Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 32, 2008, 187–197

A NATURAL FAMILY OF FACTORS FOR PRODUCT $\mathbb{Z}^2$-ACTIONS

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Abstract. It is shown that if $\mathcal{N}$ and $\mathcal{N}'$ are natural families of factors (in the sense of [5]) for minimal flows (X,T) and (X',T'), respectively, then $\{R\otimes R' : R\in\mathcal{N},\ R'\in\mathcal{N}'\}$ is a natural family of factors for the product $\mathbb{Z}^2$-action on $X\times X'$ generated by T and T'.

An example is given showing the existence of topologically disjoint minimal flows (X,T) and (X',T') for which the family of factors of the flow $(X\times X', T\times T')$ is strictly bigger than the family of factors of the product $\mathbb{Z}^2$-action on $X\times X'$ generated by T and T'.

There is also an example of a minimal distal system with no nontrivial compact subgroups in the group of its automorphisms.

By a *topological flow* we mean a triple $(X,\mathcal{T},\mathcal{U})$ where X is a compact metric space, $\mathcal{T}$ is a topological group (with the discrete topology) and $\mathcal{U}:\mathcal{T}\times X\to X$ is a continuous map such that

(1) $\mathcal{U}(e,x)=x$ (here e stands for the identity of $\mathcal{T}$) for $x\in X$;
(2) $\mathcal{U}(t,\mathcal{U}(s,x))=\mathcal{U}(ts,x)$ for $x\in X$ and $s,t\in\mathcal{T}$.

We say that $\mathcal{U}$ is a $\mathcal{T}$-action on X. If the acting group is understood we write $(X,\mathcal{U})$.

2000 *Mathematics Subject Classification.* 54H20.

Key words and phrases. Topological dynamics, structure of factors, natural families of factors.

Research partially supported by the KBN grant 1/P03A/03826.

In this note we focus on the case where $\mathcal{T} = \mathbb{Z}$ or $\mathcal{T} = \mathbb{Z}^2$. In the former case the action $\mathcal{U}$ is generated by the single homeomorphism $T\colon X \to X$ defined by $\mathcal{U}(n,x) = T^n x$ and the flow is denoted by (X,T).

The latter case will be restricted to the following situation. Given two homeomorphisms $T_i\colon X_i \to X_i$, $i = 1,2$, of compact metric spaces we consider a so called *product $\mathbb{Z}^2$-action* $\underline{\mathcal{U}} = \underline{\mathcal{U}}_{(T_1,T_2)}$ on $X_1 \times X_2$:

$$\underline{\mathcal{U}}((n,m),(x_1,x_2)) = (T_1^n x_1, T_2^m x_2).$$

In such a situation we say that $\underline{\mathcal{U}}$ is generated by (T_1,T_2).

The straightforward proof of the following lemma will be omitted.

LEMMA 1. *The action $\underline{\mathcal{U}}_{(T_1,T_2)}$ is minimal if and only if so are the homeomorphisms T_1 and T_2.*

Following [3] we recall that disjointness of minimal $\mathbb{Z}$-actions generated by two homeomorphisms means that the product $\mathbb{Z}$-action is minimal.

The following result is known and the reader may find it for instance in [1, p. 155]. However the proof in [1] uses the elaborated algebraic Ellis theory of minimal flows. Here a simple proof is presented that uses product $\mathbb{Z}^2$-actions. Recall that a homomorphism $\pi\colon (X,\mathcal{U}_1) \to (Y,\mathcal{U}_2)$ is called *proximal* if $R_\pi = \{(x_1,x_2) \in X \times X : \pi(x_1) = \pi(x_2)\}$ consists od proximal pairs (i.e. for every $(x_1,x_2) \in R_\pi$ there is a net $(u_i) \subset \mathcal{T}$ and $x \in X$ such that $(u_i x_1, u_i x_2)$ converges to (x,x)).

PROPOSITION 2. *Let $\pi_i\colon (X_i,T_i) \to (Y_i,S_i)$, $i = 1,2$, be proximal homomorphisms between minimal $\mathbb{Z}$-actions. Then topological disjointness of (Y_1,S_1) and (Y_2,S_2) implies topological disjointness of (X_1,T_1) and (X_2,T_2).*

PROOF. Since $\pi_1 \times \mathrm{id}_{Y_2}\colon X_1 \times Y_2 \to Y_1 \times Y_2$ is proximal and $(Y_1 \times Y_2, S_1 \times S_2)$ is minimal, $(X_1 \times Y_2, T_1 \times S_2)$ possesses precisely one minimal set, say M. Then $(\mathrm{id}_{X_1} \times S_2)(M) = (T_1 \times \mathrm{id}_{Y_2})(M) = M$ since $\mathrm{id}_{X_1} \times S_2$ and $T_1 \times \mathrm{id}_{Y_2}$ commute with $T_1 \times S_2$ and it follows that M is $\underline{\mathcal{U}}_{(T_1,S_2)}$-invariant. Since, by Lemma 1, $\underline{\mathcal{U}}_{(T_1,S_2)}$ is minimal, $M = X_1 \times Y_2$. It has been shown that (X_1,T_1) and (Y_2,S_2) are disjoint.

The same reasoning applied to the homomorphism $\mathrm{id}_{X_1} \times \pi_2$ shows disjointness of (X_1,T_1) and (X_2,T_2). □

Borrowing some ideas from [6] the authors of [5] introduced the concept of a natural family of factors for a minimal topological flow (see also [8]). Before we state the definition we need the following lemma.

Let $\mathcal{U}_i$, $i = 1,2$, be $\mathcal{T}$-actions on X and Y, respectively. Let M be a joining of $\mathcal{U}_1$ and $\mathcal{U}_2$, i.e. a minimal subset of the product system that projects onto both of coordinates (a *self-joining* is a joining of the system with itself).

Lemma 3 ([5], [8]). *There exist the smallest* $\mathcal{U}_i$-ICERs $R_i(M)$, $i = 1, 2$, *such that*

$$(\pi_1 \times \pi_2)(M) = \operatorname{Graph} \phi,$$

where ϕ *is some isomorphism between* $X/R_1(M)$ *and* $Y/R_2(M)$.

In such a situation we say that M *induces* ϕ. Observe that in the case where $(X, \mathcal{U}_1) = (Y, \mathcal{U}_2)$ we have

$$R_1(M) = \langle M \circ M^{-1} \rangle \quad \text{and} \quad R_2(M) = \langle M^{-1} \circ M \rangle.$$

Here and in the sequel $A \circ B = \{(x, y) \in X \times X; (x, z) \in A$ and $(z, y) \in B$ for some $z \in X\}$, $A^{-1} = \{(x, y) \in X \times X : (y, x) \in A\}$ for $A, B \subset X \times X$ and $\langle C \rangle$ denotes the smallest invariant closed equivalent relation on X containing $C \subset X \times X$.

Definition 4 ([5]). A family $\mathcal{N}$ of ICERs is said to be *natural* if

(a) $\Delta_X \in \mathcal{N}$;
(b) if $\{R_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{N}$ then $\bigvee_{\lambda\in\Lambda} R_\lambda \in \mathcal{N}$;
(c) $R_i(M) \in \mathcal{N}$ for every self-joining M;
(d) if $\Phi: X/R \to X/R'$ is an isomorphism and $R \in \mathcal{N}$ then $R' \in \mathcal{N}$.

In (b) and in the sequel $\bigvee_{\lambda\in\Lambda} R_\lambda$ stands for $\langle \bigcup_{\lambda\in\Lambda} R_\lambda \rangle$.

Remark 5. (a) For any family of factor relations $\mathcal{N}$ satisfying conditions (a) and (b) of Definition 4 and for each ICER R there exists a biggest ICER $\widetilde{R} \in \mathcal{N}$ with $\widetilde{R} \subset R$.

(b) Since the intersection of natural families is obviously a natural family and the family of all factors is natural, it follows that for any minimal $\mathcal{U}$ there exists the smallest natural family of factors.

The term "natural" is explained by the following result.

Proposition 6 ([5], [8]. *Let* $\mathcal{N}$ *be a natural family of* ICERs *for a minimal flow* $(X, \mathcal{U})$. *For each* ICER R *of* $(X, \mathcal{U})$ *the homomorphism* $\pi: X/\widetilde{R} \to X/R$ *is regular. Furthermore if* π *is distal then it is a group extension.*

The second part of the above proposition is actually Glasner's result that may be considered as a topological version of a theorem of Veech.

Theorem 7 (Glasner, [4]). *A regular and distal homomorphism between two minimal systems is a group extension.*

Recall that a homomorphism $\pi: (X, \mathcal{U}_1) \to (Y, \mathcal{U}_2)$ is called *distal* if R_π is a union of minimal sets and *regular* if every minimal subset of R_π is a graph of some element from the group of automorphisms of $(X, \mathcal{U}_1)$. (In the sequel the group of automorphisms of $(X, \mathcal{U})$ will be denoted by $\operatorname{Aut}(\mathcal{U})$. Recall that this group is usually endowed with the topology of uniform convergence of homeomorphisms and their inverse that makes it a Polish group.) Recall also that π is called a *group extension* if there is a compact subgroup $G \subset \mathrm{Aut}(\mathcal{U}_1)$ such that the quotient system $(X/G, \mathcal{U}_1/G)$ is conjugate to $(Y, \mathcal{U}_2)$.

REMARK 8. It is obvious that if $T: X \to X$, $S: X' \to X'$ are homeomorphisms then $(X \times X', \underline{\mathcal{U}}_{(T,S)})$ is distal iff (X, T) and (X', S) are so.

Let $(X \times X', \underline{\mathcal{U}})$ be a minimal $\mathbb{Z}^2$-action generated by homeomorphisms $T: X \to X$ and $S: X' \to X'$. If $A \subset X \times X$ and $B \subset X' \times X'$ then put

$$A \otimes B := \{((x_1, x_1'), (x_2, x_2')) \in (X \times X')^2 : (x_1, x_2) \in A,\ (x_1', x_2') \in B\}.$$

Thus $A \otimes B$ is an image of $A \times B$ under the isomorphisms exchanging the second and the third coordinate. We will need the following simple lemma.

Let $M \subset (X \times X')^2$ be a $\underline{\mathcal{U}}$-self-joining.

LEMMA 9. *There are a T-self-joining $N \subset X \times X$ and an S-self-joining $N' \subset X' \times X'$ such that*

$$M = N \otimes N'.$$

PROOF. Put $N = \Pi_{1,3}(M)$ and $N' = \Pi_{2,4}(M)$, where $\Pi_{i,j}$ denotes the projection onto the i-th and the j-th coordinate. □

Let us consider a family of $\underline{\mathcal{U}}$-ICERs (so called *product relations*)

$$\mathcal{P} = \mathcal{P}(\underline{\mathcal{U}}) = \{R \otimes R' : R \text{ is a } T\text{-ICER and } R' \text{ is an } S\text{-ICER}\}.$$

The following is a topological counterpart of some results from Section III of [2].

PROPOSITION 10. *The family $\mathcal{P}(\underline{\mathcal{U}})$ of product* ICER*s of a product $\mathbb{Z}^2$-action is natural.*

PROOF. (a) We have $\Delta_{X \times X'} = \Delta_X \otimes \Delta_{X'} \in \mathcal{P}$.

(b) identify the algebra $C(X \times X')$ of real continuous functions on $X \times X'$ with the algebra $C(X, C(X'))$ of continuous functions on X with values in the Banach algebra $C(X')$. The homeomorphic isomorphism is given by

$$(L(F)(x))(y) = F(x, y).$$

Let $A(\underline{R})$ denote the subalgebra of $C(X \times X')$ that consists of functions constant on co-sets of $\underline{R}$. If $\underline{R} = R \otimes R'$, then $L(A(\underline{R})) = A(R, A(R'))$, where $A(R, A(R'))$

stands for the subalgebra of $C(X, C(X'))$ consisting of functions constant on cosets of R with values in the subalgebra $A(R')$. Now we have

$$\begin{aligned} L\Big(A\Big(\bigvee_i (R_i \otimes R_i')\Big)\Big) &= L\Big(\bigcap_i A(R_i \otimes R_i')\Big) = \bigcap_i A(R_i, A(R_i')) \\ &= \bigcap_i A\Big(R_i, \bigcap_j A(R_j')\Big) = \bigcap_i A\Big(R_i, A\Big(\bigvee_j R_j'\Big)\Big) \\ &= A\Big(\bigvee_i R_i, A\Big(\bigvee_j R_j'\Big)\Big) \\ &= L\Big(A\Big(\Big(\bigvee_i R_i\Big) \otimes \Big(\bigvee_i R_i'\Big)\Big)\Big). \end{aligned}$$

Since there is one-to-one correspondence between ICERs $\underline{R}$ and subalgebras $A(\underline{R})$ we have obtained $\bigvee_i (R_i \otimes R_i') = \big(\bigvee_i R_i\big) \otimes \big(\bigvee_i R_i'\big) \in \mathcal{P}$.

(c) Using Lemma 9 for the third equality below and, for instance, Lemma 1 of [7] for the fifth one, we get

$$\begin{aligned} R_1(M) &= \langle M \circ M^{-1} \rangle = \langle M \circ M^{-1} \cup \Delta_{X \times X'} \rangle \\ &= \langle (N \otimes N') \circ (N \otimes N')^{-1} \cup \Delta_X \otimes \Delta_{X'} \rangle \\ &= \langle (N \circ N^{-1} \cup \Delta_X) \otimes (N' \circ (N')^{-1} \cup \Delta_{X'}) \rangle \\ &= \langle (N \circ N^{-1} \cup \Delta_X) \rangle \otimes \langle (N' \circ (N')^{-1} \cup \Delta_{X'}) \rangle \\ &= \langle N \circ N^{-1} \rangle \otimes \langle N' \circ (N')^{-1} \rangle = R_1(N) \otimes R_1(N') \in \mathcal{P}. \end{aligned}$$

The proof for $R_2(M)$ is analogous.

(d) It follows immediately from Corollary 11 below. □

Lemmas 3 and 9, and the proof of (c) of Proposition 10 yield the following.

Corollary 11. *Let $\underline{\phi} \in \mathrm{Aut}(\underline{\mathcal{U}})$. There exist $\phi \in \mathrm{Aut}(T)$ and $\phi' \in \mathrm{Aut}(S)$ such that $\underline{\phi} = \phi \times \phi'$.*

Proof. Obviously $M = \mathrm{Graph}\,\underline{\phi}$ is a $\underline{\mathcal{U}}$-self-joining. Then

$$R_i(N) \otimes R_i(N') = R_i(M) = \Delta_{X \times X'} = \Delta_X \otimes \Delta_{X'}$$

so $R_i(N) = \Delta_X$ and $R_i(N') = \Delta_{X'}$. The result follows from Lemma 3. □

Remark 12. One may also show Corollary 11 independently on Lemmas 3 and 9, and (c) of Proposition 10.

Indeed, let $(X_i \times X_i', \underline{\mathcal{U}}_i)$, $i = 1, 2$, be minimal product $\mathbb{Z}^2$-actions generated by (T_i, S_i), respectively, and let $\underline{\phi}$ be an isomorphism between $(X_1 \times X_1', \underline{\mathcal{U}}_1)$ and $(X_2 \times X_2', \underline{\mathcal{U}}_2)$. If $\underline{\phi}(x, x') = (\underline{\phi}_1(x, x'), \underline{\phi}_2(x, x'))$ then

$$(\underline{\phi}_1(T_1^n x, S_1^m x'), \underline{\phi}_2(T_1^n x, S_1^m x') = (T_2^n(\underline{\phi}_1(x, x')), S_2^m(\underline{\phi}_2(x, x')))$$

for every $(n,m) \in \mathbb{Z}^2$. By minimality of considered actions, $\underline{\phi}_{1\,(2)}$ does not depend on the second (first) coordinate and the result follows.

In fact the proof of Proposition 10 shows even stronger result.

PROPOSITION 13. *If $\mathcal{N}$ and $\mathcal{N}'$ are natural families of factors for minimal flows (X,T) and (X',T'), respectively, then the family*

$$\{R \otimes R' : R \in \mathcal{N},\ R' \in \mathcal{N}\}$$

is natural for $\underline{\mathcal{U}}$.

There is an interest in investigating natural families of product $\mathbb{Z}^2$-actions $\underline{\mathcal{U}}$ generated by two homeomorphisms because a product $\mathbb{Z}$-action generated by those homeomorphisms may have more factors than $\underline{\mathcal{U}}$ as the following example shows.

EXAMPLE 14. Let (Ω,σ) denote the full shift over $\{0,1\}$.

Let m and n be relatively prime positive integers. Consider two generalized Morse systems generated by substitutions of constant length m and n. Precisely, let $\varsigma_i\colon \{0,1\} \to \{0,1\}^i$, $i = n, m$, satisfy $(\varsigma_i(0))_j = (\varsigma_i(1))_j + 1 \pmod 2$. Let $\eta^{(i)}$ be a sequence generated by ς_i and take any almost periodic bisequence $\omega^{(i)}$ with $\omega^{(i)}_j = \eta^{(i)}_j$ for $j \ge 0$.

Let $X^{(i)}$ be the orbit closure of $\omega^{(i)}$. It is well-known that $(X^{(i)},\sigma)$ factors on the odometer $\mathbb{Z}(i)$ through the so called Morse–Toepliz system $(Y^{(i)},\sigma)$ in the way that the latter system is an almost 1–1 (hence proximal) extension of $\mathbb{Z}(i)$. Now Proposition 2 assures that $(Y^{(m)},\sigma)$ and $(Y^{(n)},\sigma)$ are topologically disjoint since $\mathbb{Z}(m)$ and $\mathbb{Z}(n)$ also are.

We describe the factor maps $(X^{(i)},\sigma) \to (Y^{(i)},\sigma)$. Let $\psi\colon \Omega \to \Omega$ be defined by $\psi(u)_j = u_j + u_{j+1}$ and put $\rho^{(i)}\colon X^{(i)} \to Y^{(i)}$, $\rho^{(i)} = \psi|_{X^{(i)}}$.

Put $\phi\colon \Omega\times\Omega \to \Omega$, $(\phi(u^{(1)},u^{(2)}))_j = u^{(1)}_j + u^{(2)}_j \pmod 2$. We need to show that ϕ restricted to $Y^{(m)} \times Y^{(n)}$ is not a bijection.

For this let us define two bisequences $\breve{\omega}^{(i)}$, $i = 1,2$, by

$$\breve{\omega}^{(i)}_j = \begin{cases} \omega^{(i)}_j & \text{if } j \ge 0, \\ \omega^{(i)}_j + 1 \pmod 2 & \text{if } j < 0. \end{cases}$$

It is easy to check that $\breve{\omega}^{(i)} \in X_i$ (both $\omega^{(i)}$ and $\breve{\omega}^{(i)}$ are almost periodic and the pair $(\omega^{(i)},\breve{\omega}^{(i)})$ is asymptotic) and that

$$y^{(i)} = \rho_i(\omega^{(i)}) \ne \rho_i(\breve{\omega}^{(i)}) =: \breve{y}^{(i)}. \tag{1}$$

Moreover,

$$\phi(\omega^{(m)},\omega^{(n)}) = \phi(\breve{\omega}^{(m)},\breve{\omega}^{(n)}),$$

hence $\phi(y^{(m)},y^{(n)}) = \phi(\breve{y}^{(m)},\breve{y}^{(n)})$ and this, together with (1), implies that $\phi|_{Y^{(m)}\times Y^{(n)}}$ is not a bijection.

Now we see that $\phi|_{Y^{(m)}\times Y^{(n)}}$ defines a nontrivial factor of $\mathbb{Z}$-action on $Y^{(m)} \times Y^{(n)}$ generated by $\sigma \times \sigma$ that is not a factor of a product $\mathbb{Z}^2$-action generated by (σ, σ).

In [2] the $\mathbb{Z}^2$ version of Furstenberg's filtering problem from [3] is considered. In the rest of the present note it is shown how to apply Proposition 10 to obtain some analogous result in topological dynamics.

Following [2] we define a topological counterpart of a univalence property and universal filtering.

Definition 15. The function $F\colon X \times X' \to \mathbb{R}$ has *T-pointwise univalence property* if for any distinct $x_1,\ x_2 \in X$ there exists $n \in \mathbb{Z}$ such that

$$F(T^n x_1, \cdot\,) \neq F(T^n x_2, \cdot\,).$$

Consider two equivalence relations on $X \times X'$:

$$\begin{aligned} R_X &= \{((x, x_1'), (x, x_2')) : x \in X,\ x_1', x_2' \in X'\}, \\ R_F &= \{((x_1, x_1'), (x_2, x_2')) : F(x_1, x_1') = F(x_2, x_2')\}. \end{aligned}$$

The former one is already an ICER. Since the latter one need not be an ICER we consider

$$\widehat{R}_F = \bigvee \{R \subset R_F : R \text{ is a } \underline{\mathcal{U}}\text{-ICER}\}.$$

The factor of $\underline{\mathcal{U}}$ generated by R_X is of course a $\mathbb{Z}^2$-action. Nevertheless one may naturally identify it as a $\mathbb{Z}$-action generated by T.

Definition 16. We say that the distal minimal system (X, T) is *distally filtered* if for every minimal distal system (X', S)

$$\widehat{R}_F \subset R_X,$$

for any continuous function $F\colon X \times X' \to \mathbb{R}$ that has T-pointwise univalence property.

Proposition 17. *Assume that there are no nontrivial compact subgroups in the group of automorphisms of a distal minimal system (X, T). Then (X, T) is distally filtered.*

Proof. Let $T\colon X \to X$ and $S\colon X' \to X'$ be two distal minimal homeomorphisms. Let $\underline{\mathcal{U}}$ be a product $\mathbb{Z}^2$-action generated by (T, S) and $F\colon X \times X' \to \mathbb{R}$ be a continuous function with the T-pointwise univalence property. Put $\underline{R} = \widehat{R}_F$ and, applying Proposition 13, let $\underline{\widetilde{R}} = R \otimes R'$. Let $\pi\colon X \to X/R$ and $\pi'\colon X' \to X'/R'$ be canonical factor maps. First we show that $R = \Delta_X$. Indeed, if $x_1, x_2 \in X$, $x_1 \neq x_2$ then, by univalence property of F, there are $n \in \mathbb{Z}$ and $x' \in X'$ such that $F(T^n x_1, x') \neq F(T^n x_2, x')$. Since $R \times R' \subset \widehat{R}_F \subset R_F$, the function $f\colon X \to \mathbb{R}$, $f(x) = F(T^n x, x')$ belongs to the algebra $A(R)$ and $f(x_1) \neq f(x_2)$. Therefore $A(R)$ separates points of X, so $R = \Delta_X$.

By the distality assumption, Remark 8 and Proposition 10 the extension

$$X \times X'/\underline{\widetilde{R}} \to X \times X'/\underline{R}$$

is a group, say G-extension. We intend to show that this extension is actually trivial, i.e. G is a trivial group.

Using Corollary 11 one may represent G in $\operatorname{Aut}(T)$. We only need to know that $\phi' = \mathrm{id}_{X'/R'}$, whenever $\mathrm{id}_X \times \phi' \in G$. Since

$$\begin{aligned} \Delta_X \otimes \langle (\pi')^{-1}(\operatorname{Graph}\phi')\rangle &= \langle \Delta_X \otimes (\pi')^{-1}(\operatorname{Graph}\phi')\rangle \\ &= \langle (\pi \times \pi')^{-1}(\operatorname{Graph}\mathrm{id}_X \times \phi')\rangle \subset \underline{R}, \end{aligned}$$

$(\pi')^{-1}(\operatorname{Graph}\phi') \subset R'$, hence $(\pi')^{-1}(\operatorname{Graph}\phi') = R'$. Thus $\operatorname{Graph}\phi' = \Delta_{X'/R'}$, so $\phi' = \mathrm{id}_{X'/R'}$.

Since there are no nontrivial compact subgroups in $\operatorname{Aut}(T)$,

$$G = \{\mathrm{id}_{X \times X'/\underline{\widetilde{R}}}\}.$$

It follows that $\widehat{R}_F = \underline{\widetilde{R}} = \Delta_X \otimes R' \subset R_X$. □

Here is an example of a minimal distal system with no compact subgroups in the group of its automorphisms. I thank Eli Glasner for turning my attention to the homeomorphism presented below.

First we state the following simple lemma.

LEMMA 18. *There are no nontrivial compact subgroups in the topological group* $(\mathbb{Z}^{\mathbb{N}}, \oplus)$, *where*

$$((k_1, k_2, \dots) \oplus (k'_1, k'_2, \dots))_n = \sum_{l=0}^{n} k_{n-l} k'_l,$$

for $n = 1, 2, \dots$, $k_0 = k'_0 = 1$.

PROOF. Let $K \subset \mathbb{Z}^{\mathbb{N}}$ be a compact subgroup and Π_n denote the projection onto the n'th coordinate. Since $\Pi_1\colon \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}$ is a continuous group homomorphism, $\Pi_1(K) = \{0\}$, hence $K \subset \{\underline{k} \in \mathbb{Z}^{\mathbb{N}} : k_1 = 0\}$. Assume now that $K \subset \{\underline{k} \in \mathbb{Z}^{\mathbb{N}} : k_1 = \dots = k_{n-1} = 0\}$. Then $\Pi_n|_K$ is a continuous group homomorphism and it follows that $K \subset \{\underline{k} \in \mathbb{Z}^{\mathbb{N}} : k_1 = \dots = k_n = 0\}$. By induction $K = \{(0, 0, \dots)\}$. □

EXAMPLE 19. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Put $T\colon \mathbb{T}^{\mathbb{N}} \to \mathbb{T}^{\mathbb{N}}$,

$$T(z_1, \dots, z_n, \dots) = (uz_1, z_1 z_2, \dots, z_{n-1} z_n, \dots),$$

where $u \in \mathbb{T}$ is not a root of the unity.

We will show that $(\operatorname{Aut}(T), \circ)$ as a topological group is equal to $(\mathbb{Z}^{\mathbb{N}}, \oplus)$.

Consider the restriction of T to the first n coordinates:

$$T_n(z_1, \dots, z_n) = (uz_1, z_1z_2, \dots, z_{n-1}z_n).$$

Since each factor T_n is canonical, every $S \in \mathrm{Aut}(T)$ leaves it invariant. Let S_n denote a restriction of S to T_n. We show, using the induction with respect to n, that for every $S \in \mathrm{Aut}(T)$ and $n \in \mathbb{N}$ there exist $u_n \in \mathbb{T}$ and $k_1, \dots, k_{n-1} \in \mathbb{Z}$ such that

$$\begin{aligned} (2)\quad & S_n(z_1, \dots, z_n) \\ &= (u^{k_1}z_1, u^{k_2}z_1^{k_1}z_2, \dots, u^{k_{n-1}}z_1^{k_{n-2}} \dots z_{n-2}^{k_1}z_{n-1}, u_n z_1^{k_{n-1}} \dots z_{n-1}^{k_1}z_n). \end{aligned}$$

Observe that every T_n is a cocycle extension of T_{n-1}. From results of [8] or [9] it follows that in case $n = 2$

$$S_2(z_1, z_2) = (u_1z_1, f(z_1)\upsilon(z_2)),$$

for some continuous function $f\colon \mathbb{T} \to \mathbb{T}$ and a continuous $\mathbb{T}$-automorphism υ. Let $f(z) = \sum a_m z^m$ be a Fourier series of f. Assume first that $\upsilon(z) = z^{-1}$. Since S_2 and T_2 commutes, we have $f(uz) = u_1z^2f(z)$, hence $|a_m| = |a_{m-2}|$. It follows that $f \equiv 0$, a contradiction. Thus $\upsilon = \mathrm{id}_{\mathbb{T}}$.

Now we have $f(uz) = u_1f(z)$, hence $u^m a_m = u_1 a_m$ for every $m \in \mathbb{Z}$. Since u is not a root of the unity there is $k_1 \in \mathbb{Z}$ such that $a_m = 0$ for $m \neq k_1$, $a_{k_1} \neq 0$ and $u_1 = u^{k_1}$. We have got

$$S_2(z_1, z_2) = (u^{k_1}z_1, u_2z_1^{k_1}z_2).$$

Assume now that (2) holds for $n - 1$. From the same results of [8] or [9] as above we know that

$$\begin{aligned} & S_n(z_1, \dots, z_n) \\ &= (u^{k_1}z_1, u^{k_2}z_1^{k_1}z_2, \dots, u_{n-1}z_1^{k_{n-2}} \dots z_{n-2}^{k_1}z_{n-1}, f(z_1, \dots, z_{n-1})\upsilon(z_n)), \end{aligned}$$

for some continuous function $f\colon \mathbb{T}^{n-1} \to \mathbb{T}$ and a continuous $\mathbb{T}$-automorphism υ. If $\upsilon(z) = z^{-1}$ then

$$f(uz_1, \dots, z_{n-2}z_{n-1}) = u_{n-1}z_1^{k_{n-2}}z_2^{k_{n-3}} \dots z_{n-2}^{k_1}z_{n-1}^2 f(z_1, \dots, z_{n-1}),$$

hence

$$\begin{aligned} & \sum a_{m_1, \dots, m_{n-1}} u^{m_1} z_1^{m_1+m_2} z_2^{m_2+m_3} \dots z_{n-2}^{m_{n-2}+m_{n-1}} z_{n-1}^{m_{n-1}} \\ &\qquad = \sum a_{m_1, \dots, m_{n-1}} u_{n-1} z_1^{m_1+k_{n-2}} z_2^{m_2+k_{n-3}} \dots z_{n-2}^{m_{n-2}+k_1} z_{n-1}^{m_{n-1}+2}. \end{aligned}$$

We see that

$$|a_{m_1-m_2+\dots-m_{n-1}, \dots, m_{n-2}-m_{n-1}, m_{n-1}}| = |a_{m_1-k_{n-2}, \dots, m_{n-2}-k_1, m_{n-1}-2}|,$$

for every $m_1, \ldots, m_{n-1} \in \mathbb{Z}$. It follows that $f \equiv 0$, a contradiction. Now we have $\upsilon = \mathrm{id}_{\mathbb{Z}}$, so

$$f(uz_1, \ldots, z_{n-2}z_{n-1}) = u_{n-1} z_1^{k_{n-2}} z_2^{k_{n-3}} \ldots z_{n-2}^{k_1} f(z_1, \ldots, z_{n-1}),$$

hence

$$\sum_{m_1,\ldots,m_{n-2}} a_{m_1,\ldots,m_{n-1}} u^{m_1} z_1^{m_1+m_2} z_2^{m_2+m_3} \ldots z_{n-2}^{m_{n-2}+m_{n-1}} = \sum_{m_1,\ldots,m_{n-2}} a_{m_1,\ldots,m_{n-1}} u_{n-1} z_1^{m_1+k_{n-2}} z_2^{m_2+k_{n-3}} \ldots z_{n-2}^{m_{n-2}+k_1},$$

for every $m_{n-1} \in \mathbb{Z}$. If $m_{n-1} \neq k_1$ then

$$|a_{m_1-m_2+\ldots-m_{n-1},\ldots,m_{n-2}-m_{n-1},m_{n-1}}| = |a_{m_1-k_{n-2},\ldots,m_{n-2}-k_1,m_{n-1}}|,$$

and it follows that $a_{m_1,\ldots,m_{n-2},m_{n-1}} = 0$ for every $m_1, \ldots, m_{n-2} \in \mathbb{Z}$ and $m_{n-1} \in \mathbb{Z} \setminus \{k_1\}$. If $m_{n-1} = k_1$, repeating our consideration for m_{n-2} and k_2 we get that $a_{m_1,\ldots,m_{n-2},k_1} = 0$ for every $m_1, \ldots, m_{n-3} \in \mathbb{Z}$ and $m_{n-2} \in \mathbb{Z} \setminus \{k_2\}$. After $n-2$ steps we obtain that $a_{m_1,\ldots,m_{n-1}} = 0$ for every $m_1 \in \mathbb{Z}$ and $(m_2, \ldots, m_{n-1}) \in (\mathbb{Z} \times \ldots \times \mathbb{Z}) \setminus \{(k_{n-2}, \ldots, k_1)\}$. Also we have that $u^{m_1} a_{m_1,k_{n-2},\ldots,k_1} = u_{n-1} a_{m_1,k_{n-2},\ldots,k_1}$ for every $m_1 \in \mathbb{Z}$. This forces that $a_{m_1,k_{n-2},\ldots,k_1} \neq 0$ for precisely one $m_1 = k_{n-1}$ and that $u_{n-1} = u^{k_{n-1}}$. We have shown (2) with $u_n = a_{k_{n-1},\ldots,k_1}$.

The formula (2) allows us to establish a bijection $J: \mathbb{Z}^{\mathbb{N}} \to \mathrm{Aut}(T)$ that is obviously a group isomorphism between $(\mathbb{Z}^{\mathbb{N}}, \oplus)$ and $(\mathrm{Aut}(T), \circ)$. It is easy to see that J is continuous. For inverse take an integer $N > 1$ and let $0 < \delta < 1/2^N$. Assume that $D(S, \overline{S}) < \delta$, where $J^{-1}(S) = (k_1, k_2, \ldots)$ and $J^{-1}(\overline{S}) = (\overline{k}_1, \overline{k}_2, \ldots)$. If $S = \overline{S}$ then $k_n = \overline{k}_n$ and we are done.

If $S \neq \overline{S}$ then put $l = \min\{n: k_n \neq \overline{k}_n\}$ and suppose that $l < N$. Then we have

$$\begin{aligned}\delta > D(S, \overline{S}) &= \frac{\rho(u^{k_l}, u^{\overline{k}_l})}{2^l} \\ &\quad + \sup_{z_1, z_2, \ldots \in \mathbb{T}} \sum_{n=l+1}^{\infty} \frac{\rho(u^{k_n} z_1^{k_{n-1}} \ldots z_{n-l}^{k_l}, u^{\overline{k}_n} z_1^{\overline{k}_{n-1}} \ldots z_{n-l}^{\overline{k}_l})}{2^n} \\ &\geq \frac{\rho(u^{k_l}, u^{\overline{k}_l})}{2^l} + \frac{\pi}{2^{l+1}} > \frac{1}{2^N}(\rho(u^{k_l}, u^{\overline{k}_l}) + \pi) > \delta\pi,\end{aligned}$$

a contradiction. Therefore $l \geq N$, hence $k_n = \overline{k}_n$ for $n = 1, \ldots, N-1$.

The author would like to express his gratitude to Mariusz Lemańczyk for his interest and advice during the preparation of this note.

References

[1] J. Auslander, *Minimal Flows and Their Extensions*, North-Holland Math. Stud. **153**; Notas Mat. **122** (1988), North-Holland–Amsterdam, New York, Oxford Tokyo.

[2] W. Bułatek, M. Lemańczyk and E. Lesigne, *On the filtering problem for stationary random $\mathbb{Z}^2$-fields*, IEEE Trans. Inform. Theory **51** (2005), 3586–3593.

[3] H. Furstenberg, *Disjointness in ergodic theory minimal sets, and a problem in Diophantine approximation*, Math. Systems Theory **1** (1965), 1–49.

[4] E. Glasner, *A topological version of a theorem of Veech and almost simple flows*, Ergodic Theory Dynam. Systems **10** (1990), 463–482.

[5] E. Glasner, M. K. Mentzen and A. Siemaszko, *A natural family of factors for minimal flows*, Contemp. Math. **215** (1998), 19–42.

[6] A. del Junco, M. Lemańczyk and M. K. Mentzen, *Semisimplicity, joinings and group extensions*, Studia Math **112** (1995), 141–164.

[7] K. K. Park and A. Siemaszko, *Relative topological Pinsker factors and entropy pairs*, Monatsh. Math. **134** (2001), 67–79.

[8] A. Siemaszko, *Ergodic and topological properties of distal dynamical systems*, thesis (1996), Toruń.

[9] ______, *A note on the centralizer of topological isometric extensions*, Comment. Math. Univ. Carolinae **46** (2005), 137–143.

Manuscript received May 12, 2006

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