^{2k}(X^*, [X - A_f]^*) \to H^{2k}(X^*)]. \tag{4.7}$$

Thus, from (4.6), (4.7) and by the naturality of the cup product we have

$$h^*(\eta \smile \mu) = h^*(\eta) \smile h^*(\mu) \in \mathrm{Im}\,[H^n(X^*, [X - A_f]^* \cup V^*) \to H^n(X^*)].$$

Let us observe that $\eta \smile \mu = a^{2k} \smile a^{n-2k} = a^n$ is the generator of $H^n(B\mathbb{Z}_2)$. Furthermore, $H^n(X^*, [X - A_f]^* \cup V^*) = H^n(X^*, X^*)$ is trivial and then $h^*(\eta \smile \mu) = 0 \in H^n(X^*)$ which contradicts the hypothesis that the $\mathbb{Z}_2$-index of X is greater than or equal to n. This concludes the proof of Theorem 1.2 in the case $G = H = \mathbb{Z}_p$.

For the general case, suppose that G is a finite group which acts freely on X and let $H \subset G$ be a normal cyclic subgroup of prime order p. We denote by $s = |G|/p$, the number of the left cosets of G/H and let $a_1, \ldots, a_s$ be a set of representatives of the cosets. We define the map $F\colon X \to Y^s$ by

$$F(x) = (f(a_1x), \ldots, f(a_ix), \ldots, f(a_sx)). \tag{4.8}$$

We need to show that

$$A(f, H, G) = A_F = \{x \in X : F(x) = F(hx), \text{ for all } h \in H\}.$$

Let x be a point in the set $A(f, H, G)$, then f collapses each orbit determined by the action of H on a_ix to a single point, for each $i = 1, \ldots, s$. If $h \in H$

$$F(hx) = (f(ha_1x), \ldots, f(ha_ix), \ldots, f(ha_sx)).$$

Since H is a normal subgroup of G, $ha_ix = a_i\widehat{h}x$. Furthermore, a_ix and $a_i\widehat{h}x = ha_ix$ belongs to the same H-orbit and $f(a_ix) = f(ha_ix)$, for each $i = 1, \ldots, s$ which implies that $F(x) = F(hx)$. Therefore $x \in A_F$. The proof of the another inclusion is entirely analogous.

To conclude, let us observe that $H \cong \mathbb{Z}_p$ acts freely on X by restriction and by hypothesis the $\mathbb{Z}_p$-index of X is greater than or equal to n. By using Lemma 4.1 for the map $F\colon X \to Y^s$ defined in (4.8) one has that $F^*\colon H^q(Y^s) \to H^q(X)$ is trivial for any $q \geq 1$. Since dimension of Y^s is ks and Theorem 1.2 is true for $G = \mathbb{Z}_p$ we can conclude that the $\mathbb{Z}_p$-index of $A_F = A(f, H, G)$ is greater than or equal $n - p(ks) = n - pk(|G|/p) = n - |G|k$ and this completes the proof. $\square$

Proof of Theorem 1.1. Let $\widetilde{f}\colon X_\varphi \to Y$ given by $\widetilde{f}(x_1, \ldots, x_r) = f(x_1)$, that is $\widetilde{f} = f \circ \pi_1$ where π_1 is the natural projection on the 1-th coordinate. By hypothesis f induces the zero homomorphism in each dimension, then we have that $\widetilde{f}^*\colon H^i(Y) \to H^i(X_\varphi)$ is trivial for any $i \geq 1$. Moreover, the $\mathbb{Z}_p$-index of X_φ is equal to n by Theorem 3.1. In this way, X_φ and $\widetilde{f}$ satisfy the hypotheses of Theorem 1.2 which implies that the $\mathbb{Z}_p$-index of the set $A(\widetilde{f}, H, G)$ is greater than or equal to $n - |G|k$. By Definition 2.1 $A_\varphi(f, H, G) = A(\widetilde{f}, H, G)$, and then cohom.dim $A_\varphi(f, H, G) \geq n - |G|k$. $\square$

Remark 4.3. In the particular case that $G = H = \mathbb{Z}_p$ with p prime, Volovikov in [13, Theorem 3.2] proved a version of Theorem 1.1.

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