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COMBINATORIAL LEMMAS FOR ORIENTED COMPLEXES

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80 years of the Sperner lemma

Abstract. A solid combinatorial theory is presented. The generalized Sperner lemma for chains is derived from the combinatorial Stokes formula. Many other generalizations follow from applications of an n-index of a labelling defined on chains with values in primoids. Primoids appear as the most general structure for which Sperner type theorems can be formulated. Their properties and various examples are given. New combinatorial theorems for primoids are proved. Applying them to different primoids the well-known classic results of Sperner, Fan, Shapley, Lee and Shih are obtained.

1. Introduction

In 1928 Emanuel Sperner [29] published a very simple and useful combinatorial lemma. This lemma establishes the existence of a simplex in triangulation of the n-dimensional simplex with vertices labelled by numbers from 0 to n, if some boundary conditions are satisfied. The Sperner theorem on a covering of the simplex directly follows from this lemma. Through almost eighty years this combinatorial lemma found lots of applications in various fields of mathematics, among others in nonlinear analysis, combinatorics, mathematical economics, game theory and topology. Many of its generalizations also appeared.

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One of the well-known applications of the Sperner lemma and a generalization of the Sperner covering theorem is the Knaster, Kuratowski and Mazurkiewicz [17] covering lemma published in 1929. It gives a simple proof of the Brouwer fixed point theorem.

Another version of the Sperner lemma is based on an orientation of a simplex. If the assumptions of the classic Sperner lemma are satisfied, then there exists not only a completely labelled simplex, but the difference between a number of positively oriented and negatively oriented completely labelled simplexes equals to one.

Most of generalizations went in two directions: one is due to a set of labels and the other is due to a labelled set.

Shapley [26] labelled vertices of triangulation of the simplex by subsets of the set of vertices of the simplex. His theorem was generalized by Ichiishi and Idzik [10]–[12] for labelling by vectors. Lovász [23] labelled vertices of triangulation of the simplex by elements of the matroid. Idzik [13] proved a theorem on covering of the simplex, which follows from the Lovász theorem. Tucker [35] labelled vertices of triangulation of the n-dimensional cube by numbers: $-n, \ldots, -1, 1, \ldots, n$. His lemma allowed to present a simple proof of the Borsuk–Ulam theorem [4]. In a similar way Kulpa, Turzański i Socha [19], [20] used numbers: $-n, \ldots, -1, 1, \ldots, n$, to label vertices of triangulation of the n-cube, but with different boundary conditions. A parametric generalization of the Poincaré theorem follows from their result. Ky Fan [5]–[7] used the same labels to label pseudomanifolds and van der Laan, Talman and Yang [21] labelled vertices of triangulation of polyhedrons by vectors. Todd [32], [33] and Bapat [1] used primoids to label n-pseudomanifolds.

Further generalizations involved not only pseudomanifolds, but also chains (Lindström [22]). At the conference organized to celebrate fifty years of publication of the classic Sperner lemma in Amsterdam, Sperner [30], [31] presented a generalized Sperner lemma for chains. All these previous theorems on labellings follow from this lemma.

Another step in development of this theory was made by Bapat [2]. He introduced a multilabelling by numbers. Lee and Shih [27] also studied a multilabelling. They used vectors for labels.

Many generalizations of the Sperner lemma follow from an index theory of a labelling of chains with values in primoids. New combinatorial theorems, which generalize the well-known results of Bapat [1], [2], Ichiishi and Idzik [10]–[12], Lee and Shih [27], [28], Linström [22], Lovász [23], Shapley [26], Todd [32], [33] are presented in this paper.

Related problems are considered in the paper of Björner [3]. Some proofs of related theorems known in the literature are presented for convenience of the reader.

2. Preliminaries

By $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ we denote the set of natural numbers, integer numbers and reals respectively and by $\mathbb{Z}_2$ we denote the ring $(\{0,1\}, +_{\mathrm{mod}\,2}, \cdot\,)$. For $n \in \mathbb{N}$, let $I_n = \{0, \dots, n\}$. Let V be a finite set. $\mathbf{P}(V)$ is the family of all subsets of V and $\mathbf{P}_n(V)$ is the family of all subsets of V of the cardinality $n+1$ ($n \in \mathbb{N}$). An element of $\mathbf{P}_n(V)$ is called an *n-simplex on the set V* and a nonempty family $\mathbf{C}^n \subset \mathbf{P}_n(V)$ of n-simplexes on V is called an *n-complex on the set V*. For a set A, we write $(A)^{n+1}$ to denote the Cartesian product $\prod_{i \in I_n} A_i$, where $A_i = A$ for $i \in I_n$ (for $(\mathbb{R})^n$, we write $\mathbb{R}^n$).

Let R be a commutative ring with the unity, τ be a permutation of the set I_n and $\operatorname{sgn}\tau = 1$ ($\operatorname{sgn}\tau = -1$) if the permutation τ is even (odd) (1 is the unity in R). For an n-simplex $S^n = \{v_0, \dots, v_n\} \in \mathbf{P}_n(V)$, let $\operatorname{lo} S^n = \{(\nu(0), \dots, \nu(n)) : \nu \colon I_n \to S^n$ is a one-to-one function$\}$ denote the set of all linear orders of the set S^n and for an n-complex $\mathbf{C}^n \subset \mathbf{P}_n(V)$ let $\operatorname{lo}\mathbf{C}^n = \bigcup_{S^n \in \mathbf{C}^n} \operatorname{lo} S^n$. An *orientation of an n-simplex* $S^n = \{v_0, \dots, v_n\} \in \mathbf{P}_n(V)$ is a function $\operatorname{or}_{S^n} \colon (S^n)^{n+1} \to \{-1, 0, 1\}$ fulfilling the condition $\operatorname{or}_{S^n}(w_0, \dots, w_n) = \operatorname{sgn}\tau \cdot \operatorname{or}_{S^n}(w_{\tau(0)}, \dots, w_{\tau(n)})$ for a permutation τ ($w_i \in S^n$ for $i \in I_n$, $\{-1, 0, 1\} \subset R$). In fact there are only two such functions except the zero function if the ring R has at least three elements. If the ring $R = \mathbb{Z}_2$, then there is exactly one nonzero orientation. Notice that $\operatorname{or}_{S^n}(v_0, \dots, v_n) = 0$ if $v_i = v_j$ for some $i \neq j$, $i, j \in I_n$. An *orientation of an n-complex* $\mathbf{C}^n$ is a function $\operatorname{or}_{\mathbf{C}^n} \colon (V)^{n+1} \to \{-1, 0, 1\}$ such that $\operatorname{or}_{\mathbf{C}^n} | (S^n)^{n+1}$ is an orientation of each $S^n \in \mathbf{C}^n$ and $\operatorname{or}_{\mathbf{C}^n}(v_0, \dots, v_n) = 0$ for $\{v_0, \dots, v_n\} \notin \mathbf{C}^n$. We call a pair $(S^n, \operatorname{or}_{S^n})$, $(\mathbf{C}^n, \operatorname{or}_{\mathbf{C}^n})$ an *oriented n-simplex*, an *oriented n-complex*, respectively.

There are many ways to define an orientation of an n-complex $\mathbf{C}^n$. One of them is to choose a linear order $\overline{S}^n \in \operatorname{lo} S^n$ for every $S^n \in \mathbf{C}^n$. Let $\overline{\mathbf{C}}^n = \{\overline{S}^n : S^n \in \mathbf{C}^n\}$. A function defined by

$$\operatorname{or}_{\overline{\mathbf{C}}^n}(w_0, \dots, w_n) = \begin{cases} \operatorname{sgn}\tau & \text{if there exists } \overline{S}^n \in \overline{\mathbf{C}}^n \text{ and a permutation } \tau \\ & \text{of } I_n \text{ such that } (w_{\tau(0)}, \dots, w_{\tau(n)}) = \overline{S}^n, \\ 0 & \text{if } \{w_0, \dots, w_n\} \notin \mathbf{C}^n, \end{cases}$$

is well-defined and it is an orientation of the n-complex $\mathbf{C}^n$. We say that the set $\overline{\mathbf{C}}^n$ defines the orientation $\operatorname{or}_{\overline{\mathbf{C}}^n}$ and it is a representation of the orientation $\operatorname{or}_{\overline{\mathbf{C}}^n}$. And we say that $\mathbf{C}^n$ is oriented by $\overline{\mathbf{C}}^n$ instead of $\operatorname{or}_{\overline{\mathbf{C}}^n}$. For an orientation $\operatorname{or}_{\mathbf{C}^n}$ of an n-complex $\mathbf{C}^n$ we may choose a representation $\overline{\mathbf{C}}^n$ such that $\operatorname{or}_{\mathbf{C}^n} = \operatorname{or}_{\overline{\mathbf{C}}^n}$. The choice of the representation is not necessarily unique.

From now on, for $S^n \in \mathbf{C}^n$ we denote the unique element in $\overline{\mathbf{C}}^n \cap \mathrm{lo}\, S^n$ by $\overline{S}^n$.

Let $S^n \in \mathbf{P}_n(V)$, $v \in S^n$ and $S^{n-1} = S^n \setminus \{v\}$. The $(n-1)$-simplex S^{n-1} is called a *facet of the n-simplex S^n opposite to the vertex v*. A function $\mathrm{or}_{S^n,v}\colon (S^{n-1})^n \to \{-1,0,1\}$ defined by

$$\mathrm{or}_{S^n,v}(w_1,\ldots,w_n) = \mathrm{or}_{S^n}(v,w_1,\ldots,w_n)$$

for $(w_1,\ldots,w_n) \in (S^{n-1})^n$ is an *induced orientation of the $(n-1)$-simplex S^{n-1} by an orientation* or_{S^n}. If the orientation or_{S^n} is defined by $\overline{S}^n$, then we write $\mathrm{or}_{\overline{S}^n,v}$ instead of $\mathrm{or}_{S^n,v}$.

An n-complex $\mathbf{C}^n$ on V is called an *n-pseudomanifold* if any $(n-1)$-simplex on V is contained in at most two n-simplexes of $\mathbf{C}^n$. An n-pseudo-manifold can be also defined as n-complex on the set V fulfilling the condition: for every n-simplex $S^n \in \mathbf{C}^n$ and for every $v \in S^n$ there exists at most one $v' \in V \setminus S^n$ such that an n-simplex $S^n \setminus \{v\} \cup \{v'\} \in \mathbf{C}^n$. An n-pseudomanifold $\mathbf{C}^n$ is *coherently oriented by an orientation* $\mathrm{or}_{\mathbf{C}^n}$ if for $(v,v_1,\ldots,v_n)$, $(v',v_1,\ldots,v_n) \in \mathrm{lo}\,\mathbf{C}^n$, $(v \neq v')$ we have

$$\mathrm{or}_{\mathbf{C}^n}(v,v_1,\ldots,v_n) = -\mathrm{or}_{\mathbf{C}^n}(v',v_1,\ldots,v_n).$$

Notice that the condition above is equivalent to the condition: for $i \in I_n$ and for $(v_0,\ldots,v_{i-1},v,v_{i+1},\ldots,v_n)$, $(v_0,\ldots,v_{i-1},v',v_{i+1},\ldots,u_n) \in \mathrm{lo}\,\mathbf{C}^n$ $(v \neq v')$ we have

$$\mathrm{or}_{\mathbf{C}^n}(v_0,\ldots,v_{i-1},v,v_{i+1},\ldots,v_n) = -\mathrm{or}_{\mathbf{C}^n}(v_0,\ldots,v_{i-1},v',v_{i+1},\ldots,v_n).$$

In other words, for any two n-simplexes of $\mathbf{C}^n$ with a common facet, $\mathrm{or}_{\mathbf{C}^n}$ induce the opposite orientation on their common facet. Observe that an n-pseudomanifold may be not coherently orientable if the ring R has at least three elements. In the case $R = \mathbb{Z}_2$ every n-pseudomanifold is coherently orientable since $1 = -1$ in $\mathbb{Z}_2$.

Example 2.1 (discrete Möbius strip). The family

$$\{\{1,2,4\},\{1,3,4\},\{3,4,6\},\{3,5,6\},\{1,5,6\},\{1,2,5\}\}$$

is 2-pseudomanifold and there is no coherent orientation of it if the ring R has at least three elements.

For a set $A \subset \mathbb{R}^n$, $\mathrm{co}\, A = \{\alpha_0 a_0 + \ldots + \alpha_m a_m : a_i \in A,\ \sum_{i=0}^m \alpha_i = 1,\ \alpha_i \geq 0 \text{ for } i \in I_m,\ m \in \mathbb{N}\}$ is the *convex hull* of A, $\mathrm{aff}\, A = \{\alpha_0 a_0 + \ldots + \alpha_m a_m : \sum_{i=0}^m \alpha_i = 1,\ a_i \in A,\ \alpha_i \in \mathbb{R} \text{ for } i \in I_m,\ m \in \mathbb{N}\}$ is the *affine hull* of A, $\mathrm{cone}(A,b) = \{\alpha_0(a_0-b) + \ldots + \alpha_m(a_m-b) : a_i \in A,\ \alpha_i \geq 0 \text{ for } i \in I_m,\ m \in \mathbb{N}\}$ is the *cone pointed at $b \in \mathbb{R}^n$ and span by the set A*, $\mathrm{ri}\, A$ is the *relative interior* of A, $\mathrm{bd}\, A$ is the *boundary* of A and $\mathrm{cl} A$ is the *closure* of A. Observe that $\mathrm{aff}\, A$

is a linear subspace of $\mathbb{R}^n$. A finite set $A = \{a_0, \dots, a_m\} \subset \mathbb{R}^n$ is an *affinely independent* set if the dimension of $\operatorname{aff} A$ is m ($m \le n$).

3. Chains

Let $(R, +, \cdot, 1)$ be a commutative ring with the unity, V be a finite set and $\mathbf{P}_n(V)$, $\mathbf{P}_{n-1}(V)$ be oriented by representations $\overline{\mathbf{V}}_n$, $\overline{\mathbf{V}}_{n-1}$, respectively. A function $\ell^n \colon (V)^{n+1} \to R$ fulfilling the condition $\ell^n(v_0, \dots, v_n) = \operatorname{sgn}\tau \cdot \ell^n(v_{\tau(0)}, \dots, v_{\tau(n)})$ for all $(v_0, \dots, v_n) \in (V)^{n+1}$ and for all permutations τ of the set I_n is called an *n-chain on the set V. All n-chains considered in this paper have values in the ring R.* The n-chain on V is a generalization of the orientation of $\mathbf{P}_n(V)$.

We define operations of the sum and the multiplication by an element of the ring R for n-chains in the following way: for n-chains ℓ_1^n, ℓ_2^n, $\alpha \in R$ and $(v_0, \dots, v_n) \in (V)^{n+1}$

$$\begin{aligned}
(\ell_1^n \oplus \ell_2^n)(v_0, \dots, v_n) &= \ell_1^n(v_0, \dots, v_n) + \ell_2^n(v_0, \dots, v_n),\\
(\alpha \odot \ell_1^n)(v_0, \dots, v_n) &= \alpha \cdot \ell_1^n(v_0, \dots, v_n).
\end{aligned}$$

Any n-chain on V can be formally written in the form

$$\ell^n = \bigoplus_{\overline{S}^n \in \overline{\mathbf{V}}_n} \alpha_{\overline{S}^n} \odot \chi_{\overline{S}^n},$$

where $\overline{\mathbf{V}}_n$ is the representation of an orientation of $\mathbf{P}_n(V)$, $\alpha_{\overline{S}^n} \in R$, for $\overline{S}^n \in \overline{\mathbf{V}}_n$ and $\chi_{\overline{S}^n} \colon (V)^{n+1} \to \{-1, 0, 1\}$ ($\{-1, 0, 1\} \subset R$) is an n-chain defined by

$$\chi_{\overline{S}^n}(v_0, \dots, v_n) = \begin{cases} \operatorname{sgn}\tau & \text{if there exists a permutation } \tau \text{ of the set } I_n,\\ & \text{such that } (v_{\tau(0)}, \dots, v_{\tau(n)}) = \overline{S}^n,\\ 0 & \text{in other cases.} \end{cases}$$

The set of all n-chains on V is denoted by $\mathcal{L}^n(V)$.

It is easy to prove that $(\mathcal{L}^n(V), \oplus, \odot, R)$ is a module over the ring R.

Notice that a function $1_{\overline{\mathbf{V}}_n} := \bigoplus_{\overline{S}^n \in \overline{\mathbf{V}}_n} 1 \odot \chi_{\overline{S}^n}$ is the orientation $\mathrm{or}_{\overline{\mathbf{V}}_n}$ and more generally a function $1_{\overline{\mathbf{C}}^n} := \bigoplus_{\overline{S}^n \in \overline{\mathbf{C}}^n} 1 \odot \chi_{\overline{S}^n}$ is the orientation $\mathrm{or}_{\overline{\mathbf{C}}^n}$ for any representation $\overline{\mathbf{C}}^n$ ($\mathbf{C}^n \subset \mathbf{P}_n(V)$, $1 \in R$).

Definition 3.1. An *n-index of n-chains* $\ell^n = \bigoplus_{\overline{S}^n \in \overline{\mathbf{V}}_n} \alpha_{\overline{S}^n} \odot \chi_{\overline{S}^n}$, $\ell_2^n = \bigoplus_{\overline{S}^n \in \overline{\mathbf{V}}_n} \beta_{\overline{S}^n} \odot \chi_{\overline{S}^n}$ *on the set* V is an element of the ring R and it is equal to

$$\ell^n \bullet_n \ell_2^n = \sum_{\overline{S}^n \in \overline{\mathbf{V}}_n} \alpha_{\overline{S}^n} \cdot \beta_{\overline{S}^n},$$

where $\overline{\mathbf{V}}_n$ is a representation of an orientation of $\mathbf{P}_n(V)$.

The n-index is a generalization of the Kronecker index (see [9, p. 301]).

For $\overrightarrow{S}^n \in (V)^{n+1}$ and $\overrightarrow{S}^{n-1} \in (V)^n$ an *incidence number* $[\overrightarrow{S}^n : \overrightarrow{S}^{n-1}]$ is an element of R defined by

$$[\overrightarrow{S}^n : \overrightarrow{S}^{n-1}] = \begin{cases} \operatorname{sgn}\tau & \text{if } \overrightarrow{S^n} = (v_0, \dots, v_n) \in \operatorname{lo}\mathbf{P}_n(V) \text{ and there exists} \\ & \text{a permutation } \tau \text{ of the set } I_n \text{ such that} \\ & \overrightarrow{S}^{n-1} = (v_{\tau(1)}, \dots, v_{\tau(n)}); \\ 0 & \text{in other cases.} \end{cases}$$

Proposition 3.2. *If $S^n \in \mathbf{P}_n(V)$, $v \in S^n$, $S^{n-1} = S^n \setminus \{v\}$, $\overrightarrow{S^n} \in \operatorname{lo} S^n$ and $\overrightarrow{S}^{n-1} \in \operatorname{lo} S^{n-1}$, then the incidence number $[\overrightarrow{S}^n : \overrightarrow{S}^{n-1}]$ defines whether the orientation given by $\overrightarrow{S}^{n-1}$ is the same or opposite to the induced orientation $\mathrm{or}_{\overrightarrow{S}^n, v}$. Notice, that if orientations of S^n and S^{n-1} are given by $\overline{S}^n \in \overline{\mathbf{V}}_n$ and $\overline{S}^{n-1} = (w_1, \dots, w_n) \in \overline{\mathbf{V}}_{n-1}$, respectively, then $[\overline{S}^n : \overline{S}^{n-1}] = \mathrm{or}_{\overline{\mathbf{V}}_n}(v, w_1, \dots, w_n)$.*

A *boundary operator* $\partial_n : \mathcal{L}^n(V) \to \mathcal{L}^{n-1}(V)$ is defined in the following way: for $\overline{S}^n \in \overline{\mathbf{V}}_n$ we define $\partial_n \chi_{\overline{S}^n} = \bigoplus_{\overline{S}^{n-1} \in \overline{\mathbf{V}}_{n-1}} [\overline{S}^n : \overline{S}^{n-1}] \odot \chi_{\overline{S}^{n-1}}$ and for an n-chain $\ell^n \in \mathcal{L}^n(V)$ we define

$$\partial_n \ell^n = \partial_n \bigg(\bigoplus_{\overline{S}^n \in \overline{\mathbf{V}}_n} \alpha_{\overline{S}^n} \odot \chi_{\overline{S}^n} \bigg) = \bigoplus_{\overline{S}^n \in \overline{\mathbf{V}}_n} \alpha_{\overline{S}^n} \odot \partial_n \chi_{\overline{S}^n}.$$

Observe that the boundary operator ∂_n is linear and that it does not depend on the representation $\overline{\mathbf{V}}_{n-1}$ nor the representation $\overline{\mathbf{V}}_n$.

A *coboundary operator* $\delta^V_{n-1} : \mathcal{L}^{n-1}(V) \to \mathcal{L}^n(V)$ is defined in the following way:

$$\delta^V_{n-1} \chi_{\overline{S}^{n-1}} = \bigoplus_{\overline{S}^n \in \overline{\mathbf{V}}_n} [\overline{S}^n : \overline{S}^{n-1}] \odot \chi_{\overline{S}^n},$$

$$\delta^V_{n-1} \ell^{n-1} = \delta^V_{n-1} \bigg(\bigoplus_{\overline{S}^{n-1} \in \overline{\mathbf{V}}_{n-1}} \alpha_{\overline{S}^{n-1}} \odot \chi_{\overline{S}^{n-1}} \bigg) = \bigoplus_{\overline{S}^{n-1} \in \overline{\mathbf{V}}_{n-1}} \alpha_{\overline{S}^{n-1}} \odot \delta^V_{n-1} \chi_{\overline{S}^{n-1}}.$$

Observe that the coboundary operator δ_n is linear and that it does not depend on the orientation $\mathrm{or}_{\overline{\mathbf{V}}_n}$ nor representation of the orientation $\mathrm{or}_{\overline{\mathbf{V}}_{n-1}}$. We write δ_{n-1} instead of δ^V_{n-1} for brevity.

Now, the combinatorial Stokes theorem can be formulated as

Theorem 3.3 (see J. G. Hocking, G. S. Young [9, p. 301]). *For an n-chain ℓ^n and an $(n-1)$-chain ℓ_2^{n-1} having values in the same ring R, the following equality is true*

$$(\partial_n \ell^n) \bullet_{n-1} \ell_2^{n-1} = \ell^n \bullet_n (\delta_{n-1} \ell_2^{n-1}).$$

Proof. Let V be a finite set and $\mathbf{P}_n(V)$ and $\mathbf{P}_{n-1}(V)$ be complexes oriented by $\overline{\mathbf{V}}_n$ and $\overline{\mathbf{V}}_{n-1}$, respectively. Because of the linearity of the boundary operator

∂_n it is sufficient to prove the theorem for the n-chain of the form $\ell^n = \chi_{\overline{S}_0^n}$, where $\overline{S}_0^n \in \overline{\mathbf{V}}_n$ is fixed. For $\ell_2^{n-1} = \bigoplus_{\overline{S}^{n-1} \in \overline{\mathbf{V}}_{n-1}} \beta_{\overline{S}^{n-1}} \odot \chi_{\overline{S}^{n-1}}$ the left-hand side of the equality is

$$
\begin{aligned}
&(\partial_n \chi_{\overline{S}_0^n}) \bullet_{n-1} \left(\bigoplus_{\overline{S}^{n-1} \in \overline{\mathbf{V}}_{n-1}} \beta_{\overline{S}^{n-1}} \odot \chi_{\overline{S}^{n-1}} \right) \\
&\quad = \left(\bigoplus_{\overline{S}^{n-1} \in \overline{\mathbf{V}}_{n-1}} [\overline{S}_0^n : \overline{S}^{n-1}] \odot \chi_{\overline{S}^{n-1}} \right) \bullet_{n-1} \left(\bigoplus_{\overline{S}^{n-1} \in \overline{\mathbf{V}}_{n-1}} \beta_{\overline{S}^{n-1}} \odot \chi_{\overline{S}^{n-1}} \right) \\
&\qquad = \sum_{\overline{S}^{n-1} \in \overline{\mathbf{V}}_{n-1}} [\overline{S}_0^n : \overline{S}^{n-1}] \cdot \beta_{\overline{S}^{n-1}}.
\end{aligned}
$$

And the right-hand side of the equality is

$$
\begin{aligned}
&\chi_{\overline{S}_0^n} \bullet_n \left(\delta_{n-1} \bigoplus_{\overline{S}^{n-1} \in \overline{\mathbf{V}}_{n-1}} \beta_{\overline{S}^{n-1}} \odot \chi_{\overline{S}^{n-1}} \right) \\
&\quad = \chi_{\overline{S}_0^n} \bullet_n \left(\bigoplus_{\overline{S}^{n-1} \in \overline{\mathbf{V}}_{n-1}} \beta_{\overline{S}^{n-1}} \odot \delta_{n-1} \chi_{\overline{S}^{n-1}} \right) \\
&\quad = \chi_{\overline{S}_0^n} \bullet_n \left(\bigoplus_{\overline{S}^{n-1} \in \overline{\mathbf{V}}_{n-1}} \beta_{\overline{S}^{n-1}} \odot \left(\bigoplus_{\overline{S}^{n} \in \overline{\mathbf{V}}_{n}} [\overline{S}^n : \overline{S}^{n-1}] \odot \chi_{\overline{S}^n} \right) \right) \\
&\quad = \chi_{\overline{S}_0^n} \bullet_n \left(\bigoplus_{\overline{S}^{n} \in \overline{\mathbf{V}}_{n}} \bigoplus_{\overline{S}^{n-1} \in \overline{\mathbf{V}}_{n-1}} \beta_{\overline{S}^{n-1}} \odot ([\overline{S}^n : \overline{S}^{n-1}] \odot \chi_{\overline{S}^n}) \right) \\
&\quad = \sum_{\overline{S}^{n-1} \in \overline{\mathbf{V}}_{n-1}} \beta_{\overline{S}^{n-1}} \cdot [\overline{S}_0^n : \overline{S}^{n-1}]. \qquad \square
\end{aligned}
$$

Let U be a finite set, $\mathbf{P}_n(U)$ be oriented by a representation $\overline{\mathbf{U}}_n$ and $l\colon V \to U$ be a function. The function l is called a *labelling*. For each $n \in \mathbb{N}$, let $\overrightarrow{l_n}\colon (V)^{n+1} \to (U)^{n+1}$ be a function defined by

$$\overrightarrow{l_n}(v_0, \dots, v_n) = (l(v_0), \dots, l(v_n)).$$

PROPOSITION 3.4. *For the finite sets V and U, let $S^n \in \mathbf{P}_n(V)$, $\overline{S}^n \in \overline{\mathbf{V}}_n$, $S^{n-1} \in \mathbf{P}_{n-1}(V)$, $\overline{S}^{n-1} \in \overline{\mathbf{V}}_{n-1}$, $T^n \in \mathbf{P}_n(U)$. If $S^{n-1} \subset S^n$ and a labelling $l\colon S^n \to T^n$ is one-to-one, then*

$$[\overline{S}^n : \overline{S}^{n-1}] = [\overrightarrow{l_n}(\overline{S}^n) : \overrightarrow{l}_{n-1}(\overline{S}^{n-1})].$$

DEFINITION 3.5. Let $\hbar^n = \bigoplus_{\overline{T}^n \in \overline{\mathbf{U}}_n} \beta_{\overline{T}^n} \odot \chi_{\overline{T}^n}$ be an n-chain on U, where $\overline{\mathbf{U}}_n$ is some representation of an orientation of $\mathbf{P}_n(U)$. We define an operator $\widetilde{l}_n\colon \mathcal{L}^n(U) \to \mathcal{L}^n(V)$ by

$$\widetilde{l}_n(\hbar^n) = \bigoplus_{\overline{S}^n \in \overline{\mathbf{V}}_n} \hbar^n(\overrightarrow{l_n}(\overline{S}^n)) \odot \chi_{\overline{S}^n}.$$

Observe that the operator $\widetilde{l}_n$ does not depend on the representations $\overline{\mathbf{V}}_n$ and $\overline{\mathbf{U}}_n$. The operator $\widetilde{l}_n$ is linear, i.e. for n-chains $\hbar_1^n$, $\hbar_2^n$ on U and $\alpha \in R$ we have:

$$\widetilde{l}_n(\hbar_1^n \oplus \hbar_2^n) = \widetilde{l}_n(\hbar_1^n) \oplus \widetilde{l}_n(\hbar_2^n) \quad \text{and} \quad \widetilde{l}_n(\alpha \odot \hbar_1^n) = \alpha \odot \widetilde{l}_n(\hbar_1^n).$$

LEMMA 3.6. *Let $l\colon V \to U$ be a labelling and $\hbar^{n-1}$ be an $(n-1)$-chain on U. Then*

$$\widetilde{l}_n(\delta_{n-1}^U \hbar^{n-1}) = \delta_{n-1}^V \widetilde{l}_{n-1}(\hbar^{n-1}).$$

PROOF. For finite sets V and U, let $\mathbf{P}_n(V)$, $\mathbf{P}_{n-1}(V)$, $\mathbf{P}_n(U)$, $\mathbf{P}_{n-1}(U)$ be oriented by representations $\overline{\mathbf{V}}_n$, $\overline{\mathbf{V}}_{n-1}$, $\overline{\mathbf{U}}_n$, $\overline{\mathbf{U}}_{n-1}$, respectively. Because of the linearity of operators $\widetilde{l}_n$, $\widetilde{l}_{n-1}$, δ_{n-1}^V and δ_{n-1}^U it is enough to prove our theorem for the case $\hbar^{n-1} = \chi_{\overline{T}_0^{n-1}}$, where $\overline{T}_0^{n-1} \in \overline{\mathbf{U}}_{n-1}$ only.

$$\begin{aligned}
\widetilde{l}_n(\delta_{n-1}^U \chi_{\overline{T}_0^{n-1}}) &= \widetilde{l}_n\bigg(\bigoplus_{\overline{T}^n \in \overline{\mathbf{U}}_n} [\overline{T}^n : \overline{T}_0^{n-1}] \odot \chi_{\overline{T}^n}\bigg) \\
&= \bigoplus_{\overline{S}^n \in \overline{\mathbf{V}}_n} \bigg(\bigg(\bigoplus_{\overline{T}^n \in \overline{\mathbf{U}}_n} [\overline{T}^n : \overline{T}_0^{n-1}] \odot \chi_{\overline{T}^n}\bigg)(\overrightarrow{l}_n(\overline{S}^n))\bigg) \odot \chi_{\overline{S}^n} \\
&= \bigoplus_{\overline{S}^n \in \overline{\mathbf{V}}_n} \bigg(\bigoplus_{\overline{T}^n \in \overline{\mathbf{U}}_n} [\overline{T}^n : \overline{T}_0^{n-1}] \cdot \chi_{\overline{T}^n}(\overrightarrow{l}_n(\overline{S}^n))\bigg) \odot \chi_{\overline{S}^n} \\
&= \bigoplus_{\overline{S}^n \in \overline{\mathbf{V}}_n} [\overrightarrow{l}_n(\overline{S}^n) : \overline{T}_0^{n-1}] \odot \chi_{\overline{S}^n}.
\end{aligned}$$

And for the right-hand side of the equality we have

$$\begin{aligned}
\delta_{n-1}^V \widetilde{l}_{n-1}(\chi_{\overline{T}_0^{n-1}}) &= \delta_{n-1}^V\bigg(\bigoplus_{\overline{S}^{n-1} \in \overline{\mathbf{V}}_{n-1}} \chi_{\overline{T}_0^{n-1}}(\overrightarrow{l}_{n-1}(\overline{S}^{n-1})) \odot \chi_{\overline{S}^{n-1}}\bigg) \\
&= \bigoplus_{\overline{S}^{n-1} \in \overline{\mathbf{V}}_{n-1}} \chi_{\overline{T}_0^{n-1}}(\overrightarrow{l}_{n-1}(\overline{S}^{n-1})) \odot \delta_{n-1}^V \chi_{\overline{S}^{n-1}} \\
&= \bigoplus_{\overline{S}^{n-1} \in \overline{\mathbf{V}}_{n-1}} \chi_{\overline{T}_0^{n-1}}(\overrightarrow{l}_{n-1}(\overline{S}^{n-1})) \odot \bigg(\bigoplus_{\overline{S}^n \in \overline{\mathbf{V}}_n} [\overline{S}^n : \overline{S}^{n-1}] \odot \chi_{\overline{S}^n}\bigg) \\
&= \bigoplus_{\overline{S}^n \in \overline{\mathbf{V}}_n} \bigoplus_{\overline{S}^{n-1} \in \overline{\mathbf{V}}_{n-1}} \chi_{\overline{T}_0^{n-1}}(\overrightarrow{l}_{n-1}(\overline{S}^{n-1})) \odot ([\overline{S}^n : \overline{S}^{n-1}] \odot \chi_{\overline{S}^n}) \\
&= \bigoplus_{\overline{S}^n \in \overline{\mathbf{V}}_n} \bigg(\sum_{\overline{S}^{n-1} \in \overline{\mathbf{V}}_{n-1}} \chi_{\overline{T}_0^{n-1}}(\overrightarrow{l}_{n-1}(\overline{S}^{n-1})\bigg) \cdot [\overline{S}^n : \overline{S}^{n-1}]) \odot \chi_{\overline{S}^n}.
\end{aligned}$$

Now we show that:

$$[\overrightarrow{l}_n(\overline{S}^n) : \overline{T}_0^{n-1}] = \sum_{\overline{S}^{n-1} \in \overline{\mathbf{V}}_{n-1}} \chi_{\overline{T}_0^{n-1}}(\overrightarrow{l}_{n-1}(\overline{S}^{n-1})) \cdot [\overline{S}^n : \overline{S}^{n-1}]. \tag{3.1}$$

There are three cases:

Case 1. If $T_0^{n-1} \not\subset l(S^n)$, then $[\overrightarrow{l}_n(\overline{S}^n) : \overline{T}_0^{n-1}] = 0$. And if

$$\sum_{\overline{S}^{n-1} \in \overline{\mathbf{V}}_{n-1}} \chi_{\overline{T}_0^{n-1}}(\overrightarrow{l}_{n-1}(\overline{S}^{n-1})) \cdot [\overline{S}^n : \overline{S}^{n-1}] \neq 0,$$

then there exists $S_1^{n-1} \subset S^n$ such that $T_0^{n-1} = l(S_1^{n-1})$, but $l(S_1^{n-1}) \subset l(S^n)$ and T_0^{n-1} would be contained in $l(S^n)$.

Case 2. If $T_0^{n-1} = l(S^n)$, then the left-hand side of (3.1) is equal to zero. There exists $v_0, v_1 \in S^n$ such that $l(v_0) = l(v_1)$ and thus for $S_0^{n-1} = S^n \setminus \{v_0\}$ and $S_1^{n-1} = S^n \setminus \{v_1\}$ the right-hand side of (3.1) is

$$\chi_{\overline{T}_0^{n-1}}(\overrightarrow{l}_{n-1}(\overline{S}_0^{n-1})) \cdot [\overline{S}^n : \overline{S}_0^{n-1}] + \chi_{\overline{T}_0^{n-1}}(\overrightarrow{l}_{n-1}(\overline{S}_1^{n-1})) \cdot [\overline{S}^n : \overline{S}_1^{n-1}].$$

Let $\overline{S}_0^{n-1} = (v_1, \dots, v_n)$. Therefore, by Proposition 3.2

$$\begin{aligned}[\overline{S}^n : \overline{S}_0^{n-1}] &= \mathrm{or}_{\overline{\mathbf{V}}_n}(v_0, v_1, \dots, v_n) \\ &= -\mathrm{or}_{\overline{\mathbf{V}}_n}(v_1, v_0, v_2, \dots, v_n) = -\varepsilon \cdot [\overline{S}^n : \overline{S}_1^{n-1}],\end{aligned}$$

where $\varepsilon = 1$ (-1) if $\overline{S}_1^{n-1}$ is even (odd) permutation of $(v_0, v_2, \dots, v_n)$ and $l(\overline{S}_1^{n-1}) = \varepsilon \cdot l(\overline{S}_0^{n-1})$. Thus the right-hand-side of (3.1) is

$$\begin{aligned}&\varepsilon \cdot \chi_{\overline{T}_0^{n-1}}(\overrightarrow{l}_{n-1}(\overline{S}_1^{n-1})) \cdot (-\varepsilon \cdot [\overline{S}^n : \overline{S}_1^{n-1}]) \\ &\qquad + \chi_{\overline{T}_0^{n-1}}(\overrightarrow{l}_{n-1}(\overline{S}_1^{n-1})) \cdot [\overline{S}^n : \overline{S}_1^{n-1}] = 0.\end{aligned}$$

Case 3. If $T_0^{n-1} \subset l(S^n)$ and $T_0^{n-1} \neq l(S^n)$, then there exists $v_0 \in S^n$ such that $S_0^{n-1} = S^n \setminus \{v_0\}$ and $l(S_0^{n-1}) = T_0^{n-1}$. By Proposition 3.4 on the right-hand side of (3.1), we have

$$\begin{aligned}&\chi_{\overline{T}_0^{n-1}}(\overrightarrow{l}_{n-1}(\overline{S}_0^{n-1})) \cdot [\overline{S}^n : \overline{S}_0^{n-1}] \\ &\quad = \chi_{\overline{T}_0^{n-1}}(\overrightarrow{l}_{n-1}(\overline{S}_0^{n-1})) \cdot [\overrightarrow{l}_n(\overline{S}^n) : \overrightarrow{l}_{n-1}(\overline{S}_0^{n-1})] = [\overrightarrow{l}_n(\overline{S}^n) : \overline{T}_0^{n-1}]. \quad \square\end{aligned}$$

Let $\mathbf{C}^n \subset \mathbf{P}_n(V)$ be an n-complex oriented by a representation $\overline{\mathbf{C}}^n \subset \overline{\mathbf{V}}_n$. For an n-chain $\ell^n \colon (V)^{n+1} \to R$ on V if $\ell(\overrightarrow{S}^n) = 0$ for all $\overrightarrow{S}^n \notin \mathrm{lo}\,\mathbf{C}^n$, then we say that the n-chain ℓ^n is defined *on the n-complex* $\mathbf{C}^n$. Let $\mathbf{K}^n \subset \mathbf{P}_n(U)$ be an n-complex oriented by a representation $\overline{\mathbf{K}}^n \subset \overline{\mathbf{U}}_n$.

DEFINITION 3.7. Let ℓ^n be an n-chain on V and $\hbar^n$ be an n-chain on U. An *n-index of a function $l \colon V \to U$ for an n-chain ℓ^n and an n-chain $\hbar^n$* is defined by

$$\mathrm{ind}_n l_n(\ell^n, \hbar^n) = \ell^n \bullet_n \widetilde{l}_n(\hbar^n),$$

where $\widetilde{l}_n \colon \mathcal{L}^n(V) \to \mathcal{L}^n(V)$ is defined by Definition 3.5.

Definition 3.8. An *n-index of a function $l\colon V\to U$ for oriented n-complexes $\overline{\mathbf{C}}^n$ and $\overline{\mathbf{K}}^n$* is considered as the index of the function l for the chains $1_{\overline{\mathbf{C}}^n}$ and $1_{\overline{\mathbf{K}}^n}$, and it is equal to

$$\mathrm{ind}_n l(1_{\overline{\mathbf{C}}^n}, 1_{\overline{\mathbf{K}}^n}) = 1_{\overline{\mathbf{C}}^n} \bullet_n \widetilde{l}_n(1_{\overline{\mathbf{K}}^n}).$$

The combinatorial Stokes theorem (Theorem 3.3) and the n-index of the function l are closely related to the generalized Sperner lemma, which was introduced by Sperner at the conference in Hamburg 1980. This conference was organized on the occasion of fifty years of the classic Sperner lemma.

Theorem 3.9 (Sperner, [31]). *For finite sets V and U, let $\mathbf{P}_n(V)$, $\mathbf{P}_{n-1}(V)$, $\mathbf{P}_n(U)$, $\mathbf{P}_{n-1}(U)$ be oriented by representations $\overline{\mathbf{V}}_n$, $\overline{\mathbf{V}}_{n-1}$, $\overline{\mathbf{U}}_n$, $\overline{\mathbf{U}}_{n-1}$, respectively. Let $l\colon V\to U$ be a labelling. Let ℓ^n be an n-chain on V and $\hbar^{n-1}$ be an $(n-1)$-chain on U. Then*

$$\mathrm{ind}_{n-1} l(\partial_n \ell^n, \hbar^{n-1}) = \mathrm{ind}_n l(\ell^n, \delta^U_{n-1}\hbar^{n-1}).$$

Proof. The theorem directly follows from Theorem 3.3 and Lemma 3.6. □

In the case the function l is the identity function the generalized Sperner lemma reduces to the combinatorial Stokes theorem. Tompkins [34] defined the index of a labelling function in case $\mathbf{K}^{n-1}=\{\{v_1,\dots,v_n\}\}$, $\mathbf{K}^n=\{\{v_0,\dots,v_n\}\}$ and $\mathbf{C}^n$ is an n-pseudomanifold, only. For this case Theorem 3.9 says that $\mathrm{ind}_n l(\ell^n, 1_{\overline{\mathbf{K}}^n}) = \mathrm{ind}_{n-1} l(\partial_n\ell^n, 1_{\overline{\mathbf{K}}^{n-1}})$, for some representations $\overline{\mathbf{K}}^n$ and $\overline{\mathbf{K}}^{n-1}$ of orientations of $\mathbf{K}^n$ and $\mathbf{K}^{n-1}$, respectively.

4. Primoids

Let U be a finite set. An *n-primoid $\mathbf{L}^U_n$ on U* is a nonempty n-complex on U fulfilling the following condition: for every n-simplex $T^n\in\mathbf{L}^U_n$ and for every $u\in U$ there exists exactly one $u'\in T^n$ such that an n-simplex $T^n\setminus\{u'\}\cup\{u\}\in\mathbf{L}^U_n$. An n-simplex belonging to $\mathbf{L}^U_n$ is called a complete n-simplex. For brevity we write $\mathbf{L}_n$ instead of $\mathbf{L}^U_n$.

An n-primoid $\mathbf{L}_n$ can be also defined as the family of n-simplexes on U such that every $(n+1)$-simplex on U contains either none or two n-simplexes belonging to the n-primoid $\mathbf{L}_n$.

An n-primoid $\mathbf{L}_n$ is *properly oriented by an orientation* $\mathrm{or}_{\mathbf{L}_n}\colon(U)^{n+1}\to\{-1,0,1\}$ if for $(u,u_1,\dots,u_n),(u',u_1,\dots,u_n)\in\mathrm{lo}\,\mathbf{L}_n$ we have

$$\mathrm{or}_{\mathbf{L}_n}(u,u_1,\dots,u_n) = \mathrm{or}_{\mathbf{L}_n}(u',u_1,\dots,u_n).$$

Notice that the above condition is equivalent to the condition: for $i\in I_n$ and for $(u_0,\dots,u_{i-1},u,u_{i+1},\dots,u_n),(u_0,\dots,u_{i-1},u',u_{i+1},\dots,u_n)\in\mathrm{lo}\,\mathbf{L}_n$ we have

$$\mathrm{or}_{\mathbf{L}_n}(u_0,\dots,u_{i-1},u,u_{i+1},\dots,u_n) = \mathrm{or}_{\mathbf{L}_n}(u_0,\dots,u_{i-1},u',u_{i+1},\dots u_n).$$

In other words any two n-simplexes of $\mathbf{L}_n$ induce the same orientation on their common facet. It is exactly the opposite situation to the case of a coherent orientation of the n-pseudomanifold. An n-primoid need not be properly orientable if the ring R has at least three elements. If R is isomorphic to $\mathbb{Z}_2$, then any n-primoid is properly oriented since $1 = -1$ in this ring.

Example 4.1. The family $\{\{1,2,6\},\{2,3,6\},\{3,4,6\},\{4,5,6\},\{1,5,6\},\{1,2,4\},\{2,3,5\},\{1,3,4\},\{2,4,5\},\{1,3,5\}\}$ is a 2-primoid on $\{1,2,3,4,5,6\}$ and there is no proper orientation of it.

Example 4.2. Let $U = I_n$. $\mathbf{L}_n^{I_n} = \{\{0,\dots,n\}\}$ is an n-primoid on I_n properly oriented by $\overline{\mathbf{L}}_n = \{(0,\dots,n)\}$.

Theorem 4.3. *Let U and U' be finite sets, let $\mathbf{L}_n$ be an n-primoid on U properly oriented by $\mathrm{or}_{\mathbf{L}_n}$ and let $g\colon U' \to U$ be an onto function. Then a family $\mathbf{L}(\mathbf{L}_n, g) = \{\{u_0,\dots,u_n\} \subset U' : g(\{u_0,\dots,u_n\}) \in \mathbf{L}_n\}$ is nonempty and it is an n-primoid on U' properly oriented by a function $\mathrm{or}_{\mathbf{L}(\mathbf{L}_n,g)}$ defined by $\mathrm{or}_{\mathbf{L}(\mathbf{L}_n,g)}(u'_0,\dots,u'_n) = \mathrm{or}_{\mathbf{L}_n}(g(u'_0),\dots,g(u'_n))$.*

Proof. We proved ([14, Theorem 3.2]) that $\mathbf{L}(\mathbf{L}_n, g)$ is an n-primoid on U'. Now we prove that it is properly oriented by $\mathrm{or}_{\mathbf{L}(\mathbf{L}_n,g)}$.

Take $(x,u'_1,\dots,u'_n),(y,u'_1,\dots,u'_n) \in \mathrm{lo}\,\mathbf{L}(\mathbf{L}_n,g)$. By the proper orientation of $\mathbf{L}_n$ we have

$$\begin{aligned}\mathrm{or}_{\mathbf{L}(\mathbf{L}_n,g)}(x,u'_1,\dots,u'_n) &= \mathrm{or}_{\mathbf{L}_n}(g(x),g(u'_1),\dots,g(u'_n))\\ &= \mathrm{or}_{\mathbf{L}_n}(g(y),g(u'_1),\dots,g(u'_n)) = \mathrm{or}_{\mathbf{L}(\mathbf{L}_n,g)}(y,u'_1,\dots,u'_n). \qquad \square\end{aligned}$$

Corollary 4.4. *Let M_d be a matroid with a fixed base $\{v_0,\dots,v_n\}$. Let r be the rank function of the matroid M_d and denote $\operatorname{span} A = \{x \in M_d : r(A \cup \{x\}) = r(A)\}$. Let $F_0 = \operatorname{span}\{v_0\}$ and $F_i = \operatorname{span}\{v_0,\dots,v_i\} \setminus \operatorname{span}\{v_0,\dots,v_{i-1}\}$ for $i \in I_n$. Observe, that $M_d = \bigcup_{i=0}^n F_i$ and $F_i \cap F_j = \emptyset$ for $i \neq j$. Let $g\colon M_d \to I_n$ be a function defined by $g(a) = i$ for $a \in F_i$. The function g is well-defined and the family $\mathbf{L}_n^{M_d} = \{\{u_0,\dots,u_n\} : g(\{u_0,\dots,u_n\}) = I_n\}$ is an n-primoid on M_d properly oriented by a function $\mathrm{or}_{\mathbf{L}_n^{M_d}}$ defined by $\mathrm{or}_{\mathbf{L}_n^{M_d}}(u_0,\dots,u_n) = \mathrm{or}_{\mathbf{L}_n^{I_n}}(g(u_0),\dots,g(u_n))$.*

Proof. From Theorem 4.3 and Example 4.2 we have $\mathbf{L}_n^{M_d} = \mathbf{L}(\mathbf{L}_n^{I_n}, g)$. $\square$

Proposition 4.5. *Let U be a finite set and $\mathbf{L}_1$ be a 1-primoid on U. There exists a function $g\colon U \to I_1$ such that $\mathbf{L}_1 = \mathbf{L}(\mathbf{L}_1^{I_1}, g)$.*

Proof. Notice that 1-complexes are graphs. Our thesis states that 1-primoids are complete bipartite graphs. Any graph is bipartite if and only if it contains no cycles of odd length. We show that 1-primoids contains no odd cycles. Assume that there is an odd cycle in 1-primoid $\mathbf{L}_1$. Let C be the shortest odd cycle in $\mathbf{L}_1$.

The cycle C cannot be a triangle in the primoid: for $\{u_1, u_2\} \in \mathbf{L}_1$ and $u \in U$ exactly one of two possibilities is true: $\{u, u_1\} \in \mathbf{L}_1$ or $\{u, u_2\} \in \mathbf{L}_1$. Consider $\{u_1, u_2\} \subset C$ and $u \in U$ such that $\{u, u_1\}$ and $\{u, u_2\}$ do not belong to the cycle C. Such vertex u exists since C is of length at least 5. Exactly one of the edges $\{u, u_1\}$ or $\{u, u_2\}$ belongs to $\mathbf{L}_1$. This edge creates two cycles in $\mathbf{L}_1$ and one of them would be an odd cycle and shorter than C.

Now we prove that $\mathbf{L}_1$ is a complete bipartite graph. Consider $\{u_1, u_2\} \in \mathbf{L}_1$. Let $X = \{u \in U : \{u, u_2\} \in \mathbf{L}_1\}$ and $Y = \{u \in U : \{u, u_1\} \in \mathbf{L}_1\}$. The sets X and Y are nonempty by definition and disjoint since $\mathbf{L}_1$ is bipartite. We show that for $u_3 \in X$ and $u_4 \in Y$ $\{u_3, u_4\} \in \mathbf{L}_1$. If $u_1 = u_3$ or $u_2 = u_4$, then it is obvious. Otherwise, by definition of the primoid, for $\{u_1, u_4\} \in \mathbf{L}_1$ and $u_3 \in U$ exactly one of the possibilities holds: $\{u_3, u_4\} \in \mathbf{L}_1$ or $\{u_1, u_3\} \in \mathbf{L}_1$. The second one is impossible because u_1, u_2, u_3 would form a triangle. $\square$

Let $\overrightarrow{U}^n_k = \{(u_0, \ldots, u_n) : \{u_0, \ldots, u_n\} \subset \{-k, \ldots, -1, 1, \ldots k\} : |u_0| \leq |u_1| \leq \ldots \leq |u_n|, u_i \cdot u_{i+1} < 0$ and if $|u_i| = |u_{i+1}|$, then $u_i < 0$ for $i \in I_{n-1}\}$ for $n, k \in \mathbb{N}$, $2k > n \geq 1$.

Example 4.6 (Bapat, [1, Lemma 4.1]). Let $U = \{-k, \ldots, -1, 1, \ldots, k\}$ for some k, $2k > n \geq 1$. We define an n-primoid $\mathbf{L}^k_n$ on U as follows: $\{u_0, \ldots, u_n\} \in \mathbf{L}^k_n$ if and only if there exists a permutation τ such that $(u_{\tau(0)}, \ldots, u_{\tau(n)}) \in \overrightarrow{U}^n_k$. The n-primoid $\mathbf{L}^k_n$ is properly oriented by a function

$$\mathrm{or}_{\mathbf{L}^k_n}(u_0, \ldots, u_n) = \begin{cases} \operatorname{sgn}\tau & \text{if } \{u_0, \ldots, u_n\} \in \mathbf{L}^k_n \text{ and } u_{\tau(0)} > 0, \\ (-1)^n \cdot \operatorname{sgn}\tau & \text{if } \{u_0, \ldots, u_n\} \in \mathbf{L}^k_n \text{ and } u_{\tau(0)} < 0, \\ 0 & \text{if } \{u_0, \ldots, u_n\} \notin \mathbf{L}^k_n, \end{cases}$$

where the permutation τ is such, that $(u_{\tau(0)}, \ldots, u_{\tau(n)}) \in \overrightarrow{U}^n_k$.

Proof. Take $\{u_0, \ldots, u_n\} \in \mathbf{L}^k_n$ and $u \in U$. Without loss of generality we assume that $(u_0, \ldots, u_n) \in \overrightarrow{U}^n_k$. If there exists such $k \in I_n$ that $u = u_k$, then $\{u_0, \ldots, u_{k-1}, u, u_{k+1}, \ldots, u_n\} \in \mathbf{L}^k_n$. Otherwise we have three cases:

Case 1. $|u| \leq |u_0|$. If u and u_0 have the same signs, then $\{u, u_1, \ldots, u_{n-1}, u_n\} \in \mathbf{L}^k_n$ and $\mathrm{or}_{\mathbf{L}^k_n}(u_0, u_1, \ldots, u_{n-1}, u_n) = \mathrm{or}_{\mathbf{L}^k_n}(u, u_1, \ldots, u_{n-1}, u_n)$. If they have opposite signs, then $\{u, u_0, u_1, \ldots, u_{n-1}\} \in \mathbf{L}^k_n$ and $\mathrm{or}_{\mathbf{L}^k_n}(u_0, u_1, \ldots, u_{n-1}, u_n) = (-1)^n \cdot \mathrm{or}_{\mathbf{L}^k_n}(u, u_0, u_1, \ldots, u_{n-1}) = \mathrm{or}_{\mathbf{L}^k_n}(u_0, u_1, \ldots, u_{n-1}, u)$.

Case 2. There exists $i \in \{0, \ldots, n-1\}$ such that $|u_i| < |u| \leq |u_{i+1}|$. If u and u_i have the same signs, then $\{u_0, \ldots, u_{i-1}, u, u_{i+1}, \ldots, u_{n-1}, u_n\} \in \mathbf{L}^k_n$ and $\mathrm{or}_{\mathbf{L}^k_n}(u_0, \ldots, u_n) = \mathrm{or}_{\mathbf{L}^k_n}(u_0, \ldots, u_{i-1}, u, u_{i+1}, \ldots, u_{n-1}, u_n)$. If they have the opposite signs, then $\{u_0, u_1, \ldots, u_i, u, u_{i+2}, \ldots, u_n\} \in \mathbf{L}^k_n$ and $\mathrm{or}_{\mathbf{L}^k_n}(u_0, \ldots, u_n) = \mathrm{or}_{\mathbf{L}^k_n}(u_0, \ldots, u_i, u, u_{i+2}, \ldots, u_{n-1}, u_n)$.

Case 3. The case $|u_n| < |u|$ is analogous to the Case 1. $\square$

From Example 4.6 and Theorem 4.3 we have the following

Corollary 4.7. *Let U be a finite set and let for some k, $2k > n \geq 1$ $g\colon U \to \{-k,\dots,-1,1,\dots,k\}$ be an onto function. We define an n-complex $\mathbf{L}_n^{g,k}$ as follows: $\{u_0,\dots,u_n\} \in \mathbf{L}_n^{g,k}$ if and only if there exists a permutation τ such that $(g(u_{\tau(0)}),\dots,g(u_{\tau(n)})) \in \overrightarrow{U}_k^n$. Then $\mathbf{L}_n^{g,k}$ is an n-primoid on the set U. The n-primoid $\mathbf{L}_n^{g,k}$ is properly oriented by a function*

$$\mathrm{or}_{\mathbf{L}_n^{g,k}}(u_0,\dots,u_n) = \begin{cases} \operatorname{sgn}\tau & \text{if } \{u_0,\dots,u_n\} \in \mathbf{L}_n^{g,k} \text{ and } g(u_{\tau(0)}) > 0, \\ (-1)^n \cdot \operatorname{sgn}\tau & \text{if } \{u_0,\dots,u_n\} \in \mathbf{L}_n^{g,k} \text{ and } g(u_{\tau(0)}) < 0, \\ 0 & \text{if } \{u_0,\dots,u_n\} \notin \mathbf{L}_n^{g,k}, \end{cases}$$

where the permutation τ is such that $(g(u_{\tau(0)}),\dots,g(u_{\tau(n)})) \in \overrightarrow{U}_k^n$.

Proof. Observe that $\mathbf{L}_n^{g,k} = \mathbf{L}(\mathbf{L}_n^k, g)$. □

Example 4.8 (Bapat, [1, Lemma 4.2]). Let $U \subset \mathbb{R}^n$ be a finite set with $|U| \geq n+1$ and let $b \in \mathbb{R}^n$ be a point, which is not a convex combination of less than $n+1$ elements of U. If the family $\mathbf{L}_n^b = \{\{u_0,\dots,u_n\} : b \in \operatorname{co}\{u_0,\dots,u_n\}\}$ is nonempty, then it is an n-primoid on U. The n-primoid $\mathbf{L}_n^b$ is properly oriented by

$$\mathrm{or}_{\mathbf{L}_n^b}(u_0,\dots,u_n) = \det\begin{pmatrix} u_0^1 & \dots & u_i^1 & \dots & u_n^1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ u_0^n & \dots & u_i^n & \dots & u_n^n \\ 1 & \dots & 1 & \dots & 1 \end{pmatrix},$$

where u_i^j is an j-th coordinate of the vector u_i.

Definition 4.9. Let $A = \{a_0,\dots,a_n\} \subset \mathbb{R}^n$ be a set of affinely independent points. Let $m_A = \sum_{a\in A} a/|A|$. We say that a function $\pi\colon \mathbf{P}(A) \to \mathbb{R}^n$ is in a *general position* if:

(a) $\pi(B) \in \operatorname{ri}\operatorname{co}\{a_i : i \in B\}$ for each $B \subset A$,

(b) $m_A \notin \operatorname{aff}\{\pi(\mathbf{D})\}$ for each $\mathbf{D} \subset \mathbf{P}(A)$ such that $|\mathbf{D}| < |A|$.

A family $\mathbf{D} \subset \mathbf{P}(A)$ is said to be π-*balanced* if $|\mathbf{D}| = |A|$, $m_A \in \operatorname{co}\{\pi(\mathbf{D})\}$ and π is in the general position.

Proposition 4.10. *The set of all π-balanced families is an n-primoid on $\mathbf{P}(A)$, we denote it by $\mathbf{L}_n^\pi$.*

Proof. This follows from Theorem 4.3 and Example 4.8, because $\mathbf{L}_n^\pi = \mathbf{L}(\mathbf{L}_n^b, \pi)$ for $b = m_A$. The primoid $\mathbf{L}_n^\pi$ is properly oriented by $\mathrm{or}_{\mathbf{L}_n^\pi}(A_0,\dots,A_n) = \mathrm{or}_{\mathbf{L}_n^b}(\pi(A_0),\dots,\pi(A_n))$ where $A_i \subset A$ for $i \in I_n$. □

Let $\mathbf{L}_n$ be an n-primoid on a set U ($|U| \geq n+1$). A function $\rho\colon \mathbf{P}(U) \to \{0,\dots,n+1\}$ defined by $\rho(A) = \max\{|A \cap T| : T \in \mathbf{L}_n\}$ for $A \in \mathbf{P}(U)$ is called *a rank function* of $\mathbf{L}_n$.

Notice that the number $k = n + 1 - \rho(A)$ is the minimal number such that there exists a set $\{u_1, \ldots, u_k\} \subset U$ and the set $A \cup \{u_1, \ldots, u_k\}$ contains a complete n-simplex.

Observe that a function defined by:

- $\rho(A) = |A|$ for $A \subset U$ is the rank function of $\mathbf{L}_n^{I_n}$ (see Example 4.2),
- $\rho(A) = |\{g(u) : u \in A\}|$ for $A \subset U$ is the rank function of the n-primoid $\mathbf{L}(\mathbf{L}_n^{I_n}, g)$ on U (see Theorem 4.3),
- $\rho_{\mathbf{L}(\mathbf{L}_n,g)}(A) = \rho_{\mathbf{L}_n}(\{g(u) : u \in A\})$ $(A \subset U)$ is the rank function of the n-primoid $\mathbf{L}(\mathbf{L}_n, g)$ for the rank function $\rho_{\mathbf{L}_n}$ of the n-primoid $\mathbf{L}_n$ (see Theorem 4.3).

The rank function ρ of $\mathbf{L}_n$ has the following

Properties 4.11. *For $A, B \in \mathbf{P}(U)$:*

(a) $\rho(B) = |B|$ *for* $B \subset A \in \mathbf{L}_n$,
(b) $\rho(A) \leq |A|$ *for* $A \subset U$,
(c) $\rho(A) \leq \rho(B)$ *for* $A \subset B$,
(d) $\rho(A \cup B) \leq \rho(A) + \rho(B)$,
(e) $\rho(\{u\}) = 1$ *for* $u \in U$.

Proof. Properties (a)–(c) follow directly from the definition of n-primoid. For the proof of property (d) see [14, Properties 4.1]. The property (e) says that any element of the set U belongs to some complete n-simplex. Consider $T \in \mathbf{L}_n$. If $u \notin T$, then by definition of the n-primoid there exists $u' \in T$ such that $T \setminus \{u'\} \cup \{u\} \in \mathbf{L}_n$. $\square$

Let ρ be the rank function of $\mathbf{L}_n$. A set $B \subset U$ is *a maximal set of the rank* k $(k \in \mathbb{N})$ if $\rho(B) = k$ and for each $u \in U \setminus B$, $\rho(B \cup \{u\}) = k + 1$. Now we define a subset sp$A \subset U$ spanned by elements of $A \subset U$ in the sense of the n-primoid $\mathbf{L}_n$. For $A \subset U$ *a spanned set by a set* A is $\mathrm{sp}A = \bigcap\{B : A \subset B \subset U$ and B is a maximal set of the rank $\rho(A)\}$.

Observe that the spanned set by a set A is defined:

- $\mathrm{sp}\, A = A$ for the n-primoid $\mathbf{L}_n^{I_n}$ (see Example 4.2),
- $\mathrm{sp}\, A = \{u : g(u) \in g(A)\}$ for the n-primoid $\mathbf{L}(\mathbf{L}_n^{I_n}, g)$ (see Theorem 4.3),
- $\mathrm{sp}_{\mathbf{L}(\mathbf{L}_n,g)} A = \{u : g(u) \in \mathrm{sp}_{\mathbf{L}_n} g(A)\}$ for the n-primoid $\mathbf{L}(\mathbf{L}_n, g)$, where $\mathrm{sp}_{\mathbf{L}_n}$ is in the sense of the n-primoid $\mathbf{L}_n$ (see Theorem 4.3).

Theorem 4.12. *Let $U \subset \mathbb{R}^n$ be a finite set, $b \in \mathbb{R}^n$ a point such that it does not belong to a convex hull of less than $n+1$ elements of U. For $A \subset U$ we have $U \cap \mathrm{cone}(b, A) \subset \mathrm{sp}_{\mathbf{L}_n^b} A$.*

Proof. Let $\rho(A) = k$. It is enough to show, that for $u \in U \cap \mathrm{cone}(b, A)$ we have $u \in \mathrm{sp}_{\mathbf{L}_n^b} A$ or equivalently u belongs to every maximal set of the rank k

containing A. Assume, there exists a maximal set D of the rank k containing A and such that $\rho(D \cup \{u\}) = k+1$. Thus there exists a set $T \subset U$ such that $|T| = n+1-(k+1) = n-k$ and $b \in \operatorname{co}(T \cup D \cup \{u\})$. Since $u \in \operatorname{cone}(b, A)$ and $A \subset D$ then $b \in \operatorname{co}(T \cup D)$. Hence $\rho(T \cup D) = n+1$ and $\rho(D) = k+1$, contradicting our assumption. $\square$

Properties 4.13. *For the n-primoid $\mathbf{L}_n$ and $A \subset U$:*

(a) $A \subset \operatorname{sp} A$,
(b) $\rho(A) = \rho(\operatorname{sp} A)$,
(c) *if a set A contains a complete n-simplex, then* $\operatorname{sp} A = U$.

For proofs see [14, Properties 4.2].

An m-simplex $T^m \in \mathbf{P}_m(U)$ $(m < n)$ is called *M-complete for $M \subset U$*, if $T^m \cup M \in \mathbf{L}_n$. In the case $M = \{x\}$ $(x \in U)$ we write x-complete instead of $\{x\}$-complete. Let $\mathbf{L}_n(x)$ denote the family of all x-complete $(n-1)$-simplexes. A function $\operatorname{or}_{\mathbf{L}_n(x)}\colon (U)^n \to \{-1, 0, 1\}$ defined by

$$\operatorname{or}_{\mathbf{L}_n(x)}(u_1, \ldots, u_n) = \operatorname{or}_{\mathbf{L}_n}(x, u_1, \ldots, u_n)$$

is an orientation of $\mathbf{L}_n(x)$ and we call it an *induced orientation by* $\operatorname{or}_{\mathbf{L}_n}$.

Theorem 4.14. *Let U be a finite set, $\mathbf{P}_n(U)$ be oriented by a representation $\overline{\mathbf{U}}_n$. Let $\mathbf{L}_n$ be a primoid on U properly oriented by a representation $\overline{\mathbf{L}}_n \subset \overline{\mathbf{U}}_n$ and let the family $\mathbf{L}_n(x) = \{T^{n-1} \in \mathbf{P}_{n-1}(U) : T^{n-1} \cup \{x\} \in \mathbf{L}_n\}$ be oriented by the induced orientation* $\operatorname{or}_{\overline{\mathbf{L}}_n(x)}$. *Then*

$$\delta_{n-1} 1_{\overline{\mathbf{L}}_n(x)} = 1_{\overline{\mathbf{L}}_n}.$$

Proof.

$$\begin{aligned}
\delta_{n-1} 1_{\overline{\mathbf{L}}_n(x)} &= \delta_{n-1} \bigoplus_{\overline{T}^{n-1} \in \overline{\mathbf{L}}_n(x)} \chi_{\overline{T}^{n-1}} = \bigoplus_{\overline{T}^{n-1} \in \overline{\mathbf{L}}_n(x)} \delta_{n-1} \chi_{\overline{T}^{n-1}} \\
&= \bigoplus_{\overline{T}^{n-1} \in \overline{\mathbf{L}}_n(x)} \; \bigoplus_{\overline{T}^n \in \overline{\mathbf{U}}_n} [\overline{T}^n : \overline{T}^{n-1}] \odot \chi_{\overline{T}^n} \\
&= \bigoplus_{\overline{T}^n \in \overline{\mathbf{U}}_n} \Bigg(\sum_{\overline{T}^{n-1} \in \overline{\mathbf{L}}_n(x)} [\overline{T}^n : \overline{T}^{n-1}] \Bigg) \odot \chi_{\overline{T}^n}.
\end{aligned}$$

Let us count $\gamma_{\overline{T}^n} = \sum_{\overline{T}^{n-1} \in \overline{\mathbf{L}}_n(x)} [\overline{T}^n : \overline{T}^{n-1}]$. If $T^n \in \mathbf{L}_n$, then by definition of the primoid there is exactly one element $u \in T^n$ such that $T^n \setminus \{u\} \cup \{x\} \in \mathbf{L}_n$ hence $T^n \setminus \{u\}$ is the only $(n-1)$-simplex from $\mathbf{L}_n(x)$ contained in T^n and $[\overline{T}^n : \overline{T}^{n-1}] = 1$ by definition of $\operatorname{or}_{\mathbf{L}_n(x)}$. Thus in this case $\gamma_{\overline{T}^n} = 1$.

Consider now $T^n \notin \mathbf{L}_n$. By the definition of the primoid, the set $T^n \cup \{x\}$ contains either no complete n-simplexes or exactly two of them. If $T^n \cup \{x\}$

contains no complete n-simplexes, then T^n contains no x-complete $(n-1)$-simplexes and thus $\gamma_{\overline{T}^n} = 0$. If $T^n \cup \{x\}$ contains two complete n-simplexes, then T^n contains two x-complete $(n-1)$-simplexes say T_1^{n-1}, T_2^{n-1}. In this case we have $\gamma_{\overline{T}^n} = [\overline{T}^n : \overline{T}_1^{n-1}] + [\overline{T}^n : \overline{T}_2^{n-1}]$. We will show that $[\overline{T}^n : \overline{T}_1^{n-1}] = -[\overline{T}^n : \overline{T}_2^{n-1}]$. Let $\overline{T}_1^{n-1} = (u_1, \ldots, u_n)$, $\overline{T}_2^{n-1} = (w_1, \ldots, w_n)$ and $u_0 \in T^n \setminus T_1^{n-1}$ and $T_2^{n-1} = T^n \setminus \{u_i\}$ for some $i \in \{1, \ldots, n\}$. Thus $\{w_1, \ldots, w_n\} = \{u_1, \ldots, u_{i-1}, u_0, u_{i+1}, \ldots, u_n\}$. Since orientations of T_1^{n-1} and T_2^{n-1} are induced orientations by $\overline{T}^n$ and $\mathbf{L}_n$ is properly oriented by $\mathrm{or}_{\overline{\mathbf{U}}_n}$ we have

$$\begin{aligned}\mathrm{or}_{\mathbf{L}_n(x)}(w_1, \ldots, w_n) = 1 &= \mathrm{or}_{\mathbf{L}_n(x)}(u_1, \ldots, u_n) = \mathrm{or}_{\overline{\mathbf{U}}_n}(x, u_1, \ldots, u_n)\\ &= \mathrm{or}_{\overline{\mathbf{U}}_n}(x, u_1, \ldots, u_{i-1}, u_0, u_{i+1}, \ldots, u_n)\\ &= \mathrm{or}_{\mathbf{L}_n(x)}(u_1, \ldots, u_{i-1}, u_0, u_{i+1}, \ldots, u_n).\end{aligned}$$

So $\overline{T}_2^{n-1}$ is even permutation of $(u_1, \ldots, u_{i-1}, u_0, u_{i+1}, \ldots, u_n)$ and

$$\begin{aligned}[\overline{T}^n : \overline{T}_1^{n-1}] &= \mathrm{or}_{\overline{\mathbf{U}}_n}(u_0, u_1, \ldots, u_n)\\ &= -\mathrm{or}_{\overline{\mathbf{U}}_n}(u_i, u_1, \ldots, u_{i-1}, u_0, u_{i+1}, \ldots, u_n)\\ &= -\mathrm{or}_{\overline{\mathbf{U}}_n}(u_i, w_1, \ldots, w_n) = -[\overline{T}^n : \overline{T}_2^{n-1}].\end{aligned}$$

Hence $\gamma_{\overline{T}^n} = 1$ if and only if $T^n \in \mathbf{L}_n$. □

5. Labelling

Let V, U be finite sets, $\mathbf{C}^n$ be an n-complex on V oriented by some representation $\overline{\mathbf{C}}^n$ and let $\mathbf{C}^{n-1}$ be a complex consisting of all facets of n-simplexes of $\mathbf{C}^n$ oriented by some representation $\overline{\mathbf{C}}^{n-1}$. Let $\mathbf{L}_n$ be an n-primoid on a set U properly oriented by some representation $\overline{\mathbf{L}}_n$. Let $x \in U$ and let $\mathbf{L}_n(x) = \{T^{n-1} \in \mathbf{P}_{n-1}(U) : T^{n-1} \cup \{x\} \in \mathbf{L}_n\}$ be oriented by $\overline{\mathbf{L}}_n(x)$, where $\mathrm{or}_{\overline{\mathbf{L}}_n(x)}$ is an induced orientation by $\mathrm{or}_{\overline{\mathbf{L}}_n}$. Let $l\colon V \to U$ be a labelling.

An n-simplex $S^n = \{v_0, \ldots, v_n\} \in \mathbf{C}^n$ is called *completely labelled* (c.l. n-simplex for short) if $l(S^n) \in \mathbf{L}_n$. A *signum of a c.l. n-simplex S^n* (denoted by $\operatorname{sign} S^n$) is an element of the ring R which is equal to $\operatorname{sign} S^n = \mathrm{or}_{\overline{\mathbf{C}}^n}(v_0, \ldots, v_n) \cdot \mathrm{or}_{\overline{\mathbf{L}}_n}(l(v_0), \ldots, l(v_n))$.

For $x \in U$, an $(n-1)$-simplex $S^{n-1} = \{v_1, \ldots, v_n\} \in \mathbf{C}^{n-1}$ is called an *x-completely labelled $(n-1)$-simplex* (x-c.l. $(n-1)$-simplex for short) if $l(S^{n-1}) \in \mathbf{L}_n(x)$. A *signum of an x-c.l. $(n-1)$-simplex S^{n-1}* is an element of the ring R which is equal to $\operatorname{sign} S^{n-1} = \mathrm{or}_{\overline{\mathbf{C}}^{n-1}}(v_1, \ldots, v_n) \cdot \mathrm{or}_{\overline{\mathbf{L}}_n(x)}(l(v_1), \ldots, l(v_n))$.

For $x \in U$, $S^n \in \mathbf{C}^n$ and $v \in S^n$ a pair (S^n, v) is called an *x-completely labelled facet* (x-c.l. facet for short) if $l(S^n \setminus \{v\}) \in \mathbf{L}_n(x)$. A *signum of an x-c.l. facet* is an element of the ring R which is equal to $\operatorname{sign}(S^n, v) = [\overline{S}^n, \overline{S}^{n-1}] \cdot \mathrm{or}_{\overline{\mathbf{L}}_n(x)} \overrightarrow{l}_{n-1}(\overline{S}^{n-1})$, where $\overline{S}^n \in \overline{\mathbf{C}}^n$ and $\overline{S}^{n-1} = \overline{S^n \setminus \{v\}} \in \overline{\mathbf{C}}^{n-1}$.

Observe that the definition of the signum of an c.l. n-simplex S^n, an x-c.l. $(n-1)$-simplex S^{n-1} and an x-c.l. facet do not depend on the linear order of elements $\{v_0, \dots, v_n\}$ of S^n, $\{v_1, \dots, v_n\}$ of S^{n-1}, $\{v_0, \dots, v_n\}$ of S^n, respectively.

A c.l. n-simplex, an x-c.l. $(n-1)$-simplex and an x-c.l. facet is called a *positive c.l. n-simplex, an x-c.l. $(n-1)$-simplex and an x-c.l. facet* (a *negative c.l. n-simplex, an x-c.l. $(n-1)$-simplex and an x-c.l. facet*) if $\operatorname{sign} S^n = 1$, $\operatorname{sign} S^{n-1} = 1$, $\operatorname{sign}(S^n, v) = 1$ ($\operatorname{sign} S^n = -1$, $\operatorname{sign} S^{n-1} = -1$, $\operatorname{sign}(S^n, v) = -1$), respectively.

Definitions of the c.l. labelled n-simplex and the x-c.l. facet are equivalent to the definitions of the "matched pair" and the "unmatched pair" given by Bapat ([1, Definition 2.2]) and Todd [32], [33], respectively. The signum of the c.l. n-simplex S^n ($(n-1)$-simplex S^{n-1}) informs us whether labelling l transforms S^n onto $l(S^n)$ (S^{n-1} onto $l(S^{n-1})$) preserving or reversing the orientation.

Now we present a new proof of the Bapat theorem. This theorem is a generalization of the Todd theorem ([33, Theorem 2.6]).

Theorem 5.1 (Bapat, [1, Theorem 2.6]). *Let V, U be finite sets, $\mathbf{C}^n$ be a nonempty n-complex on a set V oriented by $\overline{\mathbf{C}}^n$ and let $\mathbf{C}^{n-1}$ be a complex consisting of all facets of n-simplexes of $\mathbf{C}^n$ oriented by some representation $\overline{\mathbf{C}}^{n-1}$. Let $\mathbf{L}_n$ be an n-primoid on a set U properly oriented by $\overline{\mathbf{L}}_n$ and $l\colon V \to U$, $x \in U$. Let α^+ and α^- (β^+ and β^-) denote the number of c.l. n-simplexes in $\mathbf{C}^n$ (x-c.l. facets in $\mathbf{C}^n$) that are positive and negative, respectively. Then $\alpha^+ - \alpha^- = \beta^+ - \beta^-$.*

To prove his theorem Bapat used elementary methods concerning directed graphs. Similar methods were used by Ky Fan in [6]. We present another proof to show how this theorem is related to theorems of the previous section.

Proof of Theorem 5.1. In the case the ring R is equal to $\mathbb{Z}$ the n-index of a labelling l for $\overline{\mathbf{C}}^n$ and $\overline{\mathbf{L}}_n$: $\operatorname{ind}_n l(1_{\overline{\mathbf{C}}^n}, 1_{\overline{\mathbf{L}}_n}) = 1_{\overline{\mathbf{C}}^n} \bullet_n \widetilde{l}_{\overline{\mathbf{C}}^n}(1_{\overline{\mathbf{L}}_n})$ is equal to $\alpha^+ - \alpha^-$ since every positive (negative) c.l. n-simplex appears as 1 (-1) in the sum $\operatorname{ind}_n l(1_{\overline{\mathbf{C}}^n}, 1_{\overline{\mathbf{L}}_n}) = \sum_{\overline{S}^n \in \overline{\mathbf{C}}^n} \operatorname{or}_{\overline{\mathbf{L}}_n} \overrightarrow{l_n}(\overline{S}^n)$.

Let $\mathbf{C}^{n-1}$ be an $(n-1)$-complex consisting of all facets of all n-simplexes belonging to $\mathbf{C}^n$ oriented by some representation $\overline{\mathbf{C}}^{n-1}$. An $(n-1)$-index of the function l for $\partial_n 1_{\overline{\mathbf{C}}^n}$ and $1_{\overline{\mathbf{L}}_n(x)}$ is equal to

$$\operatorname{ind}_{n-1} l(\partial_n 1_{\overline{\mathbf{C}}^n}, 1_{\overline{\mathbf{L}}_n(x)}) = \partial_n 1_{\overline{\mathbf{C}}^n} \bullet_{n-1} \widetilde{l}_{\overline{\mathbf{C}}^{n-1}}(1_{\overline{\mathbf{L}}_n(x)}) = \beta^+ - \beta^-$$

since every positive (negative) x-c.l. facet appears as 1 (-1) in the sum

$$\operatorname{ind}_{n-1} l(\partial_n 1_{\overline{\mathbf{C}}^n}, 1_{\overline{\mathbf{L}}_n(x)}) = \sum_{\overline{S}^{n-1} \in \overline{\mathbf{V}}_{n-1}} \left(\sum_{\overline{S}^n \in \overline{\mathbf{C}}^n} [\overline{S}^n : \overline{S}^{n-1}] \right) \odot \operatorname{or}_{\overline{\mathbf{L}}_n(x)} \overrightarrow{l}_{n-1}(\overline{S}^{n-1}).$$

From Theorem 4.14 and from Theorem 3.9 we have

$$\begin{aligned}\alpha^+ - \alpha^- = \operatorname{ind}_n l(1_{\overline{\mathbf{C}}^n}, 1_{\overline{\mathbf{L}}_n}) &= \operatorname{ind}_n l(1_{\overline{\mathbf{C}}^n}, \delta^U_{n-1} 1_{\overline{\mathbf{L}}_n(x)}) \\ &= \operatorname{ind}_{n-1} l(\partial_n 1_{\overline{\mathbf{C}}^n}, 1_{\overline{\mathbf{L}}_n(x)}) = \beta^+ - \beta^-. \qquad \square\end{aligned}$$

In the case the ring R is equal to $\mathbb{Z}_2$ Theorem 5.1 reduces to non-oriented version:

Theorem 5.2 (Idzik and Junosza-Szaniawski, [14, Theorem 5.3]). *Let V, U be finite sets, $\mathbf{C}^n$ be an n-complex on V. Let $\mathbf{L}_n$ be an n-primoid on U. Let $l\colon V \to U$ be a fixed labelling and let $x \in U$ be a fixed element. Then the number of c.l. simplexes is equal to the number of x-c.l. facets modulo* 2.

An $(n-1)$-simplex S^{n-1} is a *boundary* $(n-1)$-*simplex* of an n-complex $\mathbf{C}^n$ if there is exactly one n-simplex $S^n \in \mathbf{C}^n$ such that $S^{n-1} \subset S^n$. For an n-pseudomanifold $\mathbf{C}^n$ we denote $\partial_n \mathbf{C}^n$ the $(n-1)$-complex consisting of all boundary $(n-1)$-simplexes of n-simplexes of $\mathbf{C}^n$. An orientation of $\partial\mathbf{C}^n$ is called *an induced orientation by* $\mathrm{or}_{\overline{\mathbf{C}}^n}$ if every $(n-1)$-simplex is oriented by an induced orientation from the unique n-simplex containing it. Observe that $\partial_n 1_{\mathbf{C}^n} = 1_{\partial_n \mathbf{C}^n}$. Now applying Theorem 5.1 to an n-complex which is an n-pseudomanifold we get

Theorem 5.3 (Bapat, [1, Theorem 3.3]). *Let V, U be finite sets, $\mathbf{C}^n$ be a nonempty n-pseudomanifold on V, coherently oriented by $\overline{\mathbf{C}}^n$. Let $\mathbf{L}_n$ be an n-primoid on a set U, $\mathbf{L}_n$ be properly oriented by $\overline{\mathbf{L}}_n$, $l\colon V \to U$, $x \in U$. Let α^+ and α^- (γ^+ and γ^-) denote the number of c.l. n-simplexes in $\mathbf{C}^n$ (x-c.l. boundary $(n-1)$-simplexes in $\partial_n \mathbf{C}^n$) which are positive and negative, respectively. Then $\alpha^+ - \alpha^- = \gamma^+ - \gamma^-$.*

Proof. An n-index $\operatorname{ind}_n l(1_{\overline{\mathbf{C}}^n}, 1_{\overline{\mathbf{L}}_n})$ is equal to $\alpha^+ - \alpha^-$ and $(n-1)$-index $\operatorname{ind}_{n-1} l(\partial_n 1_{\overline{\mathbf{C}}^n}, 1_{\overline{\mathbf{L}}_n(x)})$ is equal to $\gamma^+ - \gamma^-$. From the general Sperner lemma (Theorem 3.9) and Theorem 4.14 we have

$$\begin{aligned}\alpha^+ - \alpha^- = \operatorname{ind}_n l(1_{\overline{\mathbf{C}}^n}, 1_{\overline{\mathbf{L}}_n}) &= \operatorname{ind}_n l(1_{\overline{\mathbf{C}}^n}, \delta^U_{n-1} 1_{\overline{\mathbf{L}}_n(x)}) \\ &= \operatorname{ind}_n l(\partial_n 1_{\overline{\mathbf{C}}^n}, 1_{\overline{\mathbf{L}}_n(x)}) = \gamma^+ - \gamma^-. \qquad \square\end{aligned}$$

In the case the ring R is equal to $\mathbb{Z}_2$, Theorem 5.3 reduces to a non-oriented version:

Theorem 5.4 (Idzik and Junosza-Szaniawski [14, Theorem 6.1]). *Let V, U be finite sets, $\mathbf{C}^n$ be an n-pseudomanifold on V, $\mathbf{L}_n$ be an n-primoid on the set U and $x \in U$ be a fixed element. Let $l\colon V \to U$ be a labelling. Then the number of c.l. simplexes is equal to the number of boundary x-c.l. $(n-1)$-simplexes modulo* 2.

THEOREM 5.5 (Lindström [22], see also Kryński [18, Theorem 6]). *Let V be a finite set, $\mathbf{P}_n(V)$ be oriented by $\overline{\mathbf{V}}_n$ and ℓ^n be an n-chain on V such that $\partial_n \ell^n = 0$, M_d be a matroid on U of the rank $n+1$ and let $l\colon V \to U$ be a labelling. If there is an n-simplex $S^n \in \mathbf{P}_n(V)$, such that $\ell^n(\overline{S}^n) \neq 0$ and $l(S^n)$ is a base of the matroid M_d, then there is another n-simplex $S_2^n \in \mathbf{P}_n(V)$ such that $\ell^n(\overline{S}_2^n) \neq 0$.*

PROOF. From Corollary 4.4, Theorem 3.9 and Theorem 4.14 we have

$$\mathrm{ind}_n l(\ell^n, 1_{\mathbf{L}_n^{M_d}}) = \mathrm{ind}_{n-1} l(\partial_n \ell^n, 1_{\mathbf{L}_n^{M_d}(x)}) = 0. \qquad \square$$

For an n-pseudomanifold $\mathbf{C}^n$ on V, coherently oriented by $\overline{\mathbf{C}}^n$, $(k \geq n)$ and a labelling $l\colon V \to \{-k, \dots, -1, 1, \dots, k\}$ let $\alpha^+(j_0, \dots, j_n)$ $(\alpha^-(j_0, \dots, j_n))$ denote the number of elements $(v_0, \dots, v_n)$ in $\mathrm{lo}\,\mathbf{C}^n$ such that $l(v_i) = j_i$ for $i \in I_n$ and $\mathrm{or}_{\overline{\mathbf{C}}^n}(v_0, \dots, v_n) = 1$ $(\mathrm{or}_{\overline{\mathbf{C}}^n}(v_0, \dots, v_n) = -1)$ and let $\beta^+(j_0, \dots, j_{n-1})$ $(\beta^-(j_0, \dots, j_{n-1}))$ denote the number of elements $(v_0, \dots, v_{n-1})$ in $\mathrm{lo}\,\partial\mathbf{C}^n$ such that $l(v_i) = j_i$ for $i \in I_{n-1}$ and $\mathrm{or}_{\overline{\mathbf{C}}^n}(v_0, \dots, v_{n-1}) = 1$ $(\mathrm{or}_{\overline{\mathbf{C}}^n}(v_0, \dots, v_{n-1}) = -1)$. Let $\alpha(j_0, \dots, j_n) = \alpha^+(j_0, \dots, j_n) - \alpha^-(j_0, \dots, j_n)$ and $\beta(j_0, \dots, j_{n-1}) = \beta^+(j_0, \dots, j_{n-1}) - \beta^-(j_0, \dots, j_{n-1})$.

If we apply Theorem 5.3 to the primoid $\mathbf{L}_n^k$ (see Example 4.6), then we get

THEOREM 5.6 (Fan [6, Theorem 1]). *Let $\mathbf{C}^n$ be a coherently oriented n-pseudomanifold on V $(k \geq n)$ and let a labelling $l\colon V \to \{-k, \dots, -1, 1, \dots, k\}$ satisfy the condition $l(v) + l(v') \neq 0$ for v and v' belonging to some n-simplex of $\mathbf{C}^n$. Then we have*

$$\sum_{0<k_0<\dots<k_n} (\alpha(-k_0, k_1, -k_2, k_3, \dots, (-1)^{n+1}k_n) + \alpha(k_0, -k_1, k_2, -k_3, \dots, (-1)^n k_n))$$
$$= \sum_{0<k_0<\dots<k_{n-1}} \beta(k_0, -k_1, k_2, -k_3, \dots, (-1)^{n-1}k_{n-1}).$$

COROLLARY 5.7. *Applying Theorem 5.3 to the n-primoid:*

(a) $\mathbf{L}(\mathbf{L}_n^{I_n}, g)$, *we get an oriented version of the Gould and Tolle theorem* ([8, Theorem 5.2.5]),
(b) $\mathbf{L}_n^{M_d}$, *we get an oriented version of the Lovász theorem* [23], (*see also Kryński* [18, Theorem 3]).

6. Case of the geometric simplex

For a real number $r \in \mathbb{R}$ we define:

$$\mathrm{signum}\, r = \begin{cases} 1 & \text{for } r > 0, \\ -1 & \text{for } r < 0, \\ 0 & \text{for } r = 0. \end{cases}$$

Let $\{d_0, \dots, d_n\}$ be a fixed set of affinely independent vectors in $\mathbb{R}^n$, such that

$$\det\begin{pmatrix} d_0^1 & \dots & d_i^1 & \dots & d_n^1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ d_0^n & \dots & d_i^n & \dots & d_n^n \\ 1 & \dots & 1 & \dots & 1 \end{pmatrix} > 0,$$

where d_i^j is the j-th coordinate of the vector d_i. For $M \subset I_n$ we denote $\Delta^M = \operatorname{co}\{d_i : i \in M\}$. Let Tr be a *triangulation* of Δ^{I_n}. Let $\operatorname{Tr}(\Delta^M)$ $(M \subset I_n)$ be the *induced triangulation of the face* Δ^M, i.e. the family of $(|M|-1)$-dimensional simplexes $\sigma \cap \Delta^M$ for $\sigma \in \operatorname{Tr}$.

For every simplex σ in Tr, let $V(\sigma)$ denote the set of its vertices and $V = \bigcup_{\sigma\in\operatorname{Tr}} V(\sigma)$. The family $\mathbf{C}^n = \{V(\sigma) : \sigma \in \operatorname{Tr}\}$ is a pseudomanifold on V. An orientation $\operatorname{or}_{\overline{\mathbf{C}}^n}$ defined by

$$\operatorname{or}_{\overline{\mathbf{C}}^n}(v_0, \dots, v_n) = \operatorname{signum}\det\begin{pmatrix} v_0^1 & \dots & v_i^1 & \dots & v_n^1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ v_0^n & \dots & v_i^n & \dots & v_n^n \\ 1 & \dots & 1 & \dots & 1 \end{pmatrix},$$

where v_i^j is the j-th coordinate of the vector v_i, is the coherent orientation since for $\{v, v_1, \dots, v_n\}, \{v', v_1, \dots, v_n\} \in \mathbf{C}^n$ vertices v and v' lies on the opposite sides of the hyperplane $\operatorname{aff}\{v_1, \dots, v_n\}$. Thus

$$\operatorname{or}_{\overline{\mathbf{C}}^n}(v, v_1, \dots, v_n) = -\operatorname{or}_{\overline{\mathbf{C}}^n}(v', v_1, \dots, v_n).$$

This orientation we call a *geometric orientation* of the pseudomanifold $\mathbf{C}^n$.

We will say that a geometric simplex σ is completely labelled (c.l. simplex for short) or x-completely labelled (x-c.l. simplex) if the set of its vertices $V(\sigma)$ is the c.l. n-simplex or the x-c.l. $(n-1)$-simplex, respectively.

THEOREM 6.1. *Let* Tr *be a triangulation of the simplex* Δ^{I_n}*, the n-complex* $\mathbf{C}^n = \{V(\sigma) : \sigma \in \operatorname{Tr}\}$ *be oriented by the geometric orientation,* $V = \bigcup_{\sigma\in\operatorname{Tr}} V(\sigma)$, $\mathbf{L}_n$ *be an n-primoid on a set U, properly oriented by* $\overline{\mathbf{L}}_n$, $(u_0, \dots, u_n) \in \operatorname{lo}\mathbf{L}_n$ *and* $l\colon V \to U$*. If for* $M \subsetneq I_n$*, a simplex* $\sigma \subset \Delta^M$ *is not the* $\{u_i : i \notin M\}$*-completely labelled simplex, then* $\operatorname{ind}_n l(1_{\overline{\mathbf{C}}^n}, 1_{\overline{\mathbf{L}}_n}) = (-1)^n \cdot \operatorname{or}_{\overline{\mathbf{L}}_n}(u_0, \dots, u_n)$.

PROOF. We embed the simplex $\Delta^{I_n} = \operatorname{co}\{d_0, \dots, d_n\}$ in a larger simplex using the Scarf method [25, p. 192]. Without loss of generality we may assume that $0 \in \operatorname{ri}\Delta^{I_n}$. Let $\widetilde{d}_i = -a \cdot d_i$ for $i \in I_n$, where $a > 0$ is so large that $\Delta^{I_n} \subset \operatorname{ri}\operatorname{co}\{\widetilde{d}_0, \dots, \widetilde{d}_n\}$. Let us denote $\widetilde{\Delta}^{I_n} = \operatorname{co}\{\widetilde{d}_0, \dots, \widetilde{d}_n\}$. For every $i \in I_n$, $\widetilde{d}_i$ and d_i lie on the two different sides of the hyperplane $\operatorname{aff}\{d_0, \dots, d_{i-1}, d_{i+1}, \dots, d_n\}$.

Observe that

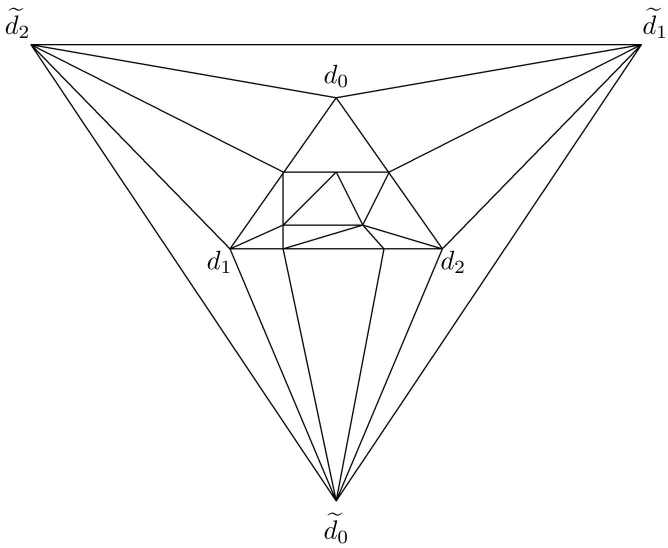
$$\begin{aligned}
\mathrm{or}_{\overline{\mathbf{C}}^n}(\widetilde{d}_0,\dots,\widetilde{d}_n) &= \operatorname{signum}\det\begin{pmatrix} \widetilde{d}_0^1 & \dots & \widetilde{d}_i^1 & \dots & \widetilde{d}_n^1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \widetilde{d}_0^n & \dots & \widetilde{d}_i^n & \dots & \widetilde{d}_n^n \\ 1 & \dots & 1 & \dots & 1 \end{pmatrix} \\
&= \operatorname{signum}\det\begin{pmatrix} -ad_0^1 & \dots & -ad_i^1 & \dots & -ad_n^1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ -ad_0^n & \dots & -ad_i^n & \dots & -ad_n^n \\ 1 & \dots & 1 & \dots & 1 \end{pmatrix} \\
&= (-1)^n \operatorname{signum}\left(a^n \det\begin{pmatrix} d_0^1 & \dots & d_i^1 & \dots & d_n^1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ d_0^n & \dots & d_i^n & \dots & d_n^n \\ 1 & \dots & 1 & \dots & 1 \end{pmatrix}\right) = (-1)^n
\end{aligned}$$

($\mathrm{or}_{\overline{\mathbf{C}}^n}$ is the geometric orientation).

Now we extend the triangulation of Δ^{I_n} by joining every vertex $v \in V \cap \Delta^{I_n \setminus \{i\}}$ with the vertex $\widetilde{d}_i$ and we get a triangulation of $\widetilde{\Delta}^{I_n}$:

$$\widetilde{\mathrm{Tr}} = \mathrm{Tr} \cup \bigcup_{M \subsetneq I_n} \bigcup_{\sigma \in \mathrm{Tr}(\Delta^M)} \mathrm{co}\,(V(\sigma) \cup \{\widetilde{d}_i : i \notin M\}).$$

For the case $n = 2$ see Picture 1.



PICTURE 1

Let $\mathbf{C}^n_\sim = \{V(\sigma) : \sigma \in \widetilde{\mathrm{Tr}}\}$. We define a labelling $\widetilde{l}$ as an extension of l on $\bigcup_{\sigma\in\widetilde{\mathrm{Tr}}} V(\sigma)$ by $\widetilde{l}(\widetilde{d}_i) = u_i$ for all $i \in I_n$.

We prove that for this new triangulation $\widetilde{\mathrm{Tr}}$ of $\widetilde{\Delta}^{I_n}$ and the labelling $\widetilde{l}$ there is no c.l. n-simplexes, which do not belong to $\mathbf{C}^n$. Consider an n-simplex $\sigma \in \widetilde{\mathrm{Tr}} \setminus \mathrm{Tr}$. Let $S = V(\sigma)$ and let Δ^M $(M \subset I_n)$ be the smallest face of Δ^{I_n} containing the set $\Delta^{I_n} \cap \sigma$. The set S is of the form $S = \{w_i \in \Delta^M : i \in M\} \cup \{\widetilde{d}_i : i \notin M\}$. By our assumption $S \cap \Delta^M$ is not $\{u_i : i \notin M\}$-completely labelled and thus S is not a c.l. n-simplex. Hence $\mathrm{ind}_n\widetilde{l}(1_{\overline{\mathbf{C}}^n_\sim}, 1_{\overline{\mathbf{L}}_n}) = \mathrm{ind}_n l(1_{\overline{\mathbf{C}}^n}, 1_{\overline{\mathbf{L}}_n})$.

By Theorem 5.3, applied to $\mathbf{C}^n_\sim$ for $x = u_0$ we know that $\mathrm{ind}_n\widetilde{l}(1_{\overline{\mathbf{C}}^n_\sim}, 1_{\overline{\mathbf{L}}_n})$ is equal to $\mathrm{ind}_{n-1}\widetilde{l}(\partial 1_{\overline{\mathbf{C}}^n_\sim}, 1_{\overline{\mathbf{L}}_n}(u_0))$ and $\mathrm{ind}_{n-1}\widetilde{l}(\partial 1_{\overline{\mathbf{C}}^n_\sim}, 1_{\overline{\mathbf{L}}_n(u_0)})$ is equal to the number of positive boundary u_0-c.l. $(n-1)$-simplexes minus the number of negative boundary u_0-c.l. $(n-1)$-simplexes. Observe that the only u_0-c.l. $(n-1)$-simplex on the boundary of $\widetilde{\Delta}^{I_n}$ is $\{\widetilde{d}_1, \dots, \widetilde{d}_n\}$. The orientation of $\{\widetilde{d}_1, \dots, \widetilde{d}_n\}$ is the induced orientation by the geometric orientation and thus $\mathrm{sign}(\widetilde{d}_1, \dots, \widetilde{d}_n) = (-1)^n$. Hence $\mathrm{ind}_n l(1_{\overline{\mathbf{C}}^n}, 1_{\overline{\mathbf{L}}_n}) = (-1)^n \cdot \mathrm{or}_{\overline{\mathbf{L}}_n}(u_0, \dots, u_n)$. $\square$

Let $\mathbf{L}_n$ be an n-primoid on a set U, properly oriented by $\overline{\mathbf{L}}_n$ and ρ be the rank of $\mathbf{L}_n$.

We say that a labelling $l\colon V \to U$ satisfies a *ρ-boundary condition for* $(u_0, \dots, u_n) \in \mathrm{lo}\,\mathbf{L}_n$ if:

- $l(d_i) = u_i$ for $i \in I_n$,
- $\rho(l(V \cap \Delta^M)) = |M|$ for $M \subset I_n$.

Theorem 6.2. *Let* Tr *be a triangulation of* Δ^{I_n}, *the n-complex* $\mathbf{C}^n = \{V(\sigma) : \sigma \in \mathrm{Tr}\}$ *be oriented by the geometric orientation,* $V = \bigcup_{\sigma\in\mathrm{Tr}} V(\sigma)$ *and* $\mathbf{L}_n$ *be an n-primoid on a set U, properly oriented by* $\overline{\mathbf{L}}_n$ *and let* $(u_0, \dots, u_n) \in \mathrm{lo}\,\mathbf{L}_n$. *If a labelling $l\colon V \to U$ satisfies the ρ-boundary condition for $(u_0, \dots, u_n)$, then* $\mathrm{ind}_n l(1_{\overline{\mathbf{C}}^n}, 1_{\overline{\mathbf{L}}_n}) = \mathrm{or}_{\overline{\mathbf{L}}_n}(u_0, \dots, u_n)$.

Proof. It is sufficient to show that l satisfies the conditions of Theorem 6.1 for $(u_1, \dots u_n, u_0)$, because

$$\mathrm{ind}_n l(1_{\overline{\mathbf{C}}^n}, 1_{\overline{\mathbf{L}}_n}) = (-1)^n \mathrm{or}_{\overline{\mathbf{L}}_n}(u_1, \dots, u_n, u_0) = \mathrm{or}_{\overline{\mathbf{L}}_n}(u_0, \dots, u_n).$$

Assume that there exists a $\{u_{i+1} : i \notin M\}$-c.l. simplex $\sigma \subset \Delta^M$ for some $M \subsetneq I_n$ $(u_{n+1} = u_0)$. Hence $l(V(\sigma)) \cup \{u_{i+1} : i \notin M\} \in \mathbf{L}_n$ $(u_{n+1} = u_0)$ and $\rho(l(V(\sigma))) = |M|$. There exists $j \notin M$ such that $j +_{\bmod (n+1)} 1 \in M$ and thus

$$\rho(l(V(\sigma)) \cup \{u_{j+1}\}) = |M| + 1.$$

But $l(d_{j+1}) = u_{j+1}$, $l(V(\sigma)) \cup \{u_{j+1}\} \subset l(V \cap \Delta^M)$ and by Property 4.11(c)

$$\rho(l(V \cap \Delta^M)) \geq |M| + 1.$$

This contradicts the ρ-boundary condition. $\square$

We say that $l: V \to U$ satisfies *sp-boundary condition for* $(u_0, \dots, u_n) \in \operatorname{lo} \mathbf{L}_n$ if:

- $l(d_i) = u_i$ for $i \in I_n$,
- for every $M \subset I_n$ and for every $v \in (V \cap \Delta^M)$, $l(v) \in sp\{u_i : i \in M\}$.

Theorem 6.3. *Let* Tr *be a triangulation of* Δ^{I_n}*, the* n*-complex* $\mathbf{C}^n = \{V(\sigma) : \sigma \in \mathrm{Tr}\}$ *be oriented by the geometric orientation,* $V = \bigcup_{\sigma \in \mathrm{Tr}} V(\sigma)$ *and* $\mathbf{L}_n$ *be an* n*-primoid on a set* U*, properly oriented by* $\overline{\mathbf{L}}_n$ *and let* $(u_0, \dots, u_n) \in \operatorname{lo} \mathbf{L}_n$*. If a labelling* $l: V \to U$ *satisfies the* sp*-boundary condition for* $(u_0, \dots u_n)$*, then* $\operatorname{ind}_n l(1_{\overline{\mathbf{C}}^n}, 1_{\overline{\mathbf{L}}_n}) = \operatorname{or}_{\overline{\mathbf{L}}_n}(u_0, \dots, u_n)$.

Proof. By Property 4.13(b) we have $\rho(l(\Delta^M \cap V)) = \rho(\mathrm{sp}\{u_i : i \in M\}) = \rho(\{u_i : i \in M\}) = |M|$ for every $M \subset I_n$ and the conditions of Theorem 6.2 are satisfied. $\square$

Corollary 6.4. *Applying Theorem* 6.3 *to the* n*-primoid:*

(a) $\mathbf{L}_n^{I_n}$*, we get an oriented version of the Sperner lemma* [29],
(b) $\mathbf{L}_n^{M_d}$*, we get an oriented version of the Lovász corollary* [23], (*see also Kryński* [18, Theorem 3]),
(c) $\mathbf{L}_n^{\pi}$*, we get an oriented version of the Shapley lemma* ([26, Lemma 7.2]).

From Theorem 6.3 applied to the n-primoid $\mathbf{L}_n^b$ and from Theorem 4.12 we get

Theorem 6.5. *Let* Tr *be a triangulation of* Δ^{I_n}*, the* n*-complex* $\mathbf{C}^n = \{V(\sigma) : \sigma \in \mathrm{Tr}\}$ *be oriented by the geometric orientation* $\operatorname{or}_{\overline{\mathbf{C}}^n}$*,* $V = \bigcup_{\sigma \in \mathrm{Tr}} V(\sigma)$*,* $U \subset \mathbb{R}^n$ *be a finite set,* $b \in \operatorname{ri} \Delta^{I_n}$ *be a point, which does not belong to the convex hull of less than* $n+1$ *elements of the set* U*. Let* $l: V \to U$ *be a labelling such that for* $M \subset I_n$*, if* $v \in V \cap \Delta^M$*, then* $l(v) \in \operatorname{cone}(b, \{d_i : i \in M\})$*. Then* $\operatorname{ind}_n l(1_{\overline{\mathbf{C}}^n}, 1_{\overline{\mathbf{L}}_n^b}) = 1$.

We say that l satisfies *dual sp-boundary condition* (*dsp-boundary condition for short*) for $(u_0, \dots, u_n) \in \operatorname{lo} \mathbf{L}_n$, if for every $M \subsetneq I_n$ and for every $v \in V \cap \operatorname{ri} \Delta^M$, $l(v) \in \mathrm{sp}\{u_i : i \notin M\}$.

Theorem 6.6. *Let* Tr *be a triangulation of* Δ^{I_n}*, the* n*-complex* $\mathbf{C}^n = \{V(\sigma) : \sigma \in \mathrm{Tr}\}$ *be oriented by the geometric orientation,* $V = \bigcup_{\sigma \in \mathrm{Tr}} V(\sigma)$*, for* $M \subsetneq I_n$ *and* $\sigma \subset \Delta^M$ *there exists* $j \in M$ *such that* $\sigma \cap \Delta^{M \setminus \{j\}} = \emptyset$ *and* $\mathbf{L}_n$ *be an* n*-primoid on a set* U*, properly oriented by* $\overline{\mathbf{L}}_n$*. If a labelling* $l: V \to U$ *satisfies the* dsp*-boundary condition for some* $(u_0, \dots u_n) \in \operatorname{lo} \mathbf{L}_n$*, then* $\operatorname{ind}_n l(1_{\overline{\mathbf{C}}^n}, 1_{\overline{\mathbf{L}}_n}) = (-1)^n \operatorname{or}_{\overline{\mathbf{L}}_n}(u_0, \dots u_n)$.

Proof. It is enough to show that the conditions of Theorem 6.1 are satisfied. For $M \subsetneq I_n$ and a simplex $\sigma \subset \Delta^M$ there exists $j \in M$ such that $\sigma \cap \Delta^{M \setminus \{j\}} = \emptyset$

and thus $l(V(\sigma)) \subset \mathrm{sp}\{u_i : i \in I_n, i \neq j\}$. Hence σ is not $\{u_i : i \notin M\}$-c.l. simplex since $l(V(\sigma)) \cup \{u_i : i \notin M\} \subset \mathrm{sp}\{u_i : i \in I_n, i \neq j\}$ and in the set $\mathrm{sp}\{u_i : i \in I_n, i \neq j\}$ there is no complete n-simplex. $\square$

A *diameter of a triangulation* Tr of Δ^{I_n} is the maximal diameter of a simplex in Tr. Observe that if the diameter of a triangulation is small enough, then the condition, for each $\sigma \subset \Delta^M$ ($M \subsetneq I_n$) there exists $j \in I_n$ such that $\sigma \cap \Delta^{M\setminus\{j\}} = \emptyset$, is satisfied.

Observe that Theorem 6.6 applied to the ring $R = \mathbb{Z}_2$ and to the primoid $\mathbf{L}_n^{I_n}$ is a generalization of the Scarf lemma ([24]; see also [21, Theorem 3.4]).

Theorem 6.7. *Let* Tr *be a triangulation of* $\Delta^{I_n} = \mathrm{co}\{d_0, \dots, d_n\}$ *and the* n*-complex* $\mathbf{C}^n = \{V(\sigma) : \sigma \in \mathrm{Tr}\}$ *be oriented by the geometric orientation,* $V = \bigcup_{\sigma\in\mathrm{Tr}} V(\sigma)$*. Let* $l\colon V \to \mathbb{R}^n$ *be a labelling such that for* $M \subset I_n$*, if* $v \in V \cap \Delta^M$*, then* $l(v) \in \mathrm{aff}\{d_i : i \in M\}$*. Let* $b \in \mathrm{ri}\,\Delta^{I_n}$ *be a point which is not a convex combination of less than* $n+1$ *elements of* $l(V)$*. Then* $\mathrm{ind}_n l(1_{\overline{\mathbf{C}}^n}, 1_{\overline{\mathbf{L}}_n^b}) = 1$.

Proof. Observe that the assumptions of Theorem 6.2 applied to the primoid $\mathbf{L}_n^b$ are satisfied. $\square$

Theorem 6.8. *Let* Tr *be a triangulation of* Δ^{I_n} *and the complex* $\mathbf{C}^n = \{V(\sigma) : \sigma \in \mathrm{Tr}\}$ *be oriented by the geometric orientation,* $V = \bigcup_{\sigma\in\mathrm{Tr}} V(\sigma)$ *and for* $M \subset I_n$ *and* $\sigma \subset \Delta^M$ *there exists* $j \in M$ *such that* $\sigma \cap \Delta^{M\setminus\{j\}} = \emptyset$*. Let* $l\colon V \to \mathbb{R}^n$ *be a labelling such that for* $M \subsetneq I_n$*, if* $v \in V \cap \mathrm{ri}\,\Delta^M$*, then* $l(v) \in \mathrm{aff}\{d_i : i \notin M\}$*. Let* $b \in \mathrm{ri}\,\Delta^{I_n}$ *be a point which is not a convex combination of less than* $n+1$ *elements of* $l(V)$*. Then* $\mathrm{ind}_n l(1_{\overline{\mathbf{C}}^n}, 1_{\overline{\mathbf{L}}_n^b}) = (-1)^n$.

Proof. Conditions of Theorem 6.1 are satisfied and the proof proceeds in a similar way as in the case of Theorem 6.6. $\square$

Applying Theorem 6.1 to the primoid $\mathbf{L}_n^{I_n}$ (see Example 4.2 for definition) and $(u_0, \dots, u_n) = (0, \dots, n)$ we get

Theorem 6.9. *Let* Tr *be a triangulation of* Δ^{I_n}*, the* n*-complex* $\mathbf{C}^n = \{V(\sigma) : \sigma \in \mathrm{Tr}\}$ *be oriented by the geometric orientation,* $V = \bigcup_{\sigma\in\mathrm{Tr}} V(\sigma)$*,* $l\colon V \to I_n$*. If for* $M \subsetneq I_n$ *and for a simplex* $\sigma \subset \Delta^M$ *we have* $l(V(\sigma)) \neq M$*, then* $\mathrm{ind}_n l(1_{\overline{\mathbf{C}}^n}, 1_{\overline{\mathbf{L}}_n^{I_n}}) = (-1)^n$.

Theorem 6.9 applied to the ring $R = \mathbb{Z}_2$ is equivalent to the van der Laan, Talman and Yang theorem ([21, Theorem 3.6]).

Applying Theorem 6.1 to the primoid $\mathbf{L}_n^{I_n}$ (see Example 4.2 for definition) and $(u_0, \dots, u_n) = (1, \dots, n, 0)$ we get

Theorem 6.10. *Let* Tr *be a triangulation of* Δ^{I_n}*, the* n*-complex* $\mathbf{C}^n = \{V(\sigma) : \sigma \in \mathrm{Tr}\}$ *be oriented by the geometric orientation,* $V = \bigcup_{\sigma\in\mathrm{Tr}} V(\sigma)$*,*

$l\colon V \to I_n$. If for $M \subsetneq I_n$ and for a simplex $\sigma \subset \Delta^M$ we have $l(V(\sigma)) \neq \{(i+1)_{\bmod (n+1)} : i \in M\}$, then $\mathrm{ind}_n l(1_{\overline{\mathbf{C}}^n}, 1_{\overline{\mathbf{L}}_n^{I_n}}) = 1$.

Theorem 6.10 is a generalization of the oriented version of the Sperner lemma [29] since for every $M \subsetneq I_n$, there is $i \in M$ such that $(i+1)_{\bmod (n+1)} \notin M$.

In the case the ring $R = \mathbb{Z}_2$, the index $\mathrm{ind}_n l(1_{\overline{\mathbf{C}}^n}, 1_{\overline{\mathbf{L}}_n})$ defines whether the number of c.l. n-simplexes is even or odd. Hence, if we consider $R = \mathbb{Z}_2$ in theorems of this section, we define an odd number of c.l. n-simplexes. This implies the existence of at least one such simplex. Some of these theorems were published in [14]–[16].

7. Multilabelling

Now we extend our definitions on n labellings. Let $\mathbf{C}^n$ be an n-complex on a finite set V ($|V| \geq n$) and let $\mathbf{C}^{n-1}$ be an $(n-1)$-complex consisting of all $(n-1)$-simplexes contained in some n-simplex of $\mathbf{C}^n$. Let $\mathbf{L}_n$ be an n-primoid on a finite set U ($|U| \geq n$) properly oriented by $\overline{\mathbf{U}}_n$. For $i \in I_n$, let $l^i\colon V \to U$ be a labelling.

Let $S^n = \{v_0, \dots, v_n\} \in \mathbf{C}^n$ and $a\colon S^n \to I_n$ be a one-to-one function. A pair (S^n, a) is called a *completely labelled n-pair* (c.l. n-pair for short) if $\{l^{a(v_0)}(v_0), \dots, l^{a(v_n)}(v_n)\} \in \mathbf{L}_n$. A *signum of a c.l. n-pair* (S^n, a) is an element of the ring R equal to

$$\mathrm{sign}\,(S^n, a) = \mathrm{or}_{\overline{\mathbf{C}}^n}(v_0, \dots, v_n) \cdot \mathrm{or}_{\overline{\mathbf{L}}_n}(l^{a(v_0)}(v_0), \dots, l^{a(v_n)}(v_n)).$$

Let $S^{n-1} = \{v_1, \dots, v_n\} \in \mathbf{C}^{n-1}$ and $a\colon S^{n-1} \to I_n$ be a one-to-one function. A pair (S^{n-1}, a) is called an *x-completely labelled $(n-1)$-pair* (x-c.l. $(n-1)$-pair for short) if $\{l^{a(v_1)}(v_1), \dots, l^{a(v_n)}(v_n)\} \in \mathbf{L}_n(x)$. A *signum of an x-c.l. $(n-1)$-pair* (S^{n-1}, a) is an element of the ring R equal to

$$\mathrm{sign}\,(S^{n-1}, a) = \mathrm{or}_{\overline{\mathbf{C}}^{n-1}}(v_1, \dots, v_n) \cdot \mathrm{or}_{\overline{\mathbf{L}}_n} x(l^{a(v_1)}(v_1), \dots, l^{a(v_n)}(v_n)).$$

Observe that the definition of the signum of a c.l. n-pair (S^n, a) (x-c.l. $(n-1)$-pair (S^{n-1}, a)) does not depend on the linear order of elements $\{v_0, \dots, v_n\}$ of S^n ($\{v_1, \dots, v_n\}$ of S^{n-1}). A c.l. n-pair and an x-c.l. $(n-1)$-pair are called a *positive c.l. n-pair and a positive x-c.l. $(n-1)$-pair* (a *negative c.l. n-pair and a negative x-c.l. $(n-1)$-pair*) if $\mathrm{sign}\,(S^n, a) = 1$ and $\mathrm{sign}\,(S^{n-1}, a) = 1$ ($\mathrm{sign}\,(S^n, a) = -1$ and $\mathrm{sign}\,(S^{n-1}, a) = -1$), respectively.

An $(n-1)$-pair (S^{n-1}, a) is called a *boundary $(n-1)$-pair* if S^{n-1} is a boundary $(n-1)$-simplex of $\mathbf{C}^n$.

We formulate a generalization of the Bapat theorem (Theorem 5.3 of this paper) on $n+1$ labellings:

Theorem 7.1. *Let $\mathbf{C}^n$ be a nonempty n-pseudomanifold on a set V coherently oriented by $\overline{\mathbf{C}}^n$. Let $\mathbf{L}_n$ be an n-primoid on a set U properly oriented by $\overline{\mathbf{L}}_n$, $x \in U$ and $l^i\colon V \to U$ for $i \in I_n$. Let α^+ and α^- (γ^+ and γ^-) denote the number of c.l. n-pairs in $\mathbf{C}^n$ (x-c.l. boundary pairs in $\mathbf{C}^n$) which are positive and negative, respectively. Then $\alpha^+ - \alpha^- = \gamma^+ - \gamma^-$.*

Proof. We construct an n-complex $\mathbf{C}^n_{I_n}$ on a set $U \times I_n$ in the following way $\{(v_0, i_0), \dots, (v_n, i_n)\} \in \mathbf{C}^n_{I_n}$ if and only if $S^n = \{v_0, \dots v_n\} \in \mathbf{C}^n$ and $i_j \neq i_k$ for $j \neq k$, $j, k \in I_n$. A function $\mathrm{or}_{\mathbf{C}^n_{I_n}}$ defined by

$$\mathrm{or}_{\overline{\mathbf{C}}^n_{I_n}}((v_0, i_0), \dots, (v_n, i_n)) = \mathrm{or}_{\overline{\mathbf{C}}^n}(v_0, \dots, v_n)$$

for $((v_0, i_0), \dots, (v_n, i_n)) \in \mathrm{lo}\,\mathbf{C}^n_{I_n}$ is an orientation of $\mathbf{C}^n_{I_n}$.

Now we will show that $\mathbf{C}^n_{I_n}$ is an n-pseudo-manifold coherently oriented by $\mathrm{or}_{\overline{\mathbf{C}}^n_{I_n}}$. Consider $\{(v_1, i_1), \dots, (v_n, i_n)\} \in \mathbf{P}_n(U \times I_n)$ such that $i_j \neq i_k$ for $j \neq k$, $j, k \in \{1, \dots n\}$. If there exists an n-simplex in $\mathbf{C}^n$ containing $S^{n-1} = \{v_1, \dots, v_n\}$, then there exist at most two such n-simplexes, say S^n_1, S^n_2. Thus there exist at most two n-simplexes $\{(w_1, j), (v_1, i_1), \dots, (v_n, i_n)\}$, $\{(w_2, j), (v_1, i_1), \dots, (v_n, i_n)\} \in \mathbf{C}^n_{I_n}$, where $w_1 \in S^n_1 \setminus S^{n-1}$, $w_2 \in S^n_2 \setminus S^{n-1}$, $j \in I_n$, $j \neq i_k$ for $k \in \{1, \dots, n\}$ that they induce opposite orientation on their common face. Thus $\mathbf{C}^n_{I_n}$ is a coherently oriented pseudomanifold. Now, observe that an n-pair $(\{v_0, \dots, v_n\}, a)$, where $a\colon S^n \to I_n$ is a one-to-one function (boundary $(n-1)$-pair $(\{v_1, \dots, v_n\}, a)$, where $a\colon S^{n-1} \to I_n$ is a one-to-one function) in $\mathbf{C}^n$ is an n-simplex $\{(v_0, a(v_0)), \dots, (v_n, a(v_n))\}$ (boundary $(n-1)$-simplex $\{(v_1, a(v_1)), \dots, (v_n, a(v_n))\}$) in $\mathbf{C}^n_{I_n}$. Hence our theorem follows from Theorem 5.3 immediately. □

If we apply Theorem 7.1 to the primoid $\mathbf{L}^k_n$ (see Example 4.6 for definition) we receive the Lee and Shih theorem ([28]), which is a generalization of the Fan theorem on $n+1$ labellings (Theorem 5.6).

Theorem 7.2 (Lee, Shih [28]). *Let $\mathbf{C}^n$ be a coherently oriented n-pseudomanifold on V, $k \geq n$ and $n+1$ labellings $l^i\colon V \to \{-k, \dots, -1, 1, \dots, k\}$ for $i \in I_n$ satisfying the condition $l^i(v) + l^j(v') \neq 0$ for vertices v and v' of an n-simplex of $\mathbf{C}^n$ and $i \neq j$, $i, j \in I_n$. Let $\alpha^+(j_0, \dots, j_n)$ ($\alpha^-(j_0, \dots, j_n)$) denotes the number of n-pairs $(\{v_0, \dots, v_n\}, a)$ in $\mathbf{C}^n$, where $a\colon \{v_0, \dots v_n\} \to I_n$ is a one-to-one function, such that*

$$\begin{aligned} l^{a(v_i)}(v_i) &= j_i \quad \textit{for } i \in I_n \\ \textit{and} \quad \mathrm{or}_{\overline{\mathbf{C}}^n}(v_0, \dots, v_n) &= 1 \quad (\mathrm{or}_{\overline{\mathbf{C}}^n}(v_0, \dots, v_n) = -1) \end{aligned}$$

and let $\beta^+(j_0, \dots, j_{n-1})$ ($\beta^-(j_0, \dots, j_{n-1})$) denote the number of boundary $(n-1)$-pairs $(\{v_0, \dots, v_{n-1}\}, a)$ in $\mathrm{lo}\,\mathbf{C}^n_{I_n}$, where $a\colon \{v_1, \dots, v_n\} \to I_n$ is one-to-one *function such that*

$$l^{a(v_i)}(v_i) = j_i \quad \textit{for } i \in I_{n-1}$$
$$\textit{and} \quad \mathrm{or}_{\overline{\mathbf{C}}^n}(v_0, \dots, v_{n-1}) = 1 \quad (\mathrm{or}_{\overline{\mathbf{C}}^n}(v_0, \dots, v_{n-1}) = -1).$$

Let

$$\alpha(j_0, \dots, j_n) = \alpha^+(j_0, \dots, j_n) - \alpha^-(j_0, \dots, j_n),$$
$$\beta(j_0, \dots, j_{n-1}) = \beta^+(j_0, \dots, j_{n-1}) - \beta^-(j_0, \dots, j_{n-1}).$$

Then we have:

$$\sum_{0<k_0<\dots<k_n} (\alpha(-k_0, k_1, -k_2, k_3, \dots, (-1)^{n+1}k_n) + \alpha(k_0, -k_1, k_2, -k_3, \dots, (-1)^n k_n)) = \sum_{0<k_0<\dots<k_{n-1}} \beta(k_0, -k_1, k_2, -k_3, \dots, (-1)^{n-1}k_{n-1}).$$

Now we apply Theorem 7.1 to the triangulation of the geometric simplex.

THEOREM 7.3. *Let* Tr *be a triangulation of* Δ^{I_n}*, the* n*-complex* $\mathbf{C}^n = \{V(\sigma) : \sigma \in \mathrm{Tr}\}$ *be oriented by the geometric orientation,* $V = \bigcup_{\sigma\in\mathrm{Tr}} V(\sigma)$*,* $\mathbf{L}_n$ *be an* n*-primoid on a set* U*, properly oriented by* $\overline{\mathbf{L}}_n$*,* $(u_0, \dots, u_n) \in \mathrm{lo}\,\mathbf{L}_n$ *and* $l^i : V \to U$ *for* $i \in I_n$*. Let* α^+*,* α^- *denote the numbers of positive, negative c.l. pairs in* $\mathbf{C}^n$*, respectively. If for* $M \subsetneq I_n$*, a simplex* $\sigma \subset \Delta^M$*,* $V(\sigma) = \{v_0, \dots, v_{|M|}\}$ *and a one-to-one function* $a : V(\sigma) \to I_n$*,* $\{l^{a(v_0)}(v_0), \dots, l^{a(v_{|M|})}(v_{|M|})\} \cup \{u_i : i \notin M\} \notin \mathbf{L}_n$*, then* $\alpha^+ - \alpha^- = (-1)^n \cdot \mathrm{or}_{\overline{\mathbf{L}}_n}(u_0, \dots, u_n) \cdot n!$.

PROOF. It is analogous to the proof of Theorem 6.1. We embed the simplex Δ^{I_n} in a larger simplex using the Scarf method [25, p. 192]. Without loss of generality we may assume that $0 \in \mathrm{ri}\,\Delta^{I_n}$. Let $\widetilde{d}_i = -a \cdot d_i$ for $i \in I_n$, where $a > 0$ is so large that $\Delta^{I_n} \subset \mathrm{ri}\,\mathrm{co}\,\{\widetilde{d}_0, \dots, \widetilde{d}_n\}$. Let us denote $\widetilde{\Delta}^{I_n} = \mathrm{co}\,\{\widetilde{d}_0, \dots, \widetilde{d}_n\}$. Observe that $\mathrm{or}_{\overline{\mathbf{C}}^n}(\widetilde{d}_0, \dots, \widetilde{d}_n) = (-1)^n$ ($\mathrm{or}_{\overline{\mathbf{C}}^n}$ is the geometric orientation).

Now we extend the triangulation of Δ^{I_n} by joining every vertex $v \in V \cap \Delta^{I_n\setminus\{i\}}$ with the vertex $\widetilde{d}_i$ and we get the triangulation of $\widetilde{\Delta}^{I_n}$:

$$\widetilde{\mathrm{Tr}} = \mathrm{Tr} \cup \bigcup_{M \subsetneq I_n} \bigcup_{\sigma \in \mathrm{Tr}(\Delta^M)} \mathrm{co}\,(V(\sigma) \cup \{\widetilde{d}_i : i \notin M\}).$$

Let $\mathbf{C}^n_\sim = \{V(\sigma) : \sigma \in \widetilde{\mathrm{Tr}}\}$. For $j \in I_n$ we define a labelling $\widetilde{l}^j$ as an extension of l^j on $\bigcup_{\sigma\in\widetilde{\mathrm{Tr}}} V(\sigma)$ by $\widetilde{l}^j(\widetilde{d}_i) = u_i$ for all $i, j \in I_n$.

Let $\widetilde{\alpha}^+$, $\widetilde{\alpha}^-$ denote the numbers of positive, negative c.l. pairs in $\mathbf{C}^n_\sim$, respectively. Because of our assumption on l^i for $i \in I_n$ there is no c.l. pair for this new triangulation $\widetilde{\mathrm{Tr}}$ of $\widetilde{\Delta}^{I_n}$ and for the labellings $\widetilde{l}^i$ ($i \in I_n$), which is not a c.l. pair in Tr. Hence $\widetilde{\alpha}^+ - \widetilde{\alpha}^- = \alpha^+ - \alpha^-$.

By Theorem 7.1 applied to $\mathbf{C}^n_\sim$ and $x = u_0$ we know that $\widetilde{\alpha}^+ - \widetilde{\alpha}^-$ is equal to $\widetilde{\gamma}^+ - \widetilde{\gamma}^-$, where $\widetilde{\gamma}^+ - \widetilde{\gamma}^-$ is the number of x-c.l. boundary pairs in $\mathbf{C}^n_\sim$.

Observe that the only u_0-c.l. $(n-1)$-pairs on the boundary of $\widetilde{\Delta}^{I_n}$ are of the type $\{(\widetilde{d}_1, a(\widetilde{d}_1)), \dots, (\widetilde{d}_n, a(\widetilde{d}_n))\}$, where $a\colon \{\widetilde{d}_1, \dots, \widetilde{d}_n\} \to I_n$ is a one-to-one function and there is exactly $n!$ such functions. The orientation of $\{\widetilde{d}_1, \dots, \widetilde{d}_n\}$ is the induced orientation by the geometric orientation and thus $\operatorname{sign}(\widetilde{d}_1, \dots, \widetilde{d}_n) = (-1)^n$. Hence $\alpha^+ - \alpha^n = (-1)^n \cdot \operatorname{or}_{\overline{\mathbf{L}}_n}(u_0, \dots, u_n) \cdot n!$. □

Theorem 7.4. *Let* Tr *be a triangulation of* Δ^{I_n}*, the n-complex* $\mathbf{C}^n = \{V(\sigma) : \sigma \in \mathrm{Tr}\}$ *be oriented by the geometric orientation* $\operatorname{or}_{\overline{\mathbf{C}}^n}$, $V = \bigcup_{\sigma\in\mathrm{Tr}} V(\sigma)$ *and* $\mathbf{L}_n$ *be an n-primoid on a set U, properly oriented by* $\overline{\mathbf{L}}_n$, $(u_0, \dots, u_n) \in \operatorname{lo}\mathbf{L}_n$. *If for* $i \in I_n$ *a labelling* $l^i : V \to U$ *satisfies the ρ-boundary condition for* $(u_0, \dots, u_n)$, *then the number of positive c.l. n-pairs minus the number of negative c.l. n-pairs is equal to* $n!$. *In particular there is at least* $n!$ *c.l. n-pairs.*

Proof. The conditions of Theorem 7.3 are satisfied for $(u_1, \dots, u_n, u_0)$. The proof proceeds in a similar way as in the case of Theorem 6.2. □

Theorem 7.5. *Let* Tr *be a triangulation of* Δ^{I_n}*, the n-complex* $\mathbf{C}^n = \{V(\sigma) : \sigma \in \mathrm{Tr}\}$ *be oriented by the geometric orientation,* $V = \bigcup_{\sigma\in\mathrm{Tr}} V(\sigma)$ *and* $\mathbf{L}_n$ *be an n-primoid on a set U, properly oriented by* $\overline{\mathbf{L}}_n$, $(u_0, \dots, u_n) \in \operatorname{lo}\mathbf{L}_n$. *If for* $i \in I_n$ *a labelling* $l^i : V \to U$ *satisfies the* sp*-boundary condition for* $(u_0, \dots, u_n)$, *then the number of positive c.l. n-pairs minus the number of negative c.l. n-pairs is equal to* $n!$. *In particular there is at least* $n!$ *c.l. n-pairs.*

Proof. By Property 4.11(b) conditions of Theorem 7.4 are satisfied. □

Corollary 7.6. *Applying Theorem* 7.5 *to the n-primoid:*

(a) $\mathbf{L}_n^{I_n}$, *we get the Bapat theorem* ([2, Theorem 1]),
(b) $\mathbf{L}_n^{\pi}$, *we get the Lee and Shih theorem* ([27, Theorem 2.1]).

In the case $\mathbf{L}_n = \mathbf{L}_n^b$ we say a b-balanced n-pair instead of a c.l. n-pair.

From Theorem 7.5 applied to the n-primoid $\mathbf{L}_n^b$ and from Theorem 4.12 we get

Corollary 7.7. *Let* Tr *be a triangulation of* $\Delta^{I_n} = \operatorname{co}\{d_0, \dots, d_n\}$, *the complex* $\mathbf{C}^n = \{V(\sigma) : \sigma \in \mathrm{Tr}\}$ *be oriented by the geometric orientation,* $V = \bigcup_{\sigma\in\mathrm{Tr}} V(\sigma)$. *Let* $b \in \operatorname{ri}\Delta^{I_n}$ *be a point which is not a convex combination of less than* $n+1$ *elements of* $\bigcup_{i\in I_n} l^i(V)$. *Let* $l^i\colon V \to \mathbb{R}^n$ *be a labelling* $(i \in I_n)$ *such that for* $M \subset I_n$, *if* $v \in V \cap \Delta^M$, *then* $l^j(v) \in \operatorname{cone}(\{d_i : i \in M\}, b)$ *for all* $j \in I_n$. *Let* α^+, α^- *denote the numbers of positive, negative b-balanced n-pairs in* $\mathbf{C}^n$, *respectively. Then* $\alpha^+ - \alpha^- = n!$.

From Theorem 7.3 we have

Corollary 7.8. *Let* Tr *be a triangulation of* Δ^{I_n}*, the n-complex* $\mathbf{C}^n = \{V(\sigma) : \sigma \in \mathrm{Tr}\}$ *be oriented by the geometric orientation and* $V = \bigcup_{\sigma\in\mathrm{Tr}} V(\sigma)$.

Let for $M \subsetneq I_n$ and a simplex $\sigma \subset \Delta^M$ there exists $j \in M$ such that $\sigma \cap \Delta^{M\setminus\{j\}} = \emptyset$ and $\mathbf{L}_n$ be an n-primoid on a set U, properly oriented by $\overline{\mathbf{L}}_n$, $(u_0, \dots, u_n) \in \operatorname{lo} \mathbf{L}_n$. Let α^+, α^- denote the numbers of positive, negative c.l. pairs in $\mathbf{C}^n$, respectively. If for every $i \in I_n$ a labelling $l^i : V \to U$ satisfies the dsp*-boundary condition for $(u_0, \dots u_n)$, then $\alpha^+ - \alpha^- = (-1)^n \cdot \operatorname{or}_{\overline{\mathbf{L}}_n}(u_0, \dots, u_n) \cdot n!$.*

Corollary 7.9. *Let* Tr *be a triangulation of $\Delta^{I_n} = \operatorname{co}\{d_0, \dots, d_n\}$ and the complex $\mathbf{C}^n = \{V(\sigma) : \sigma \in \mathrm{Tr}\}$ be oriented by the geometric orientation and $V = \bigcup_{\sigma \in \mathrm{Tr}} V(\sigma)$. Let $l^i : V \to \mathbb{R}^n$ be a labelling $(i \in I_n)$ such that for $M \subset I_n$, if $v \in V \cap \Delta^M$, then $l^j(v) \in \operatorname{aff}\{d_i : i \in M\}$ for all $j \in I_n$. Let $b \in \operatorname{ri} \Delta^{I_n}$ be a point which is not a convex combination of less than $n+1$ elements of $\bigcup_{i \in I_n} l^i(V)$. Let α^+, α^- denote the numbers of positive, negative b-balanced n-pairs in $\mathbf{C}^n$, respectively. Then the numbers $\alpha^+ - \alpha^- = n!$.*

Proof. Observe that the assumptions of Theorem 7.4 applied to the primoid $\mathbf{L}_n^b$ are satisfied. □

From Theorem 7.3 we have

Corollary 7.10. *Let* Tr *be a triangulation of Δ^{I_n} and the complex $\mathbf{C}^n = \{V(\sigma) : \sigma \in \mathrm{Tr}\}$ be oriented by the geometric orientation, $V = \bigcup_{\sigma \in \mathrm{Tr}} V(\sigma)$ and for $M \subsetneq I_n$ and a simplex $\sigma \subset \Delta^M$ there exists $j \in M$ such that $\sigma \cap \Delta^{M\setminus\{j\}} = \emptyset$. Let $l^i : V \to \mathbb{R}^n$ be a labelling $(i \in I_n)$ such that for $M \subsetneq I_n$, if $v \in V \cap \operatorname{ri} \Delta^M$, then $l^i(v) \in \operatorname{aff}\{d_i : i \notin M\}$ for all $i \in I_n$. Let $b \in \operatorname{ri} \Delta^{I_n}$ be a point which is not a convex combination of less than $n+1$ elements of $\bigcup_{i \in I_n} l(V)$. Let α^+, α^- denote the numbers of positive, negative b-balanced n-pairs in $\mathbf{C}^n$, respectively. Then $\alpha^+ - \alpha^- = (-1)^n \cdot n!$.*

Applying Theorem 7.3 to the primoid $\mathbf{L}_n^{I_n}$ (see Example 4.2 for definition) and $(u_0, \dots, u_n) = (0, \dots, n)$ we get

Corollary 7.11. *Let* Tr *be a triangulation of Δ^{I_n}, the n-complex $\mathbf{C}^n = \{V(\sigma) : \sigma \in \mathrm{Tr}\}$ be oriented by the geometric orientation, $V = \bigcup_{\sigma \in \mathrm{Tr}} V(\sigma)$, $l^i : V \to I_n$ for $i \in I_n$. Let α^+, α^- denote the numbers of positive, negative c.l. n-pairs in $\mathbf{C}^n$, respectively. If for $M \subsetneq I_n$, a simplex $\sigma \subset \Delta^M$, $V(\sigma) = \{v_1, \dots, v_{|M|}\}$ and a one-to-one function $a : V(\sigma) \to I_n$ we have $\{l^{a(v_1)}(v_1), \dots, l^{a(v_{|M|})}(v_{|M|})\} \neq M$, then $\alpha^+ - \alpha^- = (-1)^n \cdot n!$.*

Applying Theorem 7.3 to the primoid $\mathbf{L}_n^{I_n}$ and $(u_0, \dots, u_n) = (1, \dots, n, 0)$ we get

Corollary 7.12. *Let* Tr *be a triangulation of Δ^{I_n}, the n-complex $\mathbf{C}^n = \{V(\sigma) : \sigma \in \mathrm{Tr}\}$ be oriented by the geometric orientation, $V = \bigcup_{\sigma \in \mathrm{Tr}} V(\sigma)$, $l^i : V \to I_n$ for $i \in I_n$. Let α^+, α^- denote the numbers of positive, negative c.l. n-pairs in $\mathbf{C}^n$, respectively. If for $M \subsetneq I_n$, a simplex $\sigma \subset \Delta^M$, $V(\sigma) =$*

$\{v_1, \ldots, v_{|M|}\}$ *and a one-to-one function* $a \colon V(\sigma) \to I_n$ *we have* $\{l^{a(v_1)}(v_1), \ldots, l^{a(v_{|M|})}(v_{|M|})\} \neq \{(i+1) \mod (n+1) : i \in M\}$, *then* $\alpha^+ - \alpha^- = n!$.

If $l^i = l^0$ for every $i \in I_n$, then Theorems and Corollaries 7.4–7.12 reduce to Theorem 6.2, Theorem 6.3, Theorem 6.6, Theorems 6.7–6.10, respectively.

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