EQUIVARIANT PATH FIELDS
ON TOPOLOGICAL MANIFOLDS

Lucília Borsari — Fernanda Cardona — Peter Wong

ABSTRACT. A classical theorem of H. Hopf asserts that a closed connected smooth manifold admits a nowhere vanishing vector field if and only if its Euler characteristic is zero. R. Brown generalized Hopf's result to topological manifolds, replacing vector fields with path fields. In this note, we give an equivariant analog of Brown's theorem for locally smooth G-manifolds where G is a finite group.

1. Introduction

Let M be a closed connected orientable smooth manifold. A classical theorem of H. Hopf [13] states that M admits a non-singular vector field if and only if the Euler characteristic, $\chi(M)$, of M is zero. R. Brown [7] gave a generalization of Hopf's theorem for topological manifolds, by replacing vector fields with path fields, a concept first introduced by J. Nash [22]. In [7], R. Brown showed that a compact topological manifold M admits a non-singular path field if and only if $\chi(M) = 0$. Subsequently, R. Brown and E. Fadell [8] extended [7] to topological manifolds with boundary. It was shown by E. Fadell [10] that any Wecken complex of zero Euler characteristic admits a non-singular simple path field. R. Stern [24] showed the same result for topological manifolds of dimension different from four.

The existence of a path field allows one to show the so-called *Complete Invariance Property* (CIP) (see [17] and [23]). Recall that a topological space M is said to have the CIP if for any non-empty closed subset $A \subset M$, there exists a map $f : M \to M$ such that

$$A = \operatorname{Fix} f := \{x \in M \mid f(x) = x\}.$$

Similarly, M possesses the CIP with respect to deformation (denoted by CIPD) if f is homotopic to the identity 1_M. The non-singular path field problem is equivalent to the fixed point free deformation problem. That is, M admits a non-singular path field if and only if 1_M is homotopic to a fixed point free map.

In [18], [19], and [25], equivariant vector fields on compact smooth G-manifolds were studied. In particular, an equivariant analog of Hopf's theorem was proved in [18]. Furthermore, an equivariant analog of what was done for path fields on Wecken complexes in [10], was given in [26] and necessary and sufficient conditions for equivariant CIPD were given for smooth G-manifolds (see also [3] for a certain type of equivariant CIP). Similar to the non-equivariant case, the equivariant non-singular path field problem is closely related to finding an equivariant fixed point free deformation. It turns out that the existence of such a fixed point free map requires more than merely the existence of non-equivariant fixed point free deformation on the fixed point sets M^H for each isotropy type (H) (see [11]).

The objective of this paper is to prove an equivariant analog of Brown's theorem [7] for topological manifolds with locally smooth action of a finite group G. Moreover, we extend the necessary and sufficient conditions for G-CIPD found in [27] to this category of G-manifolds.

We would like to thank D. L. Gonçalves and G. Peschke for very helpful conversations and suggestions.

Throughout G will always be a finite group acting on a compact manifold M where the action is locally smooth. For the definition and basic properties of locally smooth actions, we refer the reader to [5].

2. Equivariant Euler characteristic and G-path fields

In this section, we establish the necessary definitions of path fields and Euler characteristic in the equivariant category.

Equivariant path fields were defined and studied in [26] and [27]. For our purposes, we think of G-path fields as sections of certain G-fibrations.

First, given a G-map $p : E \to B$, we say that p has the *G-Covering Homotopy Property* (G-CHP) if for all G-space X the following commutative diagram has

a solution $F \colon X \times [0,1] \to E$ where all maps are G-equivariant.

$$\begin{CD} X \times \{0\} @>f>> E \\ @V\text{incl.}VV @VVpV \\ X \times [0,1] @>>H> B \end{CD}$$

A G-*fibration* is simply a G-map $p \colon E \to B$ satisfying the G-CHP for all G-spaces.

Given a G-fibration $p \colon E \to B$, we consider $\Omega_p = \{(e,w) \in E \times B^I \mid p(e) = w(0)\}$. Then Ω_p is a G-invariant subspace of $E \times B^I$. Let $\widetilde{p} \colon E^I \to \Omega_p$ be the G-map defined by $\widetilde{p}(\tau) = (\tau(0), p(\tau))$. Consider the equivariant maps $F \colon \Omega_p \times I \to B$ defined by $F(e,w,t) = w(t)$, and $f \colon \Omega_p \to E$ by $f(e,w) = e$. Since p is a G-fibration, F can be lifted to a G-map $\widetilde{F} \colon \Omega_p \times I \to B$ which extends f. Then $\lambda \colon \Omega_p \to E^I$, defined by $\lambda(e,w)(t) = \widetilde{F}(e,w,t)$, is an equivariant lifting function for p, that is, $\widetilde{p} \circ \lambda$ is the identity on Ω_p.

A G-fibration is called regular if it admits a regular G-lifting function, meaning, a G-lifting function satisfying $\lambda(e, \overline{p(e)}) = e$, for all $e \in E$, where $\overline{p(e)}$ denotes the constant path at $p(e)$. In [14], W. Hurewicz shows that every fibration over a metric space is regular. The same proof can be adapted to the equivariant case, provided the metric d is assumed to be G-invariant, that is, $d(gx,gy) = d(x,y)$, for all $g \in G$ and $x,y \in B$.

LEMMA 2.1. *Let $p \colon E \to B$ be a regular G-fibration over a G-manifold B. Let (X,A) be a G-ANR pair and suppose that there are equivariant maps $f \colon X \times 0 \cup A \times I \to E$ and $h \colon X \times I \to M$ such that $p \circ f = h|_{(X \times 0 \cup A \times I)}$. Then, there exists a G-map $\widetilde{f} \colon X \times I \to E$ which extends f and such that $p \circ \widetilde{f} = h$.*

PROOF. This lemma is an equivariant version of Theorem 2.4 of [1]. The proof of this theorem in the non equivariant context is very constructive and it is possible to verify that, in all steps, we do obtain equivariant maps, as long as we start with the appropriate equivariant setting and make use of Corollary 2.3 of [25]. $\square$

Given a compact topological manifold M, the *Nash path space* T_M of M consists of $\overline{T_M} = \{\text{all constant paths}\}$ and the set T_M^0 of all paths α on M such that for $0 \le t \le 1$, $\alpha(t) = \alpha(0)$ if and only if $t = 0$. Consider the map $q \colon T_M \to M$ given by $q(\alpha) = \alpha(0)$. With the compact-open topology on T_M, the triple (T_M, q_M, M) is a Hurewicz fibration and the sections of q are called *path fields* on M. A path field is *non-singular* if it is a section in $(T_M^0, q_M|_{T_M^0}, M)$. A path field σ is *simple* if for any $x \in M$, $\sigma(x)$ is a simple path.

If G acts on M, then G acts on T_M via $g * \alpha(t) = g\alpha(t)$. Since $q \colon T_M \to M$ is a fibration, it is straightforward to see that it is indeed a G-fibration where the G-action on $[0,1]$ is trivial. Thus, we define a G-*path field* to be a G-section

$s\colon M \to T_M$ of q so that $q \circ s = 1_M$. Moreover, the subfibration $q_M^0\colon T_M^0 \to M$ is also a G-subfibration. The notions of non-singular and of simple G-path fields are defined in the obvious fashion.

Given a compact topological manifold M, the classical Euler characteristic of M is an integer and it coincides with the fixed point index of the identity map 1_M. When a finite group G acts on M, the appropriate equivariant Euler characteristic takes the components of the various fixed point sets M_H, $H \leq G$, into account.

We write $|\chi|(M_H) = \sum_C |\chi(C)|$, where C ranges over the connected components of $M_H = \{x \in M \mid G_x = H\}$. Here, G_x denotes the isotropy subgroup of x. Since M is compact, each $M^H = \{x \in M \mid hx = x, \text{for all } h \in M\}$ is also compact so that M_H has only a finite number of components.

3. Singularities of G-path fields

In this section, we prove our main results following the approach of [7]. Since we work in the G-manifolds category, many of the techniques employed in [7] must be modified for the equivariant setting, first of which is the following relative equivariant domination theorem for compact G-ANRs.

THEOREM 3.1 (Relative Equivariant Domination Theorem). *Let M be an n-dimensional G-manifold and A be an invariant compact submanifold of dimension k. We can find a G-complex K of dimension n, an invariant subcomplex of dimension k and equivariant maps $\varphi\colon K \to M$ and $\psi\colon M \to K$, so that ψ is barycentric, $\varphi|_L\colon L \to A$, $\psi|_A\colon A \to L$, $\varphi \circ \psi \cong_G \mathrm{id}_M$ and $\varphi|_L \circ \psi|_A \cong_G \mathrm{id}_A$*

PROOF. According to [2, Theorem 1], we can equivariantly embed M as a closed G-neighbourhood retract of a convex G-set in a Banach G-space $A(M)$ in which G acts isometrically. Now we follow the proof of the G-domination theorem (Proposition 2.3) of [20]. Let $r\colon \mathcal{O} \to M$ be the G-retraction of some G-invariant neighbourhood $\mathcal{O}$. It suffices to show that $\mathcal{O}$ can be G-dominated by a finite G-complex K. Let $\{W_\alpha\}$ be a covering of $\mathcal{O}$ by convex subsets which are open in $\mathcal{O}$. Since M is compact, we can find a finite open covering $\{\mathcal{O}_\alpha\}$ of $\mathcal{O}$ such that the convex hull of a finite union of the $\mathcal{O}_\beta$ is contained in M. Then there is a finite open pointed G-covering $\mathcal{V} = \{V_\gamma, v_\gamma\}$, a refinement of $\{\mathcal{O}_\beta\}$ such that $v_\gamma \in A$ if $V_\gamma \cap A \neq \emptyset$. Let $K = |N(\mathcal{V})|$ be the nerve of $\mathcal{V}$ with the canonical G-action.

For any $x \in M$, we let

$$\nu(x) = \sum_i d(x, M - V_i)$$

where d denotes the metric on M which is G-invariant since G acts isometrically. Let $\{v_i\}$ be the vertices of $|N(\mathcal{V})|$. Now $\{\sum_i d(x, M - V_i)/\nu(x)\}$ is a G-partition

of unity subordinate to $\mathcal{V}$. Define the G-map $\varphi\colon M \to |N(\mathcal{V})|$ by

$$\varphi(x) = \sum_i \frac{d(x, M - V_i)}{\nu(x)}\, v_i.$$

Note that φ is a barycentric mapping. Consider the G-map

$$\psi = r \circ \eta\colon |N(\mathcal{V})| \to M,$$

where η is the map ψ as in the proof of Proposition 2.3 of [20]. It is straightforward to check that the G-maps φ and ψ yield the desired G-domination. Note that K is of dimension less than or equal to n since the $\mathcal{V}$ is a refinement. It follows that K must be of dimension n otherwise K has no homology in dimension n whereas $\dim M = n$ and M is a compact manifold of dimension n. Finally, we let $\mathcal{V}_A = \{(V_i \cap A, v_i)\}$. It follows that $\mathcal{V}_A$ is a G-covering of A and the nerve $L = |N(\mathcal{V}_A)|$ is a subcomplex of K. Since A is a compact manifold of dimension k, we conclude that L is of dimension k and that L equivariantly dominates A. $\qquad\square$

REMARK 3.2. It has been noted by S. Antonyan in [2] that the equivariant embedding theorem [21, Theorem 6.2] of M. Murayama is incorrect: in that the Banach space $B(M)$ of all bounded continuous functions on M used in [21] is not a Banach G-space and the G-action defined there is not continuous. Likewise, the same mistake was also committed by S. Kwasik in [20]. Nevertheless, the G-domination theorem in both [20] and [21] is stated correctly and their proofs are valid provided one replaces $B(M)$ with the linear subspace $A(M)$ of all G-uniform functions as in [2]. We thank M. Golasiński for bringing [2] to our attention. As noted by Hanner in [12], in non-equivariant settings Borsuk showed in [4] that any compact ANR is dominated by a finite polyhedron. Then, in [6], Brooks showed, again in the non-equivariant setting, that if an n-dimensional compact ANR is dominated by a complex then it is dominated by its n-dimensional skeleton.

In order to prove the next proposition, we will need the following non-equivariant result.

LEMMA 3.3. *Let M be a n-manifold and $A \subset M$ a submanifold. Consider c an n-cell in $M - A$, with $\partial c \subset A$ and let $\sigma\colon \partial c \to T_A$ be a path field and $\mathbf{o}$ be a point in the interior of c. Then, there exists a path field $\sigma'\colon c \to T_M$, extending σ with $\mathbf{o}$ being its only singularity in the interior of c. Moreover, in case σ has singularities then we may take σ' without singularities in the interior of c.*

PROOF. Let c' be an n-cell contained in $\mathrm{Int}\, c$, with $\mathbf{o}$ in its interior. We will extend σ to $c - \mathrm{Int}\, c'$ without creating new singularities: Let $[\mathbf{o}, b_x]$ be the oriented segment through x, beginning at $\mathbf{o}$, ending at $b_x \in \partial c$. Therefore we could write any $x \in c - \mathrm{Int}\, c'$ as $x = (1 - t_x)\mathbf{o} + t_x b_x$, where $t_x \in\ ]0, 1]$.
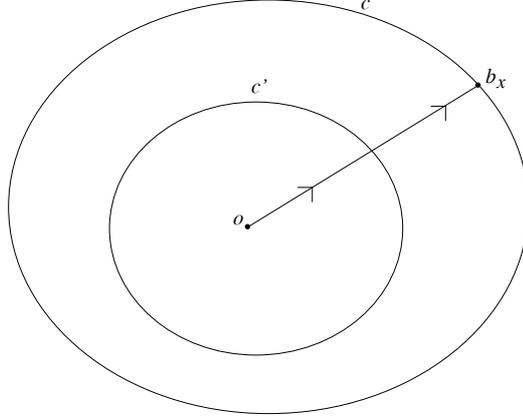
Figure 1

The extended path field σ' will be defined for each $x \in c - \operatorname{Int} c'$ as follows:

$$\sigma'(x)(s) = \begin{cases} (1 - t_x - s)\mathbf{o} + (t_x + s)\, b_x & \text{if } 0 \le s \le 1 - t_x, \\[2mm] \sigma(b_x)\left(\dfrac{s + t_x - 1}{t_x}\right) & \text{if } 1 - t_x \le s \le 1. \end{cases}$$

Observe that it is well defined because $t_x > 0$, for any $x \in c - \operatorname{Int} c'$. Also, in the first equation, for any t_x when $s = 0$ we have $(1 - t_x)\mathbf{o} + t_x b_x = x$; when $s = 1 - t_x$ we have $(t_x + 1 - t_x)\, b_x = b_x$. In the second equation, when $s = 1 - t_x$ we have $\sigma(b_x)((1 - t_x + t_x - 1)/t_x) = \sigma(b_x)(0) = b_x$; when $s = 1$ we have $\sigma(b_x)((1 + t_x - 1)/t_x) = \sigma(b_x)(1)$. Therefore σ' has no other singularities than those of σ (if it has any, they will be in the boundary of c), so σ' has no singularities in the boundary of c'. By Lemma 1.5 of [7] it can be extended to $\operatorname{Int} c'$ having only $\mathbf{o}$ as a singularity in $\operatorname{Int} c'$.

By an abuse of notation we will denote this extension of σ' to $\operatorname{Int} c'$ also by σ'. Therefore, we constructed an extension of σ, $\sigma' \colon c \to T_M$, which has only one singularity in $\operatorname{Int} c$ and in the boundary only the singularities that σ had.

If σ does have singularities in the boundary of the cell, we will eliminate the singularity of σ' in its interior:

Let $y \in \partial c$ be a singularity of σ (therefore a singularity of σ'); let $c_1 \subset c_2 \subset c$ be two cells such that $\partial c_1 \cap \partial c_2 = \{y\}$, $\partial c_i \cap \partial c = \{y\}$, for $i = 1, 2$ and $\mathbf{o} \in \operatorname{Int} c_1$ (and therefore $\mathbf{o} \in \operatorname{Int} c_2$).

Let $[b_x, y]$ be the oriented segment through x, beginning at $b_x \in \partial c_2$, ending at the singularity $y \in \partial c$. Therefore we could write any $x \in c_2$ as $x = (1 - t_x)b_x + t_x y$, where $t_x \in [0, 1]$. Also, each of these segments would determine a point $a_x \in \partial c_1$ such that $a_x = (1 - \overline{t}_x)b_x + \overline{t}_x y$, with $\overline{t}_x \in\,]0, 1[$.

The new path field $\overline{\sigma}$ will be defined in each one of the regions represented below, as follows:
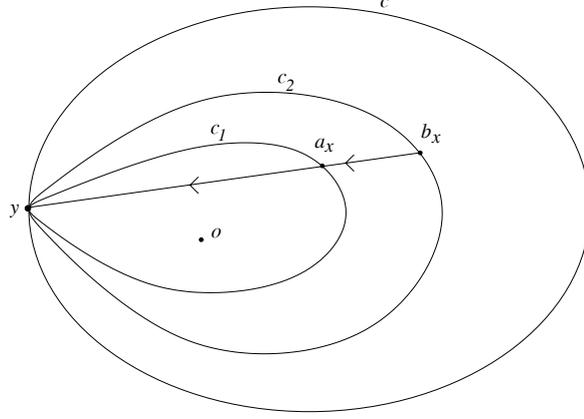
Figure 2

(a) For $x \in \text{Int } c_1$, we have $0 < t_x < 1$ and

$$\overline{\sigma}(x)(s) = \begin{cases} (1 - t_x - s)b_x + (t_x + s)y & \text{if } 0 \leq s \leq 1 - t_x, \\ \sigma'(y)\left(\dfrac{s + t_x - 1}{t_x}\right) & \text{if } 1 - t_x \leq s \leq 1. \end{cases}$$

Observe that it is well defined because $t_x > 0$, for any $x \in \text{Int } c_1$. Also, in the first equation, for any t_x when $s = 0$ we have $(1 - t_x)b_x + t_x y = x$; when $s = 1 - t_x$ we have $(t_x + 1 - t_x)\, y = y$. In the second equation, when $s = 1 - t_x$ we have $\sigma'(y)((1 - t_x + t_x - 1)/t_x) = \sigma'(y)(0) = y$.

(b) For $x \in c_2 - \text{Int } c_1$, we have $0 \leq t_x \leq \overline{t}_x$ and

$$\overline{\sigma}(x)(s) = \begin{cases} (1 - t_x - s)b_x + (t_x + s)y & \text{if } 0 \leq s \leq \dfrac{t_x}{\overline{t}_x} - t_x, \\ \sigma'\left(\left(1 - \dfrac{t_x}{\overline{t}_x}\right)b_x + \dfrac{t_x}{\overline{t}_x}y\right)\left(\dfrac{s - (t_x/\overline{t}_x - t_x)}{1 - (t_x/\overline{t}_x - t_x)}\right) & \text{if } \dfrac{t_x}{\overline{t}_x} - t_x \leq s \leq 1. \end{cases}$$

Observe that it is well defined because $\overline{t}_x > 0$, for any $x \in c_2$. Also, in the first equation, for any t_x when $s = 0$ we have $(1 - t_x)b_x + t_x y = x$; when $s = t_x/\overline{t}_x - t_x$ we have

$$\left(1 - t_x - \left(\dfrac{t_x}{\overline{t}_x} - t_x\right)\right)b_x + \left(t_x + \dfrac{t_x}{\overline{t}_x} - t_x\right)y = \left(1 - \dfrac{t_x}{\overline{t}_x}\right)b_x + \dfrac{t_x}{\overline{t}_x}\, y.$$

In the second equation, when $s = t_x/\overline{t}_x - t_x$ we have

$$\sigma'\left(\left(1 - \dfrac{t_x}{\overline{t}_x}\right)b_x + \dfrac{t_x}{\overline{t}_x}y\right)(0) = \left(1 - \dfrac{t_x}{\overline{t}_x}\right)b_x + \dfrac{t_x}{\overline{t}_x}y.$$

(c) For $x \in c - \text{Int } c_2$, $\overline{\sigma}(x) = \sigma'(x)$.

Notice that if $x \in \partial c_1$ then $t_x = \overline{t}_x$ and if $x \in \partial c_2$ then $t_x = 0$ and therefore $\overline{\sigma}$ is well-defined and continuous in ∂c_1 and ∂c_2, the boundaries of c_1 and c_2.

A simple verification will show that $\overline{\sigma}$ has no singularities in Int c and the fact that σ is a path field in A guarantees that $\overline{\sigma}$ is in fact a path field. $\qquad\square$

PROPOSITION 3.4. *Let M be a locally smooth G-manifold, $\dim M = n$, $A \subset M$ an invariant submanifold so that G acts freely on $M - A$. Given an equivariant section $\sigma_A \colon A \to T_A$, with a finite number of singular orbits, it is possible to extend σ_A to an equivariant section $\sigma \colon M \to T_M$ in such a way the closure of each component of $M - A$ intersects at most one singular orbit of σ.*

PROOF. Consider the following diagram:

Here, K and L are as in Theorem 3.1 and $q_L \colon (\varphi|_L)^*(T_L) \to L$ and $q_K \colon \varphi^*(T_M) \to K$ denote the pullbacks of q_A and q_M by φ, respectively.

Now, starting with σ_A, a G-section of $q_A \colon T_A \to A$, we define a G-section $\sigma_L \colon L \to (\varphi|_L)^*(T_A)$, by $\sigma_L(y) = (y, \sigma_A(\varphi(y)))$. A similar procedure as the one indicated in Lemma 1.6 in [7] can be used to extend σ_L to a G-section σ_K, having only a finite number of singular orbits in $K - L$. In order to extend σ_L to an m-simplex δ of $K - L$, we use Lemma 3.3 and extend it to $g\delta$ in the usual equivariant way. Define $\sigma'_M \colon M \to T_M$ by $\sigma'_M(x)\varphi(\sigma_K(\psi(x))$. Then σ'_M is an equivariant map, but it is not a section. In fact, it is a homotopy section since $q_M(\sigma_M(x)) = \varphi \circ \psi(x)$. Consider $h \colon M \times I \to M$, the G-homotopy between $\varphi \circ \psi$ and the identity on M, and $f \colon A \times I \cup M \times 0 \to T_M$ given by the G-homotopy between σ_A and $(\sigma'_M)|_A$ on $A \times I$ and by σ'_M in $M \times 0$. Since M admits an invariant metric, we may apply Lemma 2.1 to obtain an equivariant lifting f' of h extending f. Define $\sigma_M \colon M \to T_M$ by $\sigma_M(x) = f'(x, 1)$. Then σ_M is a G-section on M extending σ_A.

The first step is to change σ_M to reduce the singular set in $M - A$ to a finite one. In order to do this, consider first $\{Gx_1, \ldots, Gx_r\}$ the set of singular orbits of σ_K in $K - L$. The set of singular orbits of σ_M, which are not in A, lies in the pre-image of $\{Gx_1, \ldots, Gx_r\}$ under ψ. Since ψ is equivariant, this set is $\{G\psi^{-1}(x_1), \ldots, G\psi^{-1}(x_r)\}$. Since G acts freely in $M - A$, for each i, the sets $g\psi^{-1}(x_i)$, g in G, are disjoint. Following the proof of Theorem 1.10 of [7], we can
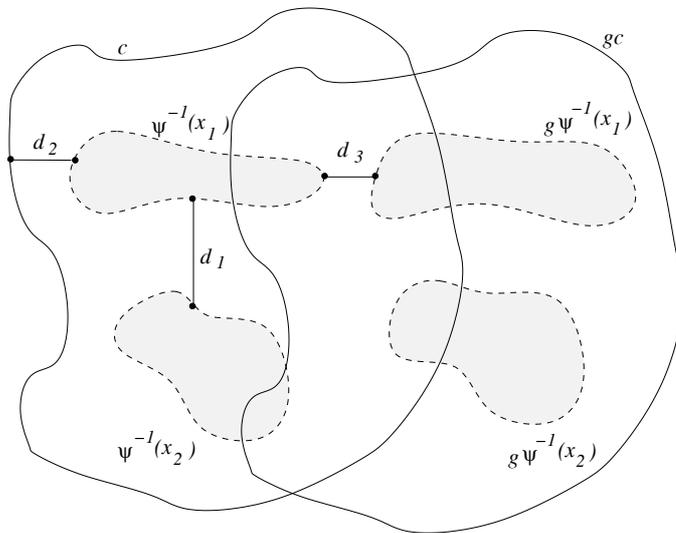
Figure 3

assume that for a connected component C of $M - A$, $\psi^{-1}(x_1) \cap C$ is contained in the interior of c, a closed topological n-cell (see the Figure 3).

Now, consider in M an invariant metric and let d_1 be the distance between $\psi^{-1}(x_1)$ and $\bigcup_{i \neq 1} G\psi^{-1}(x_i)$. Let d_2 be the distance between $\psi^{-1}(x_1)$ and the boundary of c and d_3 the distance between $\psi^{-1}(x_1)$ and $(G\psi^{-1}(x_1) - \psi^{-1}(x_1))$ $\cap C$. Finally, take d to be the minimum of $\{d_1, d_2, d_3\}$. If we consider a finite triangulation of c with mesh size less than $d/3$, then no closed simplex of c intersecting $\psi^{-1}(x_1)$ intersects a simplex which touches

$$\left[\bigcup_i (G\psi^{-1}(x_i) - \psi^{-1}(x_1)) \bigcup \partial c \right] \cap C = R_C.$$

Let P_C be the subpolyhedron of c consisting of simplices which do not intersect R_C and let Q_C be the subpolyhedron of P_C consisting of those simplices which do not intersect $\psi^{-1}(x_1)$. Then $\sigma_M|_{Q_C}$ has no singularities and again, by the same procedure used in Lemma 1.6 in [7], we may extend it to P_C with a finite number of singularities, say, $\{y_1, \dots, y_m\}$. Since the metric on M is invariant, we may triangulate $g_i c$ in the same way we triangulate c so that the complexes $g_i P_C$ and $g_i Q_C$ will be the complexes corresponding to P_C and Q_C for $g_i \psi^{-1}(x_1)$ in $g_i Q_C$. By doing so, the singularities of the extension of $\sigma_M|_{g_i Q_C}$ will be $\{g_i y_1, \dots, g_i y_m\}$. Finally, we extend this section to M by making it agree with σ_M outside $G_C P_C$, where $G_C = \{g \in G \mid gC = C\}$. Repeating this procedure for $i > 1$, we end up with an equivariant section extending σ_A with a finite number of singular orbits, $\{Gy_1, \dots, Gy_m\}$, lying in the closure of various components of $M - A$.
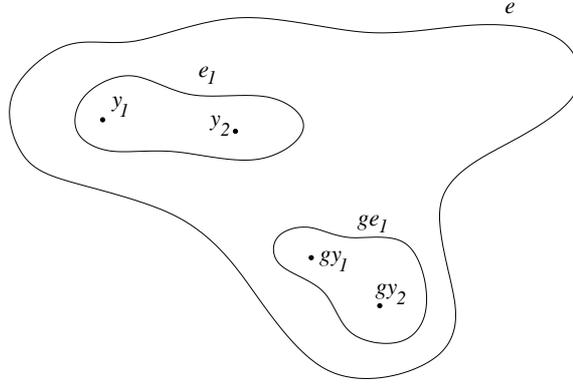
The next step is to reduce the set of singularities in such a way the closure of each component of $M - A$ meet at most one singular orbit. For this, let C be a component of $M - A$ and $G_C y_1, \dots, G_C y_r$ be all singularities in C. Consider e a closed n-cell in C containing this entire set of singularities in its interior. Let e_1 be another closed n-cell contained in the interior of e such that $y_1, \dots, y_r$ are in e_1 and $g_l e_1 \cap g_j e_1 = \emptyset$, for g_l and g_j in G_C, $l \neq j$, as in the Figure 4.

Applying once more Lemma 3.3 for the n-cell e_1 we may reduce the set $\{y_1, \dots, y_m\}$ to a single singularity, say z. Doing the same for the cells $g_j e_1$, we end up with a cross section $\tau \colon M \to T_M$ with only $G_C z$ as singularities in C. Repeating this procedure for all other components we are able to extend the path field to M in such a way that the closure of each component C of $M - A$ meet at most two singular orbits, one lying in the interior of C and the other in its boundary.
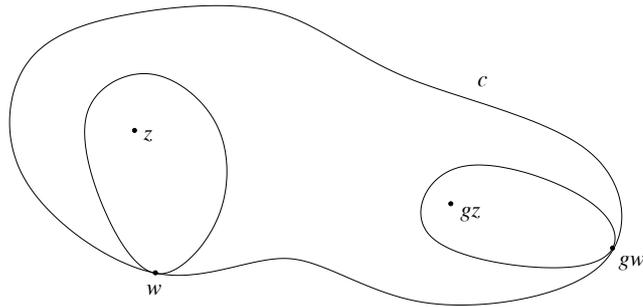
Now, for a component C of $M - A$ let $G_C z \cup G_C w$ be its set of singularities, where $z \in C$ and $w \in (\overline{C}) - C$. Then, it is possible to find $|G_C|$ cells touching

$(\overline{C}) - C$ only in gw, $g \in G_C$ and so that each of them contains only one pair of points of the form gz, gw, $g \in G_C$, as in the Figure 5.

Then applying Lemma 3.3 we see that we may change σ in the interior of the cell containing z and w so that the only singularity left is w. Then repeating the procedure to the other cells, equivariantly, we complete the proof. □

PROPOSITION 3.5. *Let M be a G-manifold, $A \subset M^n$ an invariant submanifold so that G acts freely on $M - A$. Assume that A admits a G-path field without singularities. Then M admits a G-path field with no singular orbits if and only if $|\chi|(M - A) = 0$.*

PROOF. Assume first $|\chi|(M - A) = 0$ and let σ be a G-path field on M with a single singular orbit, say Gx, x in a component C of $M - A$. Take $f : M \to M$, $f(x) = \sigma(x)(1)$. Then f has only one fixed orbit in C, namely $G_C x$. Since G is finite and $\chi(C) = 0$ we have that the sum of the fixed point indices of f at gx, g in G_C, must vanish. Since the action of G on M is locally smooth and the fixed points are isolated and lie in the same orbit, it is not hard to see that they have all the same index, and therefore index zero. Because the action is free in $M - A$, we can find an Euclidean neighbourhood, U, of x in C such that $gU \cap hU = \emptyset$, for all g and h in G_C. Applying exactly the same procedure as in the proof of Theorem 2.3 of [7], we conclude that it is possible to construct a path field σ' over M so that it agrees with σ in $M - U$ and has no singularities in U. Define $\tau : M \to T_M$ to agree with σ in $M - \bigcup_{g \in G_C} gU$ and, for $y \in gU$, $\tau(y) = g\sigma'(g^{-1}y)$. It is not difficult to see that τ is a G-path field over M without singular orbits in C. The proof is complete if we repeat the same procedure for all other components of $M - A$.

Now, suppose σ_A has no singularities and can be extended to M. Let $\{C_1, \ldots, C_r\}$ and $\{A_1, \ldots, A_l\}$ be the connected components of $M - A$ and A, respectively. Since, $\sigma|_{A_j} : A_j \to A_j$ has no singularities, we have that $\chi(A_j) = 0$, for all j. Consider D_i the union of the components of A that meet the closure of C_i. Set $\widetilde{C}_i = D_i \cup C_i$ and let U_i be a tubular neighbourhood of D_i in $\widetilde{C}_i$. Then $\chi(D_i) = 0$ and $\chi(U_i - D_i) = 0$, since $U_i - D_i$ fibers over D_i. Now, $\chi(C_i) = \chi(\widetilde{C}_i) - \chi(U_i) + \chi(U_i \cap C_i) = \chi(\widetilde{C}_i) - \chi(D_i) + \chi(U_i - D_i) = \chi(\widetilde{C}_i)$.

Using the compactness of $\widetilde{C}_i$ and the fact that $\sigma(x)$ starts at x, it is possible to find $t_i \in [0, 1]$ so that $\sigma(x)([0, t_i]) \subset \widetilde{C}_i$, for all x in $\widetilde{C}_i$. Therefore the map $f_i : \widetilde{C}_i \to \widetilde{C}_i$ given by $f_i(x) = \sigma(x)(t_i)$ is fixed point free, and this implies that $\chi(\widetilde{C}_i) = 0$. So we conclude that $\chi(C_i) = 0$. This completes the proof. □

PROPOSITION 3.6. *Suppose G acts on a manifold M with only one orbit type (H). Then it is possible to construct a G-path field on M with a single singular orbit. Moreover, M admits a G-path field with no singular orbits if and only if $|\chi|(M_H) = 0$.*

Proof. Since NH/H acts freely on M^H then, by Proposition 3.4, it is possible to construct a NH/H-path field $\sigma': M^H \to M^H$ with only one NH/H-singular orbit. Since G acts on M with only one orbit type (H), we have that M is G-homeomorphic to $G \times_{NH} M^H$, where NH is the normalizer of H (see [5]). Define $\sigma: M \to T_M$ by, $\sigma[g,x] = g\sigma'(x)$. It is not hard to see that σ is an equivariant section with a single singular orbit.

Now, assume $|\chi|(M_H) = 0$. Then $|\chi|(M^H) = 0$ and the section σ' can be taken without singularities and so does σ. Finally, if M admits a G-path field with no singular orbits, then $f: M \to M$ given by $f(x) = \sigma(x)(1)$, has no fixed orbits. Therefore $f^H: M^H \to M^H$ has no fixed points, which implies that $|\chi|(M^H) = 0$ and the proof is done.   □

Theorem 3.7. *Let G be a finite group and M a compact locally smooth G-space. Then there exists a G-path field on M having at most one singular orbit in the closure of each component of M_H. Moreover, M admits a non singular G-path field if and only if $|\chi|(M_H) = 0$, for all $H \leq G$.*

Proof. Consider $(H_1), \ldots, (H_r)$ the orbit types of the G-action on M ordered in a way that $(H_i) \subset (H_j)$ implies $j \leq i$. For each $i \in \{1, \ldots, r\}$, let $M_i = \{x \in M \mid (G_x) = (H_j), j \leq i\}$. Then $M_1 \subset \ldots \subset M_r$, $M_1 = M_{(H_1)}$, $M_r = M$ and $M_i - M_{i-1} = M_{(H_i)}$. Here, $M_{(H_j)} = \{x \in M | (G_x) = (H_j)\}$.

To prove the first part we will use induction on r. If $r = 1$, then M_1 has only one orbit type, namely, (H_1). Therefore, Proposition 3.6 implies that M_1 admits a G-path field σ_1 with only one singular orbit.

Suppose we have succeeded extending σ_1 to a G-path field, σ_{i-1}, on M_{i-1} so that the closure of each component of $M_{i-1} - M_{i-2} = M_{H_{i-1}}$ intersects at most one singular orbit. Take $N = M^{H_i} - (M_{i-1} \cap M^{H_i}) = M_{H_i}$. Since NH_i/H_i acts freely on N, Proposition 3.6 implies that we are able to extend $\sigma_{i-1}|_{M_{i-1} \cap M^{H_i}}$ to an NH_i/H_i-path field, $\overline{\sigma}_i: M^{H_i} \to T_{M^{H_i}}$, without NH_i/H_i-singular orbits.

Define $\widetilde{\sigma}_i: M^{(H_i)} \to T_{M^{(H_i)}}$ by $\widetilde{\sigma}_i(x) = l\overline{\sigma}_i(l^{-1}x)$, where $l \in G$ is such that $G_x \supset lH_il^{-1}$. Then, $\widetilde{\sigma}_i$ is a well defined G-path field extending $\sigma_{i-1}|_{M_{i-1} \cap M^{H_i}}$.

Now let $\sigma_i: M_i \to T_{M_i}$ coincide with $\widetilde{\sigma}_i$, on $M^{(H_i)}$ and with σ_{i-1} in M_{i-1}. It is not difficult to see that σ_i is a G-path field extending σ_{i-1} with the desired property.

For the second part, assume $|\chi|(M_H) = 0$, for all $H \leq G$. The G-path field σ constructed above can be taken without singular orbits by making use of Proposition 3.5, inductively on $\{(H)\}$.

Finally, if M admits a non-singular G-path field σ then looking at $M_{i-1} \cap M^{H_i}$ as an NH_i-submanifold of M^{H_i} we may repeat the proof of Proposition 3.5, to obtain that $|\chi|(M_{H_i}) = 0$. The proof is complete.   □

4. G-Complete Invariance Property

In this section, we study related problems concerning the fixed point theory for G-deformations. Recall that a G-space X is said to have the G-CIP for G-deformations (G-CIPD), if for any nonempty closed invariant subset $A \subseteq X$, there exists a G-deformation $\lambda \sim_G 1_X$ such that $\mathrm{Fix}\, \lambda = A$. In [27], necessary and sufficient conditions were given for smooth G-manifolds to possess the G-CIPD. As an application of Theorem 3.7, we obtain the following

THEOREM 4.1. *Let G be a finite group and M a compact locally smooth G-manifold. Suppose for each isotropy type (H), M^H has dimension at least 2. Let $A \subset M$ be a non-empty closed invariant subset. Then the following are equivalent:*

(a) *There exists a G-deformation $\varphi \colon M \to M$ such that $A = \mathrm{Fix}\, \varphi$.*
(b) *$A \cap \overline{C} \neq \emptyset$ whenever $\chi(C) \neq 0$ for any connected component C of M_H and $\overline{C}$ denotes the closure of C in M^H.*

PROOF. Suppose that there exists a G-deformation $\varphi \colon M \to M$ such that $\mathrm{Fix}\, \varphi = A$. Let C be a connected component of M_H such that $\chi(C) \neq 0$. By excision, we have

$$H_*(M^H, M^H - C) = H_*(M_H, M_H - C)$$

so that

$$\chi(M^H, M^H - C) = \chi(M_H, M_H - C) = \chi(C) \neq 0.$$

Using the relative Lefschetz fixed point theorem and Proposition 2.1 of [25], we conclude that φ must have a fixed point in the closure $\overline{C}$ of C in M^H. Since $\mathrm{Fix}\, \varphi = A$, it follows that $A \cap \overline{C} \neq \emptyset$.

Conversely, if $\chi(C) \neq 0$ then by Theorem 3.7 there exists a G-path field σ such that σ has one singular orbit in $G\overline{C}$. If $A \cap C \neq \emptyset$, this singular orbit lies in $A \cap GC$. If $A \cap C = \emptyset$, then the singular orbit must lie in $M^{>(H)}$ since M_H is open and dense in M^H and $M^H = M_H \cup (M^H \cap M^{>(H)})$, where $M^{>(H)}$ denotes the set of points of M of isotropy type $(K) > (H)$. Now, let $\varphi(x) = \sigma(x)(t_x)$ where $t_x = d(x, A)$ and d is a bounded G-invariant metric. Then, we have $\mathrm{Fix}\, \varphi = A$ and $\varphi \sim 1_M$. $\qquad \square$

REMARK 4.2. In the case where M is a smooth G-manifold and M^H/WH is connected for each (H), the necessary and sufficient conditions obtained in [27] can be derived from those of Theorem 4.1. Our formulation resembles closely to case (B) of Theorem 1 of [16] except that Jiang considered connected components C of M^H instead. According to case (B) of [16, Theorem 1], there is a fixed point free G-deformation if M^H is connected and $\chi(M^H) = 0$ for all (H). However, counter-examples have been found by D. Ferrario [11]. Therefore, Theorem 4.1 gives the correct necessary and sufficient conditions.

Remark 4.3. The first example of a G-space X in which each of the identity maps $1_{X^H}: X^H \to X^H$ is deformable to be fixed point free but 1_X is not G-deformable to be fixed point free was given by M. Izydorek and A. Vidal [15]. We would like to point out that one can easily modify their example (by taking the cartesion product with the unit interval) to give an example of a G-Wecken complex in the sense of [26] such that the equivariant Euler characteristic used in [26] is nonzero, that is, the identity is not equivariantly deformable to be fixed point free. The case for smooth G-manifolds was studied by D. Ferrario in [11] for more general G-maps.

References

[1]   G. Allaud and E. Fadell, *A fiber homotopy extension theorem*, Trans. Amer. Math. Soc. **104** (1962), 239–251.

[2]   S. Antonian, *Equivariant embeddings into G-AR's*, Glasnik Mat. Ser. III **22** (42) (1987), 503–533.

[3]   K. K. Azad and K. Srivastava, *On S-equivariant complete invariance property*, J. Indian Math. Soc. **62** (1996), 205–209.

[4]   K. Borsuk, *Zur kombinatorischen Eigenschaften der Retrakte*, Fund. Math. **19** (1932), 220–242.

[5]   G. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, New York, London, 1972.

[6]   R. Brooks, *Dominating polyhedra*, unpublished notes (2003).

[7]   R. Brown, *Path fields on manifolds*, Trans. Amer. Math. Soc. **118** (1965), 180–191.

[8]   R. Brown and E. Fadell, *Nonsingular path fields on compact topological manifolds*, Proc. Amer. Math. Soc. **16** (1965), 1342–1349.

[9]   E. Fadell, *Generalized normal bundles for locally-flat manifolds*, Trans. Amer. Math. Soc. **114** (1965), 488–513.

[10]   ———, *A remark on simple path fields in polyhedra of characteristic zero*, Rocky Mountain J. Math. **4** (1974), 65–68.

[11]   D. Ferrario, *A fixed point index for equivariant maps*, Topol. Methods Nonlinear Anal. **116** (2001), 57–71.

[12]   O. Hanner, *Some theorems on absolute neighborhood retracts*, Ark. Mat. **1** (1951), 389–408.

[13]   H. Hopf, *Vektorfelder in n-dimensionalen Mannigfaltigkeiten*, Math. Ann. **96** (1927), 225–250.

[14]   W. Hurewicz, *On the concept of fiber space*, PNAS **41** (1955), 956–961.

[15]   M. Izydorek and A. Vidal, *An example concerning equivariant deformation*, Topol. Methods Nonlinear Anal. **15** (2000), 187–190.

[16]   B. Jiang, *A note on equivariant vector fields*, Acta Math. Sci. (English ed.) **11** (1991), 274–282.

[17]   B. Jiang and H. Schirmer, *Fixed point sets of continuous selfmaps on polyhedra*, Lect. Notes Math. **886** (1981), Springer–Verlag, New York, 171–177.

[18]   K. Komiya, *A necessary and sufficient condition for the existence of non-singular G-vector fields on G-manifolds*, Osaka J. Math. **13** (1976), 537–546.

[19] _____ , *G-manifolds and G-vector fields with isolated zeros*, Proc. Japan Acad. Ser. A
      Math. Sci. **54** (1978), 124–127.

[20] S. Kwasik, *On the homotopy type of G-manifolds and G-ANRs*, Bull. Acad. Polon. Sci.
      S. Sci. Math. **28** (1980), 509–515.

[21] M. Murayama, *On G-ANR's and their G-homotopy types*, Osaka J. Math. **20** (1983),
      479–512.

[22] J. Nash, *A path space and the Stiefel–Whitney classes*, Proc. Nat. Acad. Sci. U.S.A. **41**
      (1955), 320–321.

[23] H. Schirmer, *Fixed point sets of continuous selfmaps*, Lect. Notes Math. **886** (1981),
      Springer–Verlag, New York, 417–428.

[24] R. Stern, *Stability implies normal and disc bundles*, Bull. Amer. Math. Soc. **79** (1973),
      133–135.

[25] D. Wilczyński, *Fixed point free equivariant homotopy classes*, Fund. Math. **123** (1984),
      47–60.

[26] P. Wong, *Equivariant path fields on G-complexes*, Rocky Mountain J. Math. **22** (1992),
      1139–1145.

[27] _____ , *On the location of fixed points of G-deformations*, Topol. Appl. **39** (1991),
      159–165.

Lucília Borsari and Fernanda Cardona
Departamento de Matemática
Instituto de Matemática e Estatística
Universidade de São Paulo
Rua do Matão, 1010
CEP 05508-090, São Paulo – SP, BRASIL

E-mail address: lucilia@ime.usp.br, cardona@ime.usp.br

Peter Wong
Bates College
Lewiston, Maine 04240, USA

E-mail address: pwong@bates.edu