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INFLUENCE OF A SMALL PERTURBATION ON POINCARÉ–ANDRONOV OPERATORS WITH NOT WELL DEFINED TOPOLOGICAL DEGREE

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(*Submitted by J. Mawhin*)

Abstract. Let $\mathcal{P}_\varepsilon \in C^0(\mathbb{R}^n, \mathbb{R}^n)$ be the Poincaré–Andronov operator over period $T > 0$ of T-periodically perturbed autonomous system $\dot{x} = f(x) + \varepsilon g(t, x, \varepsilon)$, where $\varepsilon > 0$ is small. Assuming that for $\varepsilon = 0$ this system has a T-periodic limit cycle x_0 we evaluate the topological degree $d(I - \mathcal{P}_\varepsilon, U)$ of $I - \mathcal{P}_\varepsilon$ on an open bounded set U whose boundary ∂U contains $x_0([0, T])$ and $\mathcal{P}_0(v) \neq v$ for any $v \in \partial U \setminus x_0([0, T])$. We give an explicit formula connecting $d(I - \mathcal{P}_\varepsilon, U)$ with the topological indices of zeros of the associated Malkin's bifurcation function. The goal of the paper is to prove the Mawhin's conjecture claiming that $d(I - \mathcal{P}_\varepsilon, U)$ can be any integer in spite of the fact that the measure of the set of fixed points of $\mathcal{P}_0$ on ∂U is zero.

1. Introduction

Consider the system of ordinary differential equations

$$\dot{x} = f(x) + \varepsilon g(t, x, \varepsilon), \tag{1.1}$$

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where $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $g \in C^0(\mathbb{R} \times \mathbb{R}^n \times [0,1], \mathbb{R}^n)$, $g(t+T, v, \varepsilon) \equiv g(t, v, \varepsilon)$ and $\varepsilon > 0$ is a small parameter.

We suppose that equation (1.1) defines a flow in $\mathbb{R}^n$, i.e. assume the uniqueness and global existence for the solutions of the Cauchy problems associated to (1.1). For each $v \in \mathbb{R}^n$ we denote by $x_\varepsilon(\,\cdot\,, v)$ the solution of (1.1) with $x_\varepsilon(0, v) = v$. Thus, the Poincaré–Andronov operator over the period $T > 0$ is defined by

$$\mathcal{P}_\varepsilon(v) := x_\varepsilon(T, v).$$

The problem of the existence (and even stability, see Ortega [11]) of T-periodic solutions of (1.1) with initial conditions inside an open bounded set U can be solved by evaluating the topological degree $d(I - \mathcal{P}_\varepsilon, U)$ of $I - \mathcal{P}_\varepsilon$ on U (see [6]). In the case when $\mathcal{P}_0$ has no fixed points on the boundary ∂U of U the problem is completely solved by Capietto, Mawhin and Zanolin [2] who proved that $d(I - \mathcal{P}_0, U) = (-1)^n d(f, U)$ generalizing the result by Berstein and Halanay [1] where U is assumed to be a neighbourhood of an isolated zero of f. In the case when $\mathcal{P}_0$ has fixed points on ∂U the pioneer result has been obtained by Mawhin [10] who considered the situation when $f = 0$. Mawhin proved that if $g_0(v) = \int_0^T g(\tau, v, 0)\, d\tau$ does not vanish on ∂U then $d(I - \mathcal{P}_\varepsilon, U)$ is defined for $\varepsilon > 0$ sufficiently small and it can be evaluated as $d(I - \mathcal{P}_\varepsilon, U) = d(-g_0, U)$. This paper studies an intermediate situation when the fixed points of $\mathcal{P}_0$ fill a part of ∂U. Current results on this subject deal with the case when ∂U contains a fixed number of fixed points, e.g. Feckan [4], Kamenskiĭ–Makarenkov–Nistri [5]. As a part of a wider study of this problem Jean Mawhin (his seminar, November 2005) asked a question on evaluating $d(I - \mathcal{P}_\varepsilon, U)$ in the case when ∂U contains a curve of fixed points of $\mathcal{P}_0$. He settled the following conjecture:

Mawhin's Conjecture. *For small $\varepsilon > 0$ the topological degree $d(I - \mathcal{P}_\varepsilon, U)$ can be any integer depending on the perturbation term g in spite of the fact that the measure of $\{v \in \partial U : \mathcal{P}_0(v) = v\}$ is zero.*

The goal of this paper is to evaluate $d(I - \mathcal{P}_\varepsilon, U)$ and to give a proof of the above conjecture in the case when $\{v \in \partial U : \mathcal{P}_0(v) = v\}$ forms a curve coming from a T-periodic limit cycle of the unperturbed system

$$\dot{x} = f(x). \tag{1.2}$$

Our fundamental assumption is that the algebraic multiplicity of the multiplicator $+1$ of the linearized system

$$\dot{y} = f'(x_0(t))y \tag{1.3}$$

equals to 1. In this case we say that the cycle x_0 is nondegenerate.

The paper is organized as follows. In Section 2 for a fixed point v_ε of $\mathcal{P}_\varepsilon$ satisfying $v_\varepsilon \to v_0 \in x_0([0, T])$ as $\varepsilon \to 0$ we obtain an asymptotic direction of the vector $v_\varepsilon - v_0$. By means of this result we evaluate in Section 3 the topological index of such fixed points $v_\varepsilon \to v_0 \in x_0([0,T])$ as $\varepsilon \to 0$ that $v_\varepsilon \in U$. Finally in Section 4 we give a proof of the Mawhin's conjecture provided that a technical assumption (see assumption 4.1) is satisfied.

2. Direction the fixed points of Poincaré–Andronov operator move when the perturbation increases

Since the cycle x_0 is nondegenerate we can define (see [3], Chapter IV, § 20, Lemma 1) a matrix function Z_{n-1} solving the adjoint system

$$\dot{z} = -(f'(x_0(t)))^* z \tag{2.1}$$

and having the form $Z_{n-1}(t) = \Phi(t)\mathrm{e}^{\Lambda t}$, where Φ is a continuous T-periodic $n \times n-1$ matrix function and Λ is a $n-1 \times n-1$-matrix with different from 0 eigenvalues. Let z_0 be the T-periodic solution of (2.1) satisfying $z_0(0)^*\dot{x}_0(0) = 1$. Finally, we denote by Y_{n-1} the $n \times n-1$ matrix function whose columns are solutions of the linearized system (1.3) satisfying $Y_{n-1}(0)^* Z_{n-1}(0) = I$.

The results of this paper are formulated in terms of the following auxiliary functions:

$$M(\theta) = \int_0^T z_0(\tau)^* g(\tau - \theta, x_0(\tau), 0)\, d\tau,$$

$$M^\perp(t,\theta) = (\mathrm{e}^{\Lambda T})^*((\mathrm{e}^{\Lambda T})^* - I)^{-1} \int_{t-T+\theta}^{t+\theta} (Z_{n-1}(\tau))^* g(\tau - \theta, x_0(\tau), 0)\, d\tau,$$

$$\angle(u,v) = \arccos \frac{\langle u, v\rangle}{\|u\| \cdot \|v\|}.$$

The function M was proposed by Malkin (see [9], formula 3.13) and the function $M^\perp$ is a generalization of the function $M_z^\perp$ of [8].

Next Theorem 2.1 shows that if a family $\{x_{\varepsilon,\lambda}\}_{\lambda\in\Lambda}$ of T-periodic solutions of (1.1) emanate from $x_0(\,\cdot\, + \theta_0)$ then a suitable projection of $x_{\varepsilon,\lambda}(t) - x_0(t + \theta_0)$ can be always controlled. Though motivated by the Mawhin's conjecture, Theorem 2.1 can be of a general interest in the theory of oscillations playing a role of the first approximation formula (see Loud [7], formula 1.3, Lemma 1 and formula for x at p. 510) in the case when the zeros of the bifurcation function M are not necessary isolated.

Theorem 2.1. *Let x_0 be a nondegenerate T-periodic cycle of* (1.2). *Let $\{x_{\varepsilon,\lambda}\}_{\lambda\in\Lambda}$ be a family of T-periodic solutions of* (1.1) *such that $x_{\varepsilon,\lambda}(t) \to x_0(t+\theta_0)$ as $\varepsilon \to 0$ uniformly with respect to $t \in [0,T]$ and $\lambda \in \Lambda$. Then*

$$\angle(Z_{n-1}(t+\theta_0)^*(x_{\varepsilon,\lambda}(t) - x_0(t+\theta_0)), M^\perp(t,\theta_0)) \to 0 \quad \textit{as } \varepsilon \to 0$$

uniformly with respect to $t \in [0,T]$ and $\lambda \in \Lambda$.

Proof. The proof makes use of the idea of Theorem 3.1 of [8]. In the sequel (A, B) denotes the matrix composed by columns of matrixes A and B. Let $a_\varepsilon \in C^0([0,T],\mathbb{R}^n)$ be given by

$$a_\varepsilon(t) = (z_0(t+\theta_0), Z_{n-1}(t+\theta_0))^* \,(x_\varepsilon(t) - x_0(t+\theta_0)). \tag{2.2}$$

Denoting $Y(t) = (\dot{x}_0(t), Y_{n-1}(t))$ by Perron's lemma [12] (see also Demidovich ([3, Section III, § 12]) we have

$$(z_0(t), Z_{n-1}(t))^* \, Y(t) = I, \quad \text{for any } t \in \mathbb{R}.$$

Thus

$$x_\varepsilon(t) - x_0(t+\theta_0) = Y(t+\theta_0)a_\varepsilon(t), \quad \text{for any } t \in \mathbb{R}. \tag{2.3}$$

By subtracting (1.2) where x is replaced by $x_0(\,\cdot\,+\theta_0)$ from (1.1) where x is replaced by x_ε we obtain

$$\begin{aligned}\dot{x}_\varepsilon(t) - \dot{x}_0(t+\theta_0) = {} & f'(x_0(t+\theta_0))(x_\varepsilon(t) - x_0(t+\theta_0)) \\ & + \varepsilon g(t, x_\varepsilon(t), \varepsilon) + o(t, x_\varepsilon(t) - x_0(t+\theta_0)),\end{aligned} \tag{2.4}$$

where $o(t,v)/\|v\| \to 0$ as $\mathbb{R}^n \ni v \to 0$ uniformly with respect to $t \in [0,T]$. By substituting (2.3) into (2.4) we have

$$\begin{aligned}&\dot{Y}(t+\theta_0)a_\varepsilon(t) + Y(t+\theta_0)\dot{a}_\varepsilon(t) \\ &\quad = f'(x_0(t+\theta_0))Y(t+\theta_0)a_\varepsilon(t) + \varepsilon g(t, x_\varepsilon(t), \varepsilon) + o(t, x_\varepsilon(t) - x_0(t+\theta_0)).\end{aligned}$$

Since $f'(x_0(t))Y(t) = \dot{Y}(t)$ the last relation can be rewritten as

$$Y(t+\theta_0)\dot{a}_\varepsilon(t) = \varepsilon g(t, x_\varepsilon(t), \varepsilon) + o(t, x_\varepsilon(t) - x_0(t+\theta_0)). \tag{2.5}$$

Applying $Z_{n-1}(t+\theta_0)^*$ to both sides of (2.5) we have

$$(0, I)\dot{a}_\varepsilon(t) = \varepsilon Z_{n-1}(t+\theta_0)^* \, g(t, x_\varepsilon(t), \varepsilon) + Z_{n-1}(t+\theta_0)^* \, o(t, x_\varepsilon(t) - x_0(t+\theta_0)),$$

where 0 denotes the $n-1$ dimensional zero vector and I stays for the identical $n-1 \times n-1$ matrix. So

$$\begin{aligned}(0, I)a_\varepsilon(t) = {} & (0, I)a_\varepsilon(t_0) + \varepsilon \int_{t_0}^{t} Z_{n-1}(\tau+\theta_0)^* g(\tau, x_\varepsilon(\tau), \varepsilon)\, d\tau \\ & + \int_{t_0}^{t} Z_{n-1}(\tau+\theta_0)^* o(\tau, x_\varepsilon(\tau) - x_0(\tau+\theta_0))\, d\tau.\end{aligned} \tag{2.6}$$

From the definition of Z_{n-1} we have that $Z_{n-1}(t)^* = (\mathrm{e}^{\Lambda T})^* Z_{n-1}(t-T)^*$ for any $t \in \mathbb{R}$ and so $(0, I)a_\varepsilon(t)$ satisfies

$$(0, I)a_\varepsilon(t_0) = (\mathrm{e}^{\Lambda T})^*(0, I)a_\varepsilon(t_0 - T) \quad \text{for any } t_0 \in [0,T]. \tag{2.7}$$

Solving (2.6)–(2.7) with respect to $(0,I)a_{\varepsilon,n}(t_0)$ we obtain

$$\begin{aligned}(0,I)a_\varepsilon(t_0) &= \varepsilon(\mathrm{e}^{\Lambda T})^*((\mathrm{e}^{\Lambda T})^* - I)^{-1}\int_{t_0-T}^{t_0} Z_{n-1}(\tau+\theta_0)^* g(\tau, x_\varepsilon(\tau),\varepsilon)\,d\tau\\ &\quad + (\mathrm{e}^{\Lambda T})^*((\mathrm{e}^{\Lambda T})^* - I)^{-1}\int_{t_0-T}^{t_0} Z_{n-1}(\tau+\theta_0)^* o(\tau, x_\varepsilon(\tau)-x_0(\tau+\theta_0))\,d\tau\end{aligned}$$

for any $t_0\in[0,T]$. On the other hand from (2.2) we obtain

$$Z_{n-1}(t+\theta_0)^*(x_\varepsilon(t)-x_0(t+\theta_0)) = (0,I)a_\varepsilon(t)$$

and therefore

$$\begin{aligned}(2.8)\quad & Z_{n-1}(t+\theta_0)^*(x_\varepsilon(t)-x_0(t+\theta_0)) - q_\varepsilon(t)\\ &\qquad = \varepsilon(\mathrm{e}^{\Lambda T})^*((\mathrm{e}^{\Lambda T})^* - I)^{-1}\int_{t-T}^{t} Z_{n-1}(\tau+\theta_0)^* g(\tau, x_\varepsilon(\tau),\varepsilon)\,d\tau,\end{aligned}$$

where

$$q_\varepsilon = (\mathrm{e}^{\Lambda T})^*((\mathrm{e}^{\Lambda T})^* - I)^{-1}\int_{t-T}^{t} Z_{n-1}(\tau+\theta_0)^* o(\tau, x_\varepsilon(\tau)-x_0(\tau+\theta_0))\,d\tau.$$

From (2.8) we obtain

$$\begin{aligned}&\angle(Z_{n-1}(t+\theta_0)^*(x_\varepsilon(t)-x_0(t+\theta_0)), M^\perp(t,\theta_0))\\ &= \angle\bigg(Z_{n-1}(t+\theta_0)^*\frac{x_\varepsilon(t)-x_0(t+\theta_0)}{\|x_\varepsilon - x_0(\,\cdot\,+\theta_0)\|_{[0,T]}}, M^\perp(t,\theta_0)\bigg)\\ &- \angle\bigg(Z_{n-1}(t+\theta_0)^*\frac{x_\varepsilon(t)-x_0(t+\theta_0)}{\|x_\varepsilon - x_0(\,\cdot\,+\theta_0)\|_{[0,T]}} - \frac{q_\varepsilon(t)}{\|x_\varepsilon - x_0(\,\cdot\,+\theta_0)\|_{[0,T]}}, M^\perp(t,\theta_0)\bigg)\\ &+ \angle\bigg((\mathrm{e}^{\Lambda T})^*((\mathrm{e}^{\Lambda T})^* - I)^{-1}\int_{t-T}^{t} Z_{n-1}(\tau+\theta_0)^* g(\tau, x_\varepsilon(\tau),\varepsilon)\,d\tau, M^\perp(t,\theta_0)\bigg).\end{aligned}$$

But the difference of the first two terms in the right hand part of the last equality tends to zero as $\varepsilon\to 0$ and thus the thesis follows. □

Next theorem is a reformulation of Theorem 2.1 suitable for our further considerations.

Theorem 2.2. *Let x_0 be a nondegenerate T-periodic cycle of* (1.2). *Let $\{x_{\varepsilon,\lambda}\}_{\lambda\in\Lambda}$ be a family of T-periodic solutions of* (1.1) *such that $x_{\varepsilon,\lambda}(t)\to x_0(t+\theta_0)$ as $\varepsilon\to 0$ uniformly with respect to $t\in[0,T]$ and $\lambda\in\Lambda$. Let $l\in\mathbb{R}^n$ be an arbitrary vector such that $\langle l,\dot x_0(\theta_0)\rangle = 0$. Assume that $\langle l, Y_{n-1}(\theta_0)M^\perp(0,\theta_0)\rangle\neq 0$. Then there exists $\varepsilon_0>0$ such that, for any $\lambda\in\Lambda$ and any $\varepsilon\in(0,\varepsilon_0]$,*

$$\langle l, x_{\varepsilon,\lambda}(0)-x_0(\theta_0)\rangle > 0 \quad \text{or} \quad \langle l, x_{\varepsilon,\lambda}(0)-x_0(\theta_0)\rangle < 0$$

according as

$$\langle l, Y_{n-1}(\theta_0)M^\perp(0,\theta_0)\rangle > 0 \quad \text{or} \quad \langle l, Y_{n-1}(\theta_0)M^\perp(0,\theta_0)\rangle < 0.$$

Proof. By Perron's lemma [12] (see also Demidovich ([3, Section III, § 12]) we have

$$v = Y_{n-1}(t)Z_{n-1}(t)^*v + \dot{x}_0(t)z_0(t)^*v$$

for any $v \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Therefore

$$\langle l, x_{\varepsilon,\lambda}(0) - x_0(\theta_0)\rangle = \langle l, Y_{n-1}(\theta_0)Z_{n-1}(\theta_0)^*(x_{\varepsilon,\lambda}(0) - x_0(\theta_0)) + \dot{x}_0(\theta_0)z_0(\theta_0)^*(x_{\varepsilon,\lambda}(0) - x_0(\theta_0))\rangle\langle Y_{n-1}(\theta_0)^*l, Z_{n-1}(\theta_0)^*(x_{\varepsilon,\lambda}(0) - x_0(\theta_0))\rangle.$$

Since $\langle Y_{n-1}(\theta_0)^*l, M^\perp(0,\theta_0)\rangle \neq 0$ then by Theorem 2.1 there exists $\varepsilon_0 > 0$ such that

$$\operatorname{sign}\langle Y_{n-1}(\theta_0)^*l, Z_{n-1}(\theta_0)^*(x_{\varepsilon,\lambda}(0) - x_0(\theta_0))\rangle = \operatorname{sign}\langle Y_{n-1}(\theta_0)^*l, M^\perp(0,\theta_0)\rangle$$

for any $\varepsilon \in (0, \varepsilon_0]$ and $\lambda \in \Lambda$ and thus the proof is complete. □

3. The topological degree of the perturbed Poincaré–Andronov operator

To proceed to the proof of our main Theorem 3.1 we need three additional theorems which are formulated below for the convenience of the reader.

Malkin's Theorem (see [9, p. 41]). *Assume that T-periodic solutions x_ε of* (1.1) *satisfy the property $x_\varepsilon(t) \to x_0(t + \theta_0)$ as $\varepsilon \to 0$. Then $M(\theta_0) = 0$.*

Capietto–Mawhin–Zanolin Theorem (see [2, Corollary 2]). *Let $V \subset \mathbb{R}^n$ be an open bounded set. Assume that $\mathcal{P}_0(v) \neq v$ for any $v \in \partial V$. Then $d(I - \mathcal{P}_0, V) = (-1)^n d(f, V)$.*

Kamenskiĭ–Makarenkov–Nistri Theorem (see [5, Corollary 2.8]). *Assume that $\theta_0 \in [0, T]$ is an isolated zero of the bifurcation function M. Then there exist $\varepsilon_0 > 0$ and $r > 0$ such that $\mathcal{P}_\varepsilon(v) \neq v$ for any $\|v - v_0\| = r$ and any $\varepsilon \in (0, \varepsilon_0]$. Moreover, $d(I - \mathcal{P}_\varepsilon, B_r(v_0)) = \operatorname{ind}(\theta_0, M)$.*

We will say that the set $U \subset \mathbb{R}^n$ has a smooth boundary if given any $v \in \partial U$ there exists $r > 0$ and a homeomorphism of $\{\xi \in \mathbb{R}^{n-1} : \|\xi\| \leq 1\}$ onto $\partial U \cap B_r(v)$. Thus any set U with a smooth boundary possesses a tangent plane to ∂U at any $v \in \partial U$. This tangent plane will be denoted by $L_U(v)$. Moreover, if U has a smooth boundary and $\mathbb{R}^n \ni h \notin L_U(v)$ then there exists $\lambda_0 > 0$ such that either $\lambda h + v \in U$ for any $\lambda \in (0, \lambda_0]$ or $\lambda h + v \notin U$ for any $\lambda \in (0, \lambda_0]$. In this case we will say that h centered at v is directed inward to U or outward respectively.

Theorem 3.1. *Let x_0 be a nondegenerate T-periodic cycle of* (1.2). *Let $U \subset \mathbb{R}^n$ be an open bounded set with a smooth boundary and $x_0([0, T]) \subset \partial U$. Assume that $\mathcal{P}_0(v) \neq v$ for any $v \in \partial U \setminus x_0([0, T])$. Assume that M has a finite number of zeros $0 \leq \theta_1 < \ldots < \theta_k < T$ on $[0, T]$ and $\operatorname{ind}(\theta_i, M) \neq 0$ for any*

$i \in \overline{1,k}$. Assume that $Y_{n-1}(\theta_i)M^\perp(0,\theta_i) \notin L_U(x_0(\theta_i))$ for any $i \in \overline{1,k}$. Then, for any $\varepsilon > 0$ sufficiently small, $d(I - \mathcal{P}_\varepsilon, U)$ is defined. Moreover,

$$d(I - \mathcal{P}_\varepsilon, U) = (-1)^n d(f,U) - \sum_{i=1}^{k} \operatorname{ind}(\theta_i, M) D_i,$$

where $D_i = 1$ or $D_i = 0$ according as $Y_{n-1}(\theta_i)M^\perp(0,\theta_i)$ centered at $x_0(\theta_i)$ is directed inward to U or outward.

Proof. By Kamenskiĭ–Makarenkov–Nistri theorem there exists $r > 0$ and $\varepsilon_0 > 0$ such that

$$d(I - \mathcal{P}_\varepsilon, B_r(x_0(\theta_i))) = \operatorname{ind}(\theta_i, M) \tag{3.1}$$

for any $\varepsilon \in (0,\varepsilon_0]$ and $i \in \overline{1,k}$. From Malkin's theorem we have the following "Malkin's property": $r > 0$ can be decreased, if necessary, in such a way that there exists $\varepsilon_0 > 0$ such that any T-periodic solution x_ε of (1.1) with initial condition $x_\varepsilon(0) \in B_r(x_0([0,T]))$ and $\varepsilon \in (0,\varepsilon_0]$ satisfies $x_\varepsilon(0) \in \bigcup_{i\in\overline{1,k}} B_r(x_0(\theta_i))$. Malkin's property implies that

$$d\bigg(I - \mathcal{P}_\varepsilon, \bigg(B_r(x_0([0,T])) \setminus \bigcup_{i\in\overline{1,k}} B_r(x_0(\theta_i))\bigg) \cap U\bigg) = 0 \tag{3.2}$$

for any $\varepsilon \in (0,\varepsilon_0]$. Denote by l_i the perpendicular to $L_U(x_0(\theta_i))$ directed outward away from U or inward according as $(Z_{n-1}(\theta_i)^*)^{-1}M^\perp(0,\theta_i)$ centered at $x_0(\theta_i)$ is directed outward away from U or inward. From Theorem 2.2 and Malkin's property we have that $\varepsilon_0 > 0$ can be diminished in such a way that for any $i \in \overline{1,k}$ any T-periodic solution x_ε of (1.1) with initial condition $x_\varepsilon(0) \in B_r(x_0(\theta_i))$ and $\varepsilon \in (0,\varepsilon_0]$ satisfies $x_\varepsilon(0) \in B_r(x_0(\theta_i)) \cap U$ or $x_\varepsilon(0) \notin B_r(x_0(\theta_i)) \cap U$ according as $D_i = 1$ or $D_i = 0$. This observation allows to deduce from (3.1) that

$$d(I - \mathcal{P}_\varepsilon, B_r(x_0(\theta_i)) \cap U) = \operatorname{ind}(\theta_i, M), \quad \text{if } D(\theta_i) = 1, \tag{3.3}$$

$$d(I - \mathcal{P}_\varepsilon, B_r(x_0(\theta_i)) \cap U) = 0, \quad \text{if } D(\theta_i) = 0, \tag{3.4}$$

for any $\varepsilon \in (0,\varepsilon_0]$ and $i \in \overline{1,k}$.

Observe that our choice of $r > 0$ ensures that $\mathcal{P}_0(v) \neq v$ for any $v \in \partial(U \setminus B_r(x_0([0,T])))$. Thus, by Capietto–Mawhin–Zanolin theorem we have $d(I - \mathcal{P}_0, U \setminus B_r(x_0([0,T]))) = (-1)^n d(f, U \setminus B_r(x_0([0,T])))$. Without loss of generality we can consider $r > 0$ sufficiently small such that $d(f, U \setminus B_r(x_0([0,T]))) = d(f,U)$ obtaining

$$d(I - \mathcal{P}_0, U \setminus B_r(x_0([0,T]))) = (-1)^n d(f,U). \tag{3.5}$$

Since

$$d(I-\mathcal{P}_\varepsilon,U)=d\bigg(I-\mathcal{P}_\varepsilon,\bigg(B_r(x_0([0,T]))\setminus\bigcup_{i\in\overline{1,k}}B_r(x_0(\theta_i))\bigg)\cap U\bigg)$$
$$+\,d\bigg(I-\mathcal{P}_\varepsilon,\bigcup_{i\in\overline{1,k}}B_r(x_0(\theta_i))\cap U\bigg)+d(I-\mathcal{P}_\varepsilon,U\setminus B_r(x_0([0,T])))$$

the conclusion follows from formulas (3.2)–(3.5). □

4. A proof of the Mawhin's conjecture

In this section we assume that the set $U\subset\mathbb{R}^n$ has a smooth boundary and there exists $v_{n-1}\in\mathbb{R}^{n-1}$ satisfying the following assumption

$$Y_{n-1}(t)(\mathrm{e}^{\Lambda T})^*((\mathrm{e}^{\Lambda T})^*-I)^{-1}(\mathrm{e}^{\Lambda t})^*v_{n-1}\notin L_U(t)\quad\text{for any } t\in[0,T]. \tag{4.1}$$

We note that assumption (4.1) does not depend on the perturbation term of (1.1) and relies to unperturbed system (1.2). Let $D=1$ or $D=0$ according as $Y_{n-1}(0)(\mathrm{e}^{\Lambda T})^*((\mathrm{e}^{\Lambda T})^*-I)^{-1}(\mathrm{e}^{\Lambda t})^*v_{n-1}$ centered at $x_0(0)$ is directed inward to U or outward. Given odd $m\in\mathbb{N}$ we construct the perturbation term g in such a way that $d(I-\mathcal{P}_\varepsilon,U)=(-1)^n d(f,U)-m(2D-1)$ for any $\varepsilon>0$ sufficiently small. Without loss of generality we consider $T=2\pi$.

Since $(z_0(t),Z_{n-1}(t))$ is nonsingular then $((z_0(t),\Phi(t))^*$ is nonsingular as well. Define $\Omega\colon x_0([0,2\pi])\to\mathbb{R}^n$ as $\Omega(x_0(t))=((z_0(t),\Phi(t))^*)^{-1}$ for any $t\in[0,2\pi]$. By Uryson's theorem (see [6, Chapter 1, Theorem 1.1]) Ω can be continued to the whole $\mathbb{R}^n$ in such a way that $\Omega\in C^0(\mathbb{R}^n,\mathbb{R}^n)$. Analogously, we consider $\widetilde{\Gamma}\in C^0(\mathbb{R}^n,\mathbb{R}^n)$ such that $\widetilde{\Gamma}(x_0(t))=(\arcsin(\sin t),0,\dots,0)^*$ and denote by $\Gamma\in C^0(\mathbb{R}^n,\mathbb{R})$ the first component of $\widetilde{\Gamma}$. Let us define a 2π-periodic α-approximation of $((\mathrm{e}^{\Lambda t})^*)^{-1}$ on $[-2\pi,0]$ by

$$\begin{aligned}\mathrm{e}_\alpha(t)&=((\mathrm{e}^{\Lambda t})^*)^{-1}, && \text{if } t\in[-2\pi,-\alpha],\\ \mathrm{e}_\alpha(t)&=\frac{t}{-\alpha}((\mathrm{e}^{-\Lambda\alpha})^*)^{-1}+\left(1-\frac{t}{-\alpha}\right)((\mathrm{e}^{-2\pi\Lambda})^*)^{-1}, && \text{if } t\in[-\alpha,0],\end{aligned}$$

which is continued to $(-\infty,\infty)$ by the 2π-periodicity. We are now in a position to introduce the required perturbation term, namely we consider that the perturbed system (1.1) has the following form

$$\dot{x}=f(x)+\varepsilon\Gamma(x)\Omega(x)\begin{pmatrix}D\sin(mt)+(1-D)\cos(mt)\\(D\cos(mt)+(1-D)\sin(mt))\mathrm{e}_\alpha(t)v_{n-1}\end{pmatrix}, \tag{4.2}$$

where $\alpha>0$ is sufficiently small. Consequently we denote by $\mathcal{P}_\varepsilon$ the Poincaré–Andronov operator of system (4.2) over the period 2π.

PROPOSITION 4.1. *Let $x_0([0,T]) \subset U \subset \mathbb{R}^n$ be an open bounded set with a smooth boundary and assume that there exists $v_{n-1} \in \mathbb{R}^n$ such that (4.1) is satisfied. Then given any odd $m > 0$ there exists $\alpha_0 > 0$ such that for any fixed $\alpha \in (0, \alpha_0]$ and $\varepsilon > 0$ sufficiently small $d(I - \mathcal{P}_\varepsilon, U)$ is defined and*

$$d(I - \mathcal{P}_\varepsilon, U) = \begin{cases} (-1)^n d(f, U) - m & \text{if } D = 1, \\ (-1)^n d(f, U) + m & \text{if } D = 0. \end{cases}$$

PROOF. By the definition of Ω and Γ we have

$$\begin{gathered} \begin{pmatrix} z_0(t)^* \\ Z_{n-1}(t)^* \end{pmatrix} \Omega(x_0(t)) = \begin{pmatrix} 1 & 0 \\ 0 & (\mathrm{e}^{\Lambda t})^* \end{pmatrix}, \\ \Gamma(x_0(t)) = \arcsin(\sin t). \end{gathered} \tag{4.3}$$

Therefore, taking into account that m is odd, we obtain the following formula for the bifurcation function M

$$\begin{aligned} M(\theta) &= \int_0^{2\pi} \arcsin(\sin \tau)(D \sin(m(\tau - \theta)) + (1 - D) \cos(m(\tau - \theta)))\, d\tau \\ &= (-1)^{(m-1)/2} \frac{4D \cos(m\theta) + 4(1 - D) \sin(m\theta)}{m^2} \end{aligned}$$

whose zeros are $\theta_j = (1/m)(D\pi/2 + j\pi)$, $j \in \overline{0, 2m - 1}$. Moreover,

$$\begin{aligned} &\operatorname{ind}(\theta_j, M) = \operatorname{sign}(M'(\theta_j)) \\ &= (-1)^{(m-1)/2} \operatorname{sign}\bigg(\frac{4m(-D \sin(D\pi/2 + j\pi) + (1 - D) \cos(D\pi/2 + j\pi))}{m^2}\bigg). \end{aligned} \tag{4.4}$$

Let us denote by $M_\alpha^\perp$ the function $M^\perp$ corresponding to system (4.2). From (4.3) we have that

$$\begin{aligned} M_\alpha^\perp(0, \theta) &= (\mathrm{e}^{\Lambda T})^* ((\mathrm{e}^{\Lambda T})^* - I)^{-1} \int_{-2\pi}^0 (Z_{n-1}(s + \theta))^* g(s, x_0(s + \theta), 0)\, ds \\ &= (\mathrm{e}^{\Lambda T})^* ((\mathrm{e}^{\Lambda T})^* - I)^{-1} (\mathrm{e}^{\Lambda \theta})^* \\ &\quad \circ \int_{-2\pi}^0 (\mathrm{e}^{\Lambda s})^* \mathrm{e}_\alpha(s) v_{n-1} \\ &\quad \cdot \arcsin(\sin(s + \theta))(D \cos(ms) + (1 - D) \sin(ms))\, ds. \end{aligned}$$

Since

$$\begin{aligned} &\int_{-2\pi}^0 \arcsin(\sin(s + \theta))(D \cos(ms) + (1 - D) \sin(ms))\, ds \\ &\qquad\qquad = -(-1)^{(m-1)/2} \cdot \frac{4(D \sin(m\theta) + (1 - D) \cos(m\theta))}{m^2} \end{aligned}$$

by taking into account that m is odd we have that $M_\alpha^\perp(0,\theta) \to M_0^\perp(0,\theta)$ as $\alpha \to 0$, where

$$M_0^\perp(0,\theta) = -(\mathrm{e}^{\Lambda T})^*((\mathrm{e}^{\Lambda T})^* - I)^{-1}(\mathrm{e}^{\Lambda\theta})^* v_{n-1}(-1)^{(m-1)/2} \cdot \frac{4(D\sin(m\theta) + (1-D)\cos(m\theta))}{m^2}.$$

Put $q(\theta) = -(-1)^{(m-1)/2}(D\sin(m\theta) + (1-D)\cos(m\theta))$. Then, taking any $\theta \in [0,2\pi]$ and using the definition of D we conclude that $Y_{n-1}(\theta)M_0^\perp(0,\theta)$ centered at $x_0(\theta)$ is directed inward to U or outward according as $\mathrm{sign}(q(\theta))(2D-1) = 1$ or $\mathrm{sign}(q(\theta))(2D-1) = -1$. Therefore, there exists $\alpha_0 > 0$ such that for any $\alpha \in [0,\alpha_0]$ and any $\theta \in [0,2\pi]$ we have that $Y_{n-1}(\theta)M_\alpha^\perp(0,\theta)$ centered at $x_0(\theta)$ is directed inward to U or outward according as $\mathrm{sign}(q(\theta))(2D-1) = 1$ or $\mathrm{sign}(q(\theta))(2D-1) = -1$. Thus denoting by $\mathcal{P}_{\varepsilon,\alpha}$ the Poincaré–Andronov operator of system (4.2) from Theorem 3.1 we have that

$$d(I - \mathcal{P}_{\varepsilon,\alpha}, U) = (-1)^n d(f,U) - \sum_{j\in\overline{0,2m-1}:\mathrm{sign}(q(\theta_j))(2D-1)=1} \mathrm{ind}(\theta_j, M) \tag{4.5}$$

for any $\alpha \in (0,\alpha_0]$. Consider the case when $D = 1$. Then the property $\mathrm{sign}(q(\theta_j))(2D-1) = 1$ is equivalent to

$$(-1)^{(m-1)/2}\mathrm{sign}(\sin(\pi/2 + j\pi)) = -1. \tag{4.6}$$

If $j \in \overline{0,2m-1}$ satisfies (4.6) then (4.4) implies $\mathrm{ind}(\theta_j, M) = 1$. Since there exists exactly m elements of $\overline{0,2m-1}$ satisfying (4.6) then (4.5) can be rewritten as $d(I-\mathcal{P}_\varepsilon, U) = d(f,U) - m$. Analogously, if $D = 0$ then $\mathrm{sign}(q(\theta_j))(2D-1) = 1$ is equivalent to $(-1)^{(m-1)/2}\mathrm{sign}(\cos(j\pi)) = -1$ that in combination with (4.4) gives $\mathrm{ind}(\theta_j, M) = -1$ allowing to rewrite (4.5) in the form $d(I - \mathcal{P}_\varepsilon, U) = d(f,U) + m$. □

At the end of the paper we note that system (1.2) should exhibit very complex behavior in order that assumption (4.1) be not satisfied with any $v_{n-1} \in \mathbb{R}^{n-1}$. Particularly, (4.1) holds true for the prototypic unperturbed system (1.2)

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1(x_1^2 + x_2^2 - 1), \\ \dot{x}_2 &= -x_1 - x_2(x_1^2 - x_2^2 - 1), \\ \dot{x}_3 &= -x_3 \end{aligned} \tag{4.8}$$

possessing the nondegenerate 2π-periodic cycle $x_0(t) = \binom{\sin t}{\cos t}$ and $U = B_1(0) = \{v \in \mathbb{R}^3 : \|v\| < 1\}$. Indeed, it can be easily checked that

$$\Phi(t) = \left(\begin{pmatrix}\sin t\\ 0\end{pmatrix}, \begin{pmatrix}\cos t\\ 0\end{pmatrix}, \begin{pmatrix}0\\ 1\end{pmatrix}\right)^*, \quad \mathrm{e}^{\Lambda t} = \begin{pmatrix}\mathrm{e}^{2t} & 0\\ 0 & \mathrm{e}^{t}\end{pmatrix} \quad \text{and} \quad Y_{n-1}(t) = \Phi(t)\mathrm{e}^{-\Lambda t}$$

in this case. Thus, taking $v_{n-1} = \binom{1}{0}$ we have

$$Y_{n-1}(t)(\mathrm{e}^{\Lambda T})^*((\mathrm{e}^{\Lambda T})^* - I)^{-1}(\mathrm{e}^{\Lambda t})^* v_{n-1} = \frac{\mathrm{e}^{2t}}{\mathrm{e}^{2t}-1}(\sin t, \cos t, 0)^*.$$

This last vector centered at $x_0(t)$ is perpendicular to ∂U for any $t \in [0, 2\pi]$.

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