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THE NUMBERS OF PERIODIC ORBITS HIDDEN AT FIXED POINTS OF n-DIMENSIONAL HOLOMORPHIC MAPPINGS (II)

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Abstract. Let Δ^n be the ball $|x| < 1$ in the complex vector space $\mathbb{C}^n$, let $f\colon \Delta^n \to \mathbb{C}^n$ be a holomorphic mapping and let M be a positive integer. Assume that the origin $0 = (0, \ldots , 0)$ is an isolated fixed point of both f and the M-th iteration f^M of f. Then the (local) Dold index $P_M(f, 0)$ at the origin is well defined, which can be interpreted to be the number of periodic points of period M of f hidden at the origin: any holomorphic mapping $f_1\colon \Delta^n \to \mathbb{C}^n$ sufficiently close to f has exactly $P_M(f, 0)$ distinct periodic points of period M near the origin, provided that all the fixed points of f_1^M near the origin are simple. Therefore, the number $\mathcal{O}_M(f, 0) = P_M(f, 0)/M$ can be understood to be the number of periodic orbits of period M hidden at the fixed point.

According to Shub–Sullivan [18] and Chow–Mallet-Paret–Yorke [2], a necessary condition so that there exists at least one periodic orbit of period M hidden at the fixed point, say, $\mathcal{O}_M(f, 0) \geq 1$, is that the linear part of f at the origin has a periodic point of period M. It is proved by the author in [21] that the converse holds true.

In this paper, we continue to study the number $\mathcal{O}_M(f, 0)$. We will give a sufficient condition such that $\mathcal{O}_M(f, 0) \geq 2$, in the case that all eigenvalues of $Df(0)$ are primitive m_1-th, $\ldots$, m_n-th roots of unity, respectively, and $m_1, \ldots , m_n$ are distinct primes with $M = m_1 \ldots m_n$.

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1. Introduction

Let $\mathbb{C}^n$ be the complex vector space of dimension n with the Euclidean norm, let U be an open subset of $\mathbb{C}^n$ and let $g\colon U \to \mathbb{C}^n$ be a holomorphic mapping.

If $p \in U$ is an isolated zero of g, say, there exists a ball B centered at p with $\overline{B} \subset U$ such that p is the unique solution of the equation $g(x) = 0$ ($0 = (0, \dots, 0)$ is the origin) in $\overline{B}$, then we can define the *zero order* of g at p by

$$\pi_g(p) = \#(g^{-1}(v) \cap B) = \#\{x \in B; g(x) = v\},$$

where v is a regular value of g such that $|v|$ is small enough and $\#$ denotes the cardinality. $\pi_g(p)$ is a well defined integer (see [14] or [19] for the details).

If $q \in U$ is an isolated fixed point of g, then q is an isolated zero of the mapping $\mathrm{id} - g\colon U \to \mathbb{C}^n$, which puts each $x \in U$ into $x - g(x) \in \mathbb{C}^n$, and then the *fixed point index* $\mu_g(q)$ of g at q is defined to be the zero order of $\mathrm{id} - g$ at q:

$$\mu_g(q) = \pi_{\mathrm{id}-g}(q) = \pi_{g-\mathrm{id}}(q).$$

The zero order defined here is the (local) topological degree, and the fixed point index defined here is the (local) Lefschetz fixed point index, if g is regarded as a continuous mapping of real variables (see the appendix of [21] for the details).

If q is a fixed point of g such that $\mathrm{id} - g$ is regular at q, say, the Jacobian matrix $Dg(q)$ of g at q has no eigenvalue 1, q is called a *simple* fixed point of g. By Lemma 2.1, a fixed point of a holomorphic mapping has index 1 if and only if it is simple.

We denote by $\mathcal{O}(\mathbb{C}^n, 0, 0)$ the space of all germs of holomorphic mappings f between two neighbourhoods of the origin $0 = (0, \dots, 0)$ in $\mathbb{C}^n$ such that

$$f(0) = 0.$$

Then, for each $f \in \mathcal{O}(\mathbb{C}^n, 0, 0)$, 0 is a fixed point of f and for each $m \in \mathbb{N}$ (the set of positive integers), the m-th iteration f^m of f is well defined in a neighbourhood of 0, which is defined as

$$f^1 = f, f^2 = f \circ f, \dots, f^m = f \circ f^{m-1},$$

inductively.

Let $f \in \mathcal{O}(\mathbb{C}^n, 0, 0)$ and assume that the origin $0 = (0, \dots, 0)$ is an isolated fixed point of both f and the M-th iteration f^M of f. Then for each factor m of M, the origin is again an isolated fixed point of f^m and the fixed point index $\mu_{f^m}(0)$ of f^m at the origin is well defined, and so is the (local) Dold index (see [5]) at the origin:

$$P_M(f, 0) = \sum_{\tau \subset P(M)} (-1)^{\#\tau} \mu_{f^{M:\tau}}(0), \tag{1.1}$$

where $P(M)$ is the set of all primes dividing M, the sum extends over all subsets τ of $P(M)$, $\#\tau$ is the cardinal number of τ and $M:\tau = M(\prod_{k\in\tau} k)^{-1}$. Note that the sum includes the term $\mu_{f^M}(0)$ which corresponds to the empty subset $\tau = \emptyset$. If $M = 12 = 2^2 \cdot 3$, for example, then $P(M) = \{2,3\}$, and

$$P_{12}(f,0) = \mu_{f^{12}}(0) - \mu_{f^4}(0) - \mu_{f^6}(0) + \mu_{f^2}(0).$$

The formula (1.1) is known as the Möbius inversion formula (see [11] and [21] for more interpretations of the formulae of type (1.1)).

$P_M(f,0)$ can be interpreted to be the number of (*virtual*) periodic points of period M of f hidden at the origin:

For any ball B centered at the origin, such that f^M is well defined on $\overline{B}$ and has no fixed point in $\overline{B}$ other than the origin, any holomorphic mapping $g\colon \overline{B} \to \mathbb{C}^n$ has exactly $P_M(f,0)$ mutually distinct periodic points of period M in B, provided that all fixed points of g^M in B are simple and that g is sufficiently close to f, in the sense that

$$||f-g||_{\overline{B}} = \sup_{x\in\overline{B}} |f(x)-g(x)|$$

is small enough (*see Lemma* 2.4(c) *and* 2.5(b)).

p is called a *periodic point* of g of period m if $g^m(p) = p$ but $g^j(p) \neq p$ for $j = 1,\dots,m-1$. When p is a periodic point of g of period m, the set $\{p, g(p), g^2(p),\dots, g^{m-1}(p)\}$ is called a *periodic orbit* of periodic m.

It is easy to see that any two periodic orbits either coincide, or do not intersect and any periodic orbit of period M contains exactly M distinct points. On the other hand, in the above interpretation of $P_M(f,0)$, if g is close to f enough and g has a periodic point p of period M in B, then the whole periodic orbit containing p is in B. Therefore, the number

$$\mathcal{O}_M(f,0) = P_M(f,0)/M$$

is an integer and can be understood to be the number of (*virtual*) periodic orbits of period M hidden at the fixed point 0.

REMARK 1.1. The important local index $P_M(f,0)$ and the global index $P_M(f)$ (see (2.1) in Section 2) were first introduced by Dold [5], via fixed point indices of iterations of real mappings. Some interesting topics are related to these indices and the Dold's relation [5] stating that the global index $P_M(f)$ is divided by M (the reader is referred to the references [6], [7]–[10], [12], [13], [16] and [17]).

According to Shub–Sullivan [18] and Chow–Mallet-Paret–Yorke [2], a necessary condition such that $\mathcal{O}_M(f,0) \neq 0$, say, there exists at least one periodic orbit of period M hidden at the fixed point 0 of f, is that the linear part of f

at the origin has a periodic point of period M (see Lemma 2.9 and its consequence Lemma 2.10(a)). The term "linear part" indicates the linear mapping $l\colon \mathbb{C}^n \to \mathbb{C}^n$,

$$l(x_1, \dots, x_n) = \left(\sum_{j=1}^{n} a_{1j} x_j, \dots, \sum_{j=1}^{n} a_{nj} x_j \right),$$

where

$$(a_{ij}) = Df(0) = \left(\frac{\partial f_i}{\partial x_j} \right)\bigg|_0$$

is the Jacobian matrix of $f = (f_1, \dots, f_n)$ at the origin.

We have proved in [21] that the above necessary condition is sufficient. Thus, one has

THEOREM 1.2. *Let $M \in \mathbb{N}$, $f \in \mathcal{O}(\mathbb{C}^n, 0, 0)$ and assume that the origin is an isolated fixed point of f^M. Then, $\mathcal{O}_M(f, 0) \neq 0$ if and only if the linear part of f at 0 has a periodic point of period M.*

If $M > 1$, then by Lemma 2.8, the linear part of f at 0 has a periodic point of period M if and only if the following condition holds.

CONDITION 1.3. *$Df(0)$ has eigenvalues $\lambda_1, \dots, \lambda_s, s \leq n$, that are primitive m_1-th, $\dots$, m_s-th roots of unity, respectively, such that M is the least common multiple of $m_1, \dots, m_s$.*

Thus, if $M > 1$, by the above theorem, $\mathcal{O}_M(f, 0) \geq 1$ if and only if Condition 1.3 holds. This gives rise to the following problem.

PROBLEM 1.4. Assume that Condition 1.3 holds. Under which additional condition, one has $\mathcal{O}_M(f, 0) \geq 2$?

There are two aspects to study this problem. One is to study $Df(0)$ alone and we have proved the following theorems.

THEOREM 1.5 ([22]). *Let $M > 1$ be a positive integer and let A be a 2 by 2 matrix. Then the following conditions are equivalent:*

(A) *For any holomorphic mapping germ $f \in \mathcal{O}(\mathbb{C}^2, 0, 0)$ such that $Df(0) = A$ and that 0 is an isolated fixed point of both f and f^M,*

$$\mathcal{O}_M(f, 0) \geq 2.$$

(B) *The two eigenvalues λ_1 and λ_2 of A are primitive m_1-th and m_2-th roots of unity, respectively, and one of the following conditions holds.*
  (b1) *$m_1 = m_2 = M$, $\lambda_1 = \lambda_2$ and A is diagonalizable,*
  (b2) *$m_1 = m_2 = M$ and there exist positive integers α and β such that $1 < \alpha < M$, $1 < \beta < M$ and*

$$\lambda_1^\alpha = \lambda_2, \quad \lambda_2^\beta = \lambda_1, \quad \alpha\beta > M + 1.$$

(b3) $m_1|m_2$, $m_2 = M$, *and* $\lambda_2^{m_2/m_1} \neq \lambda_1$.
(b4) $M = [m_1, m_2]$, $(m_1, m_2) > 1$ *and* $\max\{m_1, m_2\} < M$.

Here, $[m_1, m_2]$ denotes the least common multiple and (m_1, m_2) denotes the greatest common divisor, of m_1 and m_2, and $m_1|m_2$ means that m_1 divides m_2.

Theorem 1.6 ([23]). *Let $M > 1$ be a positive integer, and let A be an $n \times n$ matrix ($n \geq 3$) such that all eigenvalues of A are the same primitive M-th root of unity. Then the following conditions are equivalent:*

(A) *For any holomorphic mapping germ $f \in \mathcal{O}(\mathbb{C}^n, 0, 0)$ such that $Df(0) = A$ and that 0 is an isolated fixed point of both f and f^M,*

$$\mathcal{O}_M(f, 0) \geq 2.$$

(B) *A has at least two distinct nontrivial invariant space.*

In this paper, we study the above problem in another aspect, to consider the higher order terms, in the case that the mapping f is given by

$$f(x_1, \dots, x_n) = (\lambda_1 x_1, \lambda_2 x_2, \dots, \lambda_n x_n) + o(|x|),$$

where, $x = (x_1, \dots, x_n)$, $\lambda_1, \dots, \lambda_n$ are primitive m_1-th, $\dots, m_n$-th roots of unity, respectively, and $m_1, \dots, m_n$ are distinct primes. Then, $\mathcal{O}_{m_1 \dots m_n}(f, 0) \geq 1$ if 0 is an isolated fixed point of $f^{m_1, \dots, m_n}$, by Theorem 1.2.

By the theory of normal forms (see Corollary 3.2 in Section 3), there exists a local biholomorphic transform $h \in \mathcal{O}(\mathbb{C}^n, 0, 0)$ in the form of

$$h(y_1, \dots, y_n) = (y_1, \dots, y_n) + o(|y|),$$

where $y = (y_1, \dots, y_n)$, such that

$$g(y) = h^{-1} \circ f \circ h(y)$$

can be expressed as

$$g(y) = \begin{pmatrix} \lambda_1 y_1 + y_1 \displaystyle\sum_{i=1}^{n} a_{1i} y_i^{m_i} + h_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \lambda_n y_n + y_n \displaystyle\sum_{i=1}^{n} a_{ni} y_i^{m_i} + h_n \end{pmatrix}^T + o(|y|^{m_1 \dots m_n}),$$

where, a_{ji} are complex numbers and each h_j is a polynomial in $y_1^{m_1}, \dots, y_n^{m_n}$ without constant and linear terms, that is, in a neighbourhood of the origin, h_j has a power series expansion in the form of

$$\sum_{i_1 + \dots + i_n = 2}^{N} c_{i_1 \dots i_n} (x_1^{m_1})^{i_1} \dots (x_n^{m_n})^{i_n}, \tag{1.2}$$

for some positive integer N. We will call the matrix $\mathcal{A}_h(f) = (a_{ji})$ the *first resonant matrix* of f determined by h at 0. By Lemma 2.11, for any positive integer M, $\mathcal{O}_M(f,0) = \mathcal{O}_M(g,0)$.

Now, we can state our main result.

THEOREM 1.7. *Let $m_1, \dots, m_n$ be distinct primes and let $f \in \mathcal{O}(\mathbb{C}^n, 0, 0)$. Assume that the origin is an isolated fixed point of both f and f^M and that $Df(0) = (\lambda_1, \dots, \lambda_n)$ is diagonal, where $\lambda_1, \dots, \lambda_n$ are primitive m_1-th, ..., m_n-th roots of unity. Then*

$$\mathcal{O}_{m_1 \dots m_n}(f,0) \geq 2 \tag{1.3}$$

if one of the first resonant matrices $\mathcal{A}_h(f)$ of f at 0 is singular, say,

$$\det \mathcal{A}_h(f) = 0. \tag{1.4}$$

Here and throughout this paper, when we use the vector notation $(\lambda_1, \dots, \lambda_n)$ to denote a square matrix, it denotes the $n \times n$ diagonal matrix with $\lambda_1, \dots, \lambda_n$ down its main diagonal. But when there is no extra explanation, the notation $(\lambda_1, \dots, \lambda_n)$ always denotes a vector.

Sections 2–4 are arranged for proving the main theorem, and the proof will be completed in Section 5.

EXAMPLE 1.8. For one variable holomorphic functions, it is relatively easy to understand the number $\mathcal{O}_M(f,0)$ and Theorem 1.7.

We first introduce an iteration formula of one variable functions. Let $M \in \mathbb{N}$, let

$$f(z) = \lambda z + o(z)$$

be a germ of holomorphic function at the origin and assume that $\lambda = f'(0)$ is a primitive M-th root of unity. Then it is well known (see [15]) that either (i) $f^M(z) \equiv z$, or (ii) there exist an $\alpha \in \mathbb{N}$ and a constant $a \neq 0$ such that

$$f^M(z) = z + az^{\alpha M+1} + o(z^{\alpha M+1}).$$

In the later case (ii), we show that $\mathcal{O}_M(f,0) = \alpha$.

In fact, in case (ii), we have $\mu_{f^M}(0) = \alpha M + 1$ and $\mu_{f^j}(0) = 1$ for any $j \in \mathbb{N}$ with $j \not\equiv 0 \pmod{M}$. Thus, (1.1) becomes

$$\begin{aligned} P_M(f,0) &= \mu_{f^M}(0) + \sum_{\substack{\tau \subset P(M) \\ \tau \neq \emptyset}} (-1)^{\#\tau} \mu_{f^{M:\tau}}(0) \\ &= \mu_{f^M}(0) + \sum_{\substack{\tau \subset P(M) \\ \tau \neq \emptyset}} (-1)^{\#\tau} \cdot 1 \\ &= \mu_{f^M}(0) + \sum_{k=1}^{\#P(M)} (-1)^k \binom{\#P(M)}{k} = \mu_{f^M}(0) - 1, \end{aligned}$$

and then one has

$$\mathcal{O}_M(f,0) = \frac{\alpha M}{M} = \alpha.$$

On the other hand, in case (ii), by Corollary 3.2, there exists a holomorphic function $h(z) = z + o(z)$ defined in a neighbourhood of the origin such that

$$g(z) = h^{-1} \circ f \circ h(z) \equiv \lambda z + \mathcal{A}_h z^{M+1} + o(z^{M+1}),$$

where $\mathcal{A}_h$ is a constant, and then, it is easy to see that

$$g^M(z) = z + M\mathcal{A}_h \lambda^{M-1} z^{M+1} + o(z^{M+1}),$$

and, repeating the above argument, we have that $\mathcal{O}_M(g,0) \geq 1$, and $\mathcal{O}_M(g,0) > 1$ if and only if $\mathcal{A}_h = 0$. Thus by Lemma 2.11, $\mathcal{O}_M(f,0) = \mathcal{O}_M(g,0) > 1$ if and only if $\mathcal{A}_h = 0$. So, in the case $n = 1$, we have already proved Theorem 1.7, and moreover, we see that the sufficient condition in Theorem 1.7 is also necessary, namely, (1.3) implies (1.4).

Example 1.9. Let $f \in \mathcal{O}(\mathbb{C}^2, 0, 0)$ be given by

$$f(x,y) = (\lambda_1 x + o(x), \lambda_2 y + o(y)),$$

such that λ_1 is a primitive m_1-th root of unity, λ_2 is a primitive m_2-th root of unity, m_1 and m_2 are distinct primes, and 0 is an isolated fixed point of $f^{m_1 m_2}$. Then there exist nonzero constants a, b, and positive integers α and β such that

$$f^{m_1}(x,y)^T = \begin{pmatrix} x + a x^{\alpha m_1 + 1}(1 + o(1)) \\ \lambda_2^{m_1} y (1 + o(1)) \end{pmatrix},$$

$$f^{m_2}(x,y)^T = \begin{pmatrix} \lambda_1^{m_2} x (1 + o(1)) \\ y + b y^{\beta m_2 + 1}(1 + o(1)) \end{pmatrix},$$

and

$$f^{m_1 m_2}(x,y)^T = \begin{pmatrix} x + a m_2 x^{\alpha m_1 + 1}(1 + o(1)) \\ y + b m_1 x^{\beta m_2 + 1}(1 + o(1)) \end{pmatrix}.$$

Thus, by Cronin Theorem introduced in Section 4 we have

$$\mu_{f^{m_1}}(0) = \pi_{f^{m_1} - \mathrm{id}} = \alpha m_1 + 1, \tag{1.5}$$

$$\mu_{f^{m_2}}(0) = \pi_{f^{m_2} - \mathrm{id}} = \beta m_2 + 1, \tag{1.6}$$

$$\mu_{f^{m_1 m_2}}(0) = \pi_{f^{m_1 m_2} - \mathrm{id}} = (\alpha m_1 + 1)\,(\beta m_2 + 1). \tag{1.7}$$

On the other hand, it is easy to see that

$$\mu_f(0) = 1. \tag{1.8}$$

Thus, by (1.1), (1.5) and (1.8) we have

$$P_{m_1}(f,0) = \mu_{f^{m_1}}(0) - \mu_f(0) = \alpha m_1,$$

by (1.1), (1.6) and (1.8) we have

$$P_{m_2}(f,0) = \mu_{f^{m_2}}(0) - \mu_f(0) = \beta m_2,$$

and by (1.5)–(1.8) and (1.1) we have

$$P_{m_1m_2}(f,0) = \mu_{f^{m_1m_2}}(0) - \mu_{f^{m_1}}(0) - \mu_{f^{m_2}}(0) + 1 = \alpha\beta m_1 m_2.$$

Hence, we have

$$\mathcal{O}_{m_1}(f,0) = \alpha, \mathcal{O}_{m_2}(f,0) = \beta, \mathcal{O}_{m_1m_2}(f,0) = \alpha\beta,$$

and

$$\mathcal{O}_{m_1m_2}(f,0) = \mathcal{O}_{m_1}(f,0)\mathcal{O}_{m_2}(f,0). \tag{1.9}$$

By this example, one may guess that there is a relation between the numbers $\mathcal{O}_{m_1}(f,0)$, $\mathcal{O}_{m_2}(f,0)$ and $\mathcal{O}_{m_1m_2}(f,0)$ similar to the above equality (1.9). But see the next example.

EXAMPLE 1.10. Let $k > 1$ be any given positive integer and $f \in \mathcal{O}(\mathbb{C}^2, 0, 0)$ be given by

$$(f(x,y))^T = \begin{pmatrix} -x + x^{2k+1} + xy^3 \\ e^{2\pi i/3}y + x^2y + y^{3k+1} \end{pmatrix}.$$

We show that $\mathcal{O}_2(f,0) = \mathcal{O}_3(f,0) = k$, but $\mathcal{O}_6(f,0) = 1$.

After a careful computation, we have

$$(f^2(x,y))^T = \begin{pmatrix} x - 2x^{2k+1}(1+o(1)) - 2xy^3(1+o(1)) \\ e^{4\pi i/3}y(1+o(1)) \end{pmatrix}, \tag{1.10}$$

$$(f^3(x,y))^T = \begin{pmatrix} -x(1+o(1)) \\ y + 3e^{4\pi i/3}x^2y(1+o(1)) + 3e^{4\pi i/3}y^{3k+1}(1+o(1)) \end{pmatrix}, \tag{1.11}$$

and

$$(f^6(x,y))^T = \begin{pmatrix} x + xh_1(x,y) \\ y + yh_2(x,y) \end{pmatrix},$$

with

$$\begin{aligned} h_1(x,y) &= -6x^{2k}(1+o(1)) - 6y^3(1+o(1)), \\ h_2(x,y) &= 6e^{4\pi i/3}x^2(1+o(1)) + 6e^{4\pi i/3}y^{3k}(1+o(1)). \end{aligned}$$

By (1.10) and Cronin theorem, we have

$$\mu_{f^2}(0) = \pi_{f^2-\mathrm{id}}(0) = 2k+1.$$

Similarly, by (1.11) and Cronin theorem, we have

$$\mu_{f^3}(0) = \pi_{f^3-\mathrm{id}}(0) = 3k+1.$$

On the other hand, $\mu_f(0) = 1$. Therefore, by the formula (1.1), we have

$$P_2(f,0) = \mu_{f^2}(0) - 1 = 2k \quad \text{and} \quad P_3(f,0) = \mu_{f^3}(0) - 1 = 3k;$$

and then, we have

$$\mathcal{O}_2(f,0) = P_2(f,0)/2 = k \quad \text{and} \quad \mathcal{O}_3(f,0) = P_3(f,0)/3 = k.$$

Next, we show that $\mathcal{O}_6(f,0) = 1$. It is clear that $\mu_{f^6}(0)$ equals the zero order of the mapping

$$f^6 - \mathrm{id}\colon (x,y) \mapsto (xh_1(x,y), yh_2(x,y)),$$

and by Lemma 2.12, the zero order of $f^6 - \mathrm{id}$ at 0 is the sum of the zero orders of the four mappings putting (x,y) into (x,y), $(x,h_2(x,y))$, $(h_1(x,y),y)$ and $(h_1(x,y),h_2(x,y))$, which are 1, $3k$, $2k$ and 6, respectively, by Cronin Theorem. Thus $\mu_{f^6}(0) = 5k+7$, and then, by the formula (1.1), we have

$$P_6(f,0) = \mu_{f^6}(0) - \mu_{f^2}(0) - \mu_{f^3}(0) + 1 = 5k + 7 - 2k - 1 - 3k - 1 + 1,$$

and then $P_6(f,0) = 6$, and $\mathcal{O}_6(f,0) = 1$.

Remark 1.11. By Theorem 1.2, the numbers $\mathcal{O}_{m_1}, \dots, \mathcal{O}_{m_n}$, $\mathcal{O}_{[m_1,\dots,m_n]}$ have the relation that

$$\mathcal{O}_{[m_1,\dots,m_n]} \geq 1 \quad \text{if } \mathcal{O}_{m_1} \geq 1, \dots, \mathcal{O}_{m_n} \geq 1,$$

where $[m_1,\dots,m_n]$ denotes the least common multiple.

2. Some basic results of fixed point indices and zero indices

In this section we introduce some results for later use. Most of them are known.

Let U be an open and bounded subset of $\mathbb{C}^n$ and let $f\colon \overline{U} \to \mathbb{C}^n$ be a holomorphic mapping. If f has no fixed point on the boundary ∂U, then the fixed point set $\mathrm{Fix}(f)$ of f is a compact analytic subset of U, and then it is finite (see [3]); and therefore, we can define the global fixed point index $L(f)$ of f as:

$$L(f) = \sum_{p \in \mathrm{Fix}(f)} \mu_f(p),$$

which is just the number of all fixed points of f, counting indices. $L(f)$ is, in fact, the Lefschetz fixed point index of f (see the appendix section in [21] for the details).

For each $m \in \mathbb{N}$, the m-th iteration f^m of f is understood to be defined on

$$K_m(f) = \bigcap_{k=0}^{m-1} f^{-k}(\overline{U}) = \{x \in \overline{U}; f^k(x) \in \overline{U} \text{ for all } k = 1, \dots, m-1\},$$

which is the largest set where f^m is well defined. Since U is bounded, $K_m(f)$ is a compact subset of $\overline{U}$. Here, $f^0 = \mathrm{id}$.

Now, let us introduce the global Dold index. Let $M \in \mathbb{N}$ and assume that f^M has no fixed point on the boundary ∂U. Then, for each factor m of M, f^m

again has no fixed point on ∂U, and then the fixed point set $\mathrm{Fix}(f^m)$ of f^m is a compact subset of U. Thus, there exists an open subset V_m of U such that $\mathrm{Fix}(f^m) \subset V_m \subset \overline{V_m} \subset U$ and f^m is well defined on $\overline{V_m}$, and thus $L(f^m|_{\overline{V_m}})$ is well defined and we write $L(f^m) = L(f^m|_{\overline{V_m}})$, where $f^m|_{\overline{V_m}}$ is the restriction of f^m to $\overline{V_m}$. In this way, we can define the global Dold index (see [5]) as (1.1):

$$P_M(f) = \sum_{\tau \subset P(M)} (-1)^{\#\tau} L(f^{M:\tau}). \tag{2.1}$$

Let $m \in \mathbb{N}$. It is clear that, for any compact subset K of U with $\bigcup_{j=1}^m f^j(K) \subset U$, there is a neighbourhood $V \subset U$ of K, such that for any holomorphic mapping $g: \overline{U} \to \mathbb{C}^n$ sufficiently close to f, the iterations $g^j, j = 1, \dots, m$, are well defined on $\overline{V}$ and

$$\max_{x \in \overline{U}} |g(x) - f(x)| \to 0 \ \Rightarrow \ \max_{1 \le j \le m} \max_{x \in \overline{V}} |g^j(x) - f^j(x)| \to 0.$$

We shall use these facts frequently and tacitly.

LEMMA 2.1 ([14]). *Let $f \in \mathcal{O}(\mathbb{C}^n, 0, 0)$ and assume that the origin is an isolated fixed point. Then $\mu_f(0) \ge 1$, and the equality holds if and only if 1 is not an eigenvalue of $Df(0)$.*

We denote by Δ^n a ball in $\mathbb{C}^n$ centered at the origin.

LEMMA 2.2 ([14]). (a) *Let $f: \overline{\Delta^n} \to \mathbb{C}^n$ be a holomorphic mapping such that f has no fixed point on the boundary $\partial\Delta^n$. Then there exists a $\delta > 0$ such that any holomorphic mapping $g: \overline{\Delta^n} \to \mathbb{C}^n$ with $\max_{x \in \overline{\Delta^n}} |g(x) - f(x)| < \delta$ has finitely many fixed points in Δ^n and satisfies*

$$L(g) = \sum_{p \in \mathrm{Fix}(g)} \mu_g(p) = \sum_{p \in \mathrm{Fix}(f)} \mu_f(p) = L(f).$$

(b) *In particular, if 0 is the unique fixed point of f in $\overline{\Delta^n}$, then for any holomorphic mapping $g : \overline{\Delta^n} \to \mathbb{C}^n$ with $\max_{x \in \overline{\Delta^n}} |g(x) - f(x)| < \delta$,*

$$\mu_f(0) = \sum_{p \in \mathrm{Fix}(g)} \mu_g(p),$$

and if in addition all fixed points of g are simple, then

$$\mu_f(0) = \#\mathrm{Fix}(g) = \#\{y \in \Delta^n; g(y) = y\}.$$

This result is another version of Rouché Theorem which is stated as follows (see [14]).

THEOREM 2.3 (Rouché Theorem). *Let $f\colon \overline{\Delta^n} \to \mathbb{C}^n$ be a holomorphic mapping such that f has no zero on $\partial\Delta^n$, Then there exists a $\delta > 0$ such that any holomorphic mapping $g\colon \overline{\Delta^n} \to \mathbb{C}^n$ with $\max_{x\in\partial\Delta^n} |g(x)-f(x)| < \delta$ has the same number of zeros in Δ^n as f, counting zero orders, say,*

$$\sum_{f(x)=0} \pi_f(x) = \sum_{g(x)=0} \pi_g(x).$$

LEMMA 2.4 ([21]). *Let M be a positive integer, let U be a bounded open subset of $\mathbb{C}^n$, let $f\colon \overline{U} \to \mathbb{C}^n$ be a holomorphic mapping and assume that f^M has no fixed point on ∂U. Then:*

(a) *There exists an open subset V of U, such that f^M is well defined on $\overline{V}$, has no fixed point outside V, and has only finitely many fixed points in V.*

(b) *For any holomorphic mapping $g : \overline{U} \to \mathbb{C}^n$ sufficiently close to f, g^M is well defined on $\overline{V}$, has no fixed point outside V and has only finitely many fixed points in V; and furthermore,*

$$L(g^M) = L(f^M), \quad P_M(g) = P_M(f).$$

(c) *In particular, if $p_0 \in U$ is the unique fixed point of both f and f^M in $\overline{U}$, then for any holomorphic mapping $g\colon \overline{U} \to \mathbb{C}^n$ sufficiently close to f,*

$$L(g^M) = L(f^M) = \mu_{f^M}(p_0), \quad P_M(g) = P_M(f) = P_M(f, p_0).$$

LEMMA 2.5. *Let M be a positive integer and let $f\colon \overline{\Delta^n} \to \mathbb{C}^n$ be a holomorphic mapping such that f^M has no fixed point on $\partial\Delta^n$ and each fixed point of f^M is simple. Then,* $\mathrm{Fix}(f^M)$ *is finite, and*

(a) $L(f^M) = \#\mathrm{Fix}(f^M) = \sum_{m|M} P_m(f)$;

(b) *$P_M(f)/M$ is the number of distinct periodic orbits of f of period M.*

PROOF. This is proved in [6] (see [21] for a very simple proof). □

A fixed point p of f is called *hyperbolic* if $Df(p)$ has no eigenvalue of modulus 1. If p is a hyperbolic fixed point of f, then it is a hyperbolic fixed point of all iterations $f^j, j \in \mathbb{N}$. A hyperbolic fixed point is a simple fixed point, and so it has index 1 by Lemma 2.1.

LEMMA 2.6 ([23]). *Let M be a positive integer, let U be a bounded open subset of $\mathbb{C}^n$, let $f\colon \overline{U} \to \mathbb{C}^n$ be a holomorphic mapping, and assume that f^M has no fixed point on ∂U. Then $P_M(f) \geq 0$. In particular, if $p \in U$ is an isolated fixed point of both f and f^M, then $P_M(f, p) \geq 0$.*

LEMMA 2.7 ([23]). *Let k and M be positive integers and let $f \in \mathcal{O}(\mathbb{C}^n, 0, 0)$. Assume that 0 is an isolated fixed point of both f and f^M, and there exists a sequence of holomorphic mappings $f_j \in \mathcal{O}(\mathbb{C}^n, 0, 0)$, uniformly converging to f in a neighbourhood of 0, such that $\mathcal{O}_M(f_j, 0) \geq k$. Then $\mathcal{O}_M(f, 0) \geq k$.*

LEMMA 2.8. *Let $L: \mathbb{C}^n \to \mathbb{C}^n$ be a linear mapping and let $M > 1$ be a positive integer. Then L has a periodic point of period M if and only if L has eigenvalues $\lambda_1, \dots, \lambda_s$, $s \leq n$, that are primitive m_1-th,…,m_s-th roots of unity, respectively, such that M is the least common multiple of $m_1, \dots, m_s$.*

This is a basic knowledge of elementary linear algebra. The following result is due to Shub and Sullivan.

LEMMA 2.9 ([18]). *Let $m > 1$ be a positive integer and let $f \in \mathcal{O}(\mathbb{C}^n, 0, 0)$. Assume that the origin is an isolated fixed point of f and that, for each eigenvalue λ of $Df(0)$, either $\lambda = 1$ or $\lambda^m \neq 1$. Then the origin is still an isolated fixed point of f^m and*

$$\mu_f(0) = \mu_{f^m}(0).$$

LEMMA 2.10. *Let $f \in \mathcal{O}(\mathbb{C}^n, 0, 0)$ and let*

$$\mathfrak{M}_f = \{m \in \mathbb{N} \text{ the linear part of } f \text{ at } 0 \text{ has periodic points of period } m\}.$$

Then,

(a) *For each $m \in \mathbb{N} \setminus \mathfrak{M}_f$ such that 0 is an isolated fixed point of f^m,*

$$P_m(f, 0) = 0;$$

(b) *For each positive integer M such that 0 is an isolated fixed point of f^M,*

$$\mu_{f^M}(0) = \sum_{\substack{m \in \mathfrak{M}_f \\ m | M}} P_m(f, 0).$$

PROOF. (a) and (b) are essentially proved in [2] (see [21] for a simple proof).□

LEMMA 2.11. *Let k be a positive integer, let f and h be germs in $\mathcal{O}(\mathbb{C}^n, 0, 0)$ such that 0 is an isolated fixed point of both f and f^k and $\det Dh(0) \neq 0$, and let $g = h \circ f \circ h^{-1}$. Then 0 is still an isolated fixed point of both g and g^k,*

$$\mu_{f^k}(0) = \mu_{g^k}(0) \quad \text{and} \quad P_k(f, 0) = P_k(g, 0).$$

Therefore,

$$\mathcal{O}_k(f, 0) = \mathcal{O}_k(g, 0).$$

PROOF. The first equality is well known. The second and the last follow from the first equality and the definition of Dold's indices. □

Lemma 2.12 ([14]). *Let h_1 and h_2 be germs in $\mathcal{O}(\mathbb{C}^n, 0, 0)$. If 0 is an isolated zero of both h_1 and h_2, then the zero order of $h_1 \circ h_2$ at 0 equals the product of the zero orders of h_1 and h_2 at 0, say, $\pi_{h_1 \circ h_2}(0) = \pi_{h_1}(0)\pi_{h_2}(0)$.*

3. Normal forms and iteration formulae

The following lemma is known as a basic result in the theory of normal forms (see [1, p. 84–85]).

Lemma 3.1. *Let $f \in \mathcal{O}(\mathbb{C}^n, 0, 0)$ and assume that $Df(0) = (\lambda_1, \dots, \lambda_n)$ is a diagonal matrix. Then for any positive integer r, there exists a biholomorphic coordinate transform in the form of*

$$(y_1, \dots, y_n) = H(x_1, \dots, x_n) = (x_1, \dots, x_n) + \textit{higher terms}$$

in a neighbourhood of the origin such that each component g_j of $g = (g_1, \dots, g_n) = H^{-1} \circ f \circ H$ has a power series expansion

$$g_j(x_1, \dots, x_n) = \lambda_j x_j + \sum_{i_1 + \dots + i_n = 2}^{r} c^j_{i_1 \dots i_n} x_1^{i_1} \dots x_n^{i_n} + \textit{higher terms, } j = 1, \dots, n,$$

in a neighbourhood of the origin, where the sum extends over all n-tuples $(i_1, \dots, i_n)$ of nonnegative integers with

$$2 \le i_1 + \dots + i_n \le r \quad \textit{and} \quad \lambda_j = \lambda_1^{i_1} \dots \lambda_n^{i_n}.$$

Corollary 3.2. *Let $f \in \mathcal{O}(\mathbb{C}^n, 0, 0)$. If all eigenvalues $\lambda_1, \dots, \lambda_n$ of $Df(0)$ are primitive m_1-th,$\dots, m_n$-th roots of unity, respectively, and $m_1, \dots, m_n$ are distinct primes, then for any positive integer N, there exists a biholomorphic coordinate transform in the form of*

$$(y_1, \dots, y_n) = h(x_1, \dots, x_n)$$

in a neighbouhood of the origin such that $g = h^{-1} \circ f \circ h = (g_1, \dots, g_n)$ has the form

$$g_j(x_1, \dots, x_n) = \lambda_j x_j + x_j h_j(x_1^{m_1}, \dots, x_n^{m_n}) + o(|x|^N), \quad j = 1, \dots, n,$$

where each h_j is a polynomial in $x_1^{m_1}, \dots, x_n^{m_n}$ without constant term.

This corollary follows from the previous lemma. See [21] for a simple proof.

Lemma 3.3 ([21]). *Let $f \in \mathcal{O}(\mathbb{C}^n, 0, 0)$ be a holomorphic mapping given by*

$$f_j(x_1, \dots, x_n) = \lambda_j x_j + x_j \sum_{i=1}^{n} a_{ji} x_i^{m_i} + x_j p_j + o(|x|^{m_1 \cdots m_n}), j = 1, \dots, n,$$

where $\lambda_1, \dots, \lambda_n$ are primitive m_1-th,$\dots, m_n$-th roots of unity, respectively, $m_1, \dots, m_n$ are mutually distinct primes and each p_j is a polynomial in $x_1^{m_1}, \dots, x_n^{m_n}$

without constant terms and linear terms. Then the k-th iteration $f^k = (f_1^{(k)}, \dots, f_n^{(k)})$ *of* f *is given by*

$$f_j^{(k)}(x_1, \dots, x_n) = \lambda_j^k x_j + k\lambda_j^{k-1} x_j \sum_{i=1}^{s} a_{ji} x_i^{m_i} + x_j p_j^{(k)} + o(|x|^{m_1 \dots m_n}),$$

for $1 \le j \le n$, *where each* $p_j^{(k)}$ *is a polynomial in* $x_1^{m_1}, \dots, x_n^{m_n}$ *without constant terms and linear terms.*

Recall that the condition about p_j and $p_j^{(k)}$ means that p_j and $p_j^{(k)}$ have power series expansions of the form (1.2).

4. Cronin Theorem and a consequence

THEOREM 4.1 (Cronin, [4]). *Let* $P = (P_1, \dots, P_n) \in \mathcal{O}(\mathbb{C}^n, 0, 0)$ *be given by*

$$P_j(x_1, \dots, x_n) = \sum_{k=m_j}^{\infty} P_{jk}(x_1, \cdots, x_n), \quad j = 1, \cdots, n,$$

where each P_{jk} *is a homogeneous polynomial of degree* k *in* $x_1, \dots, x_n$. *If* 0 *is an isolated solution of the system of the* n *equations*

$$P_{jm_j}(x_1, \dots, x_n) = 0, \quad j = 1, \dots, n, \tag{4.1}$$

then 0 *is an isolated zero of the mapping* P *with zero order* $\pi_P(0) = m_1 \dots m_n$. *If* 0 *is an isolated zero of* P *but is not an isolated solution of the system* (4.1), *then*

$$\pi_P(0) > m_1 \dots m_n.$$

To apply this theorem, we first prove a lemma.

LEMMA 4.2. *Let* M *and* k_j *be positive integers,* $j = 1, \dots, n$. *Then* $0 = (0, \dots, 0)$ *is an isolated solution of the system*

$$z_j^{k_j} \sum_{i=1}^{n} a_{ji} z_i^M = 0, \quad 1 \le j \le n, \tag{4.2}$$

if and only if all principal submatrices of (a_{ij}) *are nonsingular* (1), *where each* a_{ji} *is a complex number and* $z_1, \dots, z_n$ *are unknowns.*

PROOF. It is clear that 0 is not an isolated solution of (4.2) if and only if there exist a positive integer $k \le n$ and a permutation $\{i_1, \dots, i_n\}$ of $\{1, \dots, n\}$ such that 0 is not an isolated solution of the system

$$\begin{aligned} a_{i_l 1} z_1^M + \dots + a_{i_l n} z_n^M &= 0, \quad l = 1, \dots, k, \\ z_{i_l} &= 0, \quad l = k+1, \dots, n, \end{aligned}$$

(1) For the $n \times n$ matrix $A = (a_{ij})$, a $k \times k$ matrix $B = (b_{\rm st}) = (a_{i_s j_t})$ is called a principal submatrix of A if it is obtained from A by deleting some $n - k$ rows and deleting the same columns of A, say, $i_1 = j_1, \dots, i_k = j_k$.

which is equivalent to the system

$$\begin{aligned} a_{i_l i_1} z_{i_1}^M + \ldots + a_{i_l i_k} z_{i_k}^M &= 0, \quad l = 1, \ldots, k, \\ z_{i_l} &= 0, \quad l = k+1, \ldots, n. \end{aligned}$$

It is also clear that the previous system has no isolated solution at 0 if and only if the system

$$a_{i_l i_1} z_{i_1}^M + \ldots + a_{i_l i_k} z_{i_k}^M = 0, \quad l = 1, \ldots, k,$$

has no isolated solution at 0, which is equivalent to the condition that the principal submatrix

$$\begin{pmatrix} a_{i_1 i_1} & \ldots & a_{i_1 i_k} \\ \vdots & \ddots & \vdots \\ a_{i_k i_1} & \ldots & a_{i_k i_k} \end{pmatrix}$$

of (a_{ji}) is singular. This completes the proof. $\square$

We now apply Cronin theorem to a special case.

PROPOSITION 4.3. *Let $m_1, \ldots, m_n$ be positive integers and let*

$$P(x_1, \ldots, x_n) = \begin{pmatrix} x_1(a_{11} x_1^{m_1} + \ldots + a_{1n} x_n^{m_n} + p_1) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n(a_{n1} x_1^{m_1} + \ldots + a_{nn} x_n^{m_n} + p_n) \end{pmatrix}^T + o(|x|^{m_1 \ldots m_n})$$

be a holomorphic mapping defined in a neighbourhood the origin of $\mathbb{C}^n$, where each p_j is a polynomial in $x_1^{m_1}, \ldots, x_n^{m_n}$, without constant and linear terms. If the origin is an isolated zero of P, then

$$\pi_P(0) \geq (m_1 + 1) \ldots (m_n + 1),$$

and the equality holds if and only if all principal submatrices of (a_{ij}) are non-singular.

PROOF. Let $H : \mathbb{C}^n \to \mathbb{C}^n$ be the mapping

$$(x_1, \ldots, x_n) = (z_1^{M/m_1}, \ldots, z_n^{M/m_n}),$$

where $M = m_1 \ldots m_n$. Then, in the new coordinates $(z_1, \ldots, z_n)$ the mapping $P \circ H = (g_1, \ldots, g_n)$ has the form

$$g_j(z_1, \ldots, z_n) = z_j^{M/m_j} \sum_{i=1}^n a_{ji} z_i^M + \text{higher terms}, \quad 1 \leq j \leq n,$$

If 0 is an isolated zero of P, then by Cronin theorem, 0 is an isolated zero of $P \circ H$ with zero order

$$\pi_{P \circ H}(0) \geq \prod_{j=1}^n \left(\frac{M}{m_j} + M \right) = M^{n-1} \prod_{j=1}^n (1 + m_j),$$

and by Lemma 4.2 the equality holds if and only if all principal submatrices of (a_{ji}) are nonsingular. On the other hand, the zero order of H at 0 is $\pi_H(0) = M^n/(m_1 \dots m_t) = M^{n-1}$. Thus by Lemma 2.12,

$$\pi_P(0) = \frac{\pi_{P\circ H}(0)}{\pi_H(0)} \geq \prod_{j=1}^{n}(1+m_j),$$

and the equality holds if and only if all principal submatrices of (a_{ji}) are nonsingular. This completes the proof. $\square$

PROPOSITION 4.4. *Let $f \in \mathcal{O}(\mathbb{C}^n, 0, 0)$ be given by*

$$f_j(x_1, \dots, x_n) = \lambda_j x_j + x_j \sum_{i=1}^{n} a_{ji} x_i^{m_i} + x_j p_j + o(|x|^{m_1 \dots m_n}), \quad j = 1, \dots, n,$$

where $\lambda_1, \dots, \lambda_n$ are primitive m_1-th, $\dots$, m_n-th roots of unity, respectively, m_1,$\dots$,m_n are distinct primes and each p_j is a polynomial in $x_1^{m_1}, \dots, x_n^{m_n}$ without constant and linear terms. Then, for any t-tuple $(i_1, \dots, i_t)$ of positive integers with $1 \leq i_1 < \dots < i_t \leq n$, if all the principal submatrices of the principal submatrix $(a_{i_r i_s})_{t\times t}$ are nonsingular, then 0 is an isolated fixed point of $f^{m_{i_1} \cdots m_{i_t}}$ and the following two formulae hold.

$$\mu_{f^{m_{i_1} \cdots m_{i_t}}}(0) = (m_{i_1} + 1) \dots (m_{i_t} + 1),$$
$$P_{m_{i_1} \dots m_{i_t}}(f, 0) = m_{i_1} \dots m_{i_t}.$$

PROOF. This is proved in [21] in a more general version (see Section 3 in [21]). $\square$

5. Proof of the main theorem

We first state a property of matrices.

LEMMA 5.1. *Let A be an $n \times n$ matrix. If $\det A = 0$, then there exists a sequence of matrices A_k converges to A such that, for each k, $\det A_k = 0$ but all $s \times s$ submatrices of A_k with $s < n$ are nonsingular.*

PROOF. This follows from the fact that any square matrix can be arbitrarily approximated by nonsingular matrix and any sufficiently small perturbation does not change the nonsingularity of a nonsingular matrix. We omit the standard argument. $\square$

PROOF OF THEOREM 1.7. Assume that there is a local biholomorphic transform

$$x = h(y) = y + o(|y|),$$

where $y = (y_1, \dots, y_n)$, such that the first resonant matrix $\mathcal{A}_h(f)$ of f at 0 is not invertible, say, the mapping $g = h^{-1} \circ f \circ h = (g_1, \dots, g_n)$ has the form

$$g_j = \lambda_j y_j + y_j \sum_{i=1}^{n} a_{ji} y_i^{m_i} + y_j p_j + o(|y|^{m_1 \dots m_n}), \quad 1 \le j \le n, \tag{5.1}$$

with

$$\det \mathcal{A}_h(f) = \det(a_{ji}) = 0, \tag{5.2}$$

where each p_j is a polynomial in $y_1^{m_1}, \dots, y_n^{m_n}$, without constant and linear terms.

By (5.2) and Lemma 5.1, there exists a sequence $A_k = (a_{k,ji})$ of $n \times n$ matrices converging to A, as $k \to \infty$, such that for each k, all proper square submatrices of A_k are invertible but $\det A_k = 0$. We then consider the holomorphic mapping $\widetilde{g}_k = (g_{k1}, \dots, g_{kn})$ given by

$$g_{kj} = \lambda_j y_j + y_j \sum_{i=1}^{n} a_{k,ji} y_i^{m_i} + y_j p_j + o(|y|^{m_1 \dots m_n}), \quad 1 \le j \le n,$$

which is obtained from $g = (g_1, \dots, g_n)$ by just replacing the numbers a_{ji} in (5.1) by the numbers $a_{k,ji}$. Then the mappings $\widetilde{g}_k$ converges to g uniformly in a neighbourhood of the origin and by Lemma 3.3, for $M = m_1 \dots m_n$, the M-th iteration $\widetilde{g}_k^M = (g_{k1}^{(M)}, \dots, g_{kn}^{(M)})$ of $\widetilde{g}_k$ has the form

$$g_{kj}^{(M)} = y_j + M \lambda_j^{M-1} y_j \sum_{i=1}^{s} a_{k,ji} y_i^{m_i} + p_j^{(M)} + o(|y|^{m_1 \dots m_n}), \quad 1 \le j \le n, \tag{5.3}$$

where each $p_j^{(M)}$ is a polynomial in $y_1^{m_1}, \dots, y_n^{m_n}$, without constant and linear terms. By (5.3) and Proposition 4.3, we have

$$\mu_{\widetilde{g}_k^M}(0) = \pi_{\widetilde{g}_k^M - \mathrm{id}}(0) > (m_1 + 1) \dots (m_n + 1). \tag{5.4}$$

On the other hand, by Proposition 4.4, for any t-tuple $(i_1, \dots, i_t)$ with $1 \le i_1 < \dots < i_t \le n, t < n$, we have

$$P_{m_{i_1} \dots m_{i_t}}(\widetilde{g}_k, 0) = m_{i_1} \dots m_{i_t}. \tag{5.5}$$

But by Lemmas 2.8 and 2.10,

$$\mu_{\widetilde{g}_k^M}(0) = \sum_{m | m_1 \dots m_n} P_m(\widetilde{g}_k, 0),$$

and then, by (5.4) and the hypothesis that $m_1, \dots, m_n$ are distinct primes, we have

$$\begin{aligned} \mu_{\widetilde{g}_k^M}(0) = P_{m_1 \dots m_n}(\widetilde{g}_k, 0) + \sum_{\substack{1 \le i_1 < \dots < i_t \le n \\ 1 \le t < n}} P_{m_{i_1} \dots m_{i_t}}(\widetilde{g}_k, 0) + P_1(\widetilde{g}_k, 0) \\ > (m_1 + 1) \dots (m_n + 1), \end{aligned}$$

and then, considering that $P_1(\widetilde{g}_k, 0) = \mu_{\widetilde{g}_k}(0) = 1$, by (5.5) we have

$$P_{m_1 \dots m_n}(\widetilde{g}_k, 0) > (m_1 + 1) \dots (m_n + 1) - \sum_{\substack{1 \le i_1 < \dots < i_t \le n \\ 1 \le t < n}} P_{m_{i_1} \dots m_{i_t}}(\widetilde{g}_k, 0) + 1$$
$$= (m_1 + 1) \dots (m_n + 1) - \sum_{\substack{1 \le i_1 < \dots < i_t \le n \\ 1 \le t < n}} m_{i_1} \dots m_{i_t} + 1 = m_1 \dots m_n.$$

Thus, $\mathcal{O}_M(\widetilde{g}_k, 0) > 1$. But $\mathcal{O}_M(\widetilde{g}_k, 0)$ is an integer, we have

$$\mathcal{O}_M(\widetilde{g}_k, 0) \ge 2.$$

and thus, by Lemma 2.7,

$$\mathcal{O}_M(g, 0) \ge 2,$$

and then by Lemma 2.11 we have

$$\mathcal{O}_M(f, 0) \ge 2. \qquad \square$$

6. An open problem

PROBLEM 6.1. Is the sufficient condition in Theorem 1.7 necessary? In other words, does (1.3) imply (1.4)?

If the answer is affirmative, then one can easily see that the nonsingularity of the first resonant matrix $\mathcal{A}_h(f)$ is independent of the transform h.

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