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ALMOST HOMOCLINIC SOLUTIONS FOR THE SECOND ORDER HAMILTONIAN SYSTEMS

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Abstract. The second order Hamiltonian system $\ddot{q} + V_q(t,q) = f(t)$, where $t \in \mathbb{R}$ and $q \in \mathbb{R}^n$, is considered. We assume that a potential $V \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ is of the form $V(t,q) = -K(t,q) + W(t,q)$, where K satisfies the pinching condition and $W_q(t,q) = o(|q|)$, as $|q| \to 0$ uniformly with respect to t. It is also assumed that $f \in C(\mathbb{R}, \mathbb{R}^n)$ is non-zero and sufficiently small in $L^2(\mathbb{R}, \mathbb{R}^n)$. In this case $q \equiv 0$ is not a solution. Therefore there are no orbits homoclinic to 0 in a classical sense. However, we show that there is a solution emanating from 0 and terminating at 0. We are to call such a solution almost homoclinic to 0. It is obtained here as a weak limit in $W^{1,2}(\mathbb{R}, \mathbb{R}^n)$ of a sequence of almost critical points.

1. Introduction

In this paper the existence of almost homoclinic orbits for some time-dependent Hamiltonian systems will be studied. Consider

$$\ddot{q} + V_q(t,q) = f(t), \tag{1.1}$$

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where $t \in \mathbb{R}$, $q \in \mathbb{R}^n$ and functions $V\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}^n$ satisfy the following conditions:

(H_1) $V(t,q) = -K(t,q) + W(t,q)$, where $K, W\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are C^1-maps, and f is non-zero and continuous,

(H_2) there are constants $b_1, b_2 > 0$ such that for all $(t,q) \in \mathbb{R} \times \mathbb{R}^n$,

$$b_1|q|^2 \leq K(t,q) \leq b_2|q|^2,$$

(H_3) $K_q(t,q)$ is Lipschitzian in q in a neighbourhood of $0 \in \mathbb{R}^n$ uniformly with respect to t,

(H_4) $W_q(t,q) = o(|q|)$, as $|q| \to 0$ uniformly with respect to t,

(H_5) there are $M > 0$, $\mu \geq 2$ and $\varrho > 0$ such that for every $t \in \mathbb{R}$ and $0 < |q| \leq \varrho$,

$$0 < W(t,q) \leq M|q|^{\mu}.$$

Here and subsequently, $(\,\cdot\,,\,\cdot\,)\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denotes the standard inner product in $\mathbb{R}^n$ and $|\cdot|\colon \mathbb{R}^n \to [0,\infty)$ is the induced norm. Set

$$\overline{b}_1 := \min\{1, 2b_1\} \quad \text{and} \quad r := \min\{1, \varrho\}.$$

Finally, suppose that

(H_6) $\overline{b}_1 - 2M > 0$ and f satisfies the inequality:

$$\left(\int_{-\infty}^{\infty} |f(t)|^2\, dt\right)^{1/2} < \frac{\sqrt{2}}{4} r(\overline{b}_1 - 2M).$$

As an easy example of K and W satisfying conditions (H_1)–(H_5) and such that K is not a quadratic function, we can take $W(t,x) = x^4/16$ and $K(t,x) = x^2 + \ln(1+x^2)$, where $t, x \in \mathbb{R}$. One can immediately check that in this case $b_1 = 1$, $b_2 = 2$, $\mu = 4$ and $M = 1/16$. Another example is the following. W is as above and K is given by

$$K(t,x) = \begin{cases} \left(1 + \dfrac{1}{1+x^2}\right)x^2 & \text{if } x \geq 0, \\ \left(1 + \dfrac{2}{1+x^2}\right)x^2 & \text{if } x < 0. \end{cases}$$

Then $b_1 = 1$ and $b_2 = 3$.

The existence of homoclinic orbits both for the second order Hamiltonian systems and for the first order ones has been studied by many authors and the literature on this subject is vast. In particular, the second order systems were considered in [1], [2], [4], [5], [11], [12], and those of the first order in [3], [6], [7], [13], [14]. This work joins up with our earlier ones written together with M. Izydorek (see [8] and [9]). We studied there the system (1.1) with a potential V which was T-periodic in t.

The Hamiltonian system (1.1) does not possess a solution homoclinic to 0 in a classical meaning, because $q \equiv 0$ is not a solution of this system. However, we can still ask about the existence of solutions of (1.1) emanating from 0 and terminating at 0. We will call such solutions *almost homoclinic* (to 0).

Definition 1.1. We will say that a solution $q\colon \mathbb{R} \to \mathbb{R}^n$ of (1.1) is almost homoclinic (to 0) if $q(t) \to 0$, as $t \to \pm\infty$.

Our main theorem states as follows.

Theorem 1.2. *If the assumptions* (H_1)–(H_6) *are satisfied then the Hamiltonian system* (1.1) *possesses an almost homoclinic solution* $q_0 \in W^{1,2}(\mathbb{R},\mathbb{R}^n)$ *such that*

$$\dot{q}_0(t) \to 0, \quad \text{as } t \to \pm\infty.$$

This result is proved in Section 2 by studying the corresponding to (1.1) action functional $I\colon W^{1,2}(\mathbb{R},\mathbb{R}^n) \to \mathbb{R}$. Applying Ekeland's principle we get a sequence $\{q_k\}_{k\in\mathbb{N}}$ such that $\{I(q_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(q_k) \to 0$, as $k \to \infty$. We show that $\{q_k\}_{k\in\mathbb{N}}$ has a weakly convergent subsequence and its weak limit is a desired almost homoclinic solution.

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2. Proof of Theorem 1.2

Let E be the Sobolev space $W^{1,2}(\mathbb{R},\mathbb{R}^n)$ with the standard norm

$$\|q\|_E := \left(\int_{-\infty}^{\infty} (|q(t)|^2 + |\dot{q}(t)|^2)\, dt \right)^{1/2}.$$

We first recall some auxiliary properties of functions from E.

Fact 2.1. *Let* $q\colon \mathbb{R} \to \mathbb{R}^n$ *be a continuous mapping such that* $\dot{q} \in L^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n)$. *For every* $t \in \mathbb{R}$ *the following inequality holds*

$$|q(t)| \le \sqrt{2} \left(\int_{t-1/2}^{t+1/2} (|q(s)|^2 + |\dot{q}(s)|^2)\, ds \right)^{1/2}. \tag{2.1}$$

The proof of Fact 2.1 can be found in [8]. (See Fact 2.8, p. 385.)

Fact 2.2. *For each* $q \in E$,

$$\|q\|_{L^\infty(\mathbb{R},\mathbb{R}^n)} \le \sqrt{2}\|q\|_E. \tag{2.2}$$

Fact 2.2 is a direct consequence of the inequality (2.1).

Fact 2.3. *For each $q \in E$, if $p \in [2,\infty)$, then*

$$\|q\|_{L^p(\mathbb{R},\mathbb{R}^n)} \le 2^{(p-2)/(2p)}\|q\|_E.$$

Moreover, if $\|q\|_{L^\infty(\mathbb{R},\mathbb{R}^n)} \le 1$, then

$$\|q\|^p_{L^p(\mathbb{R},\mathbb{R}^n)} \le \|q\|^2_{L^2(\mathbb{R},\mathbb{R}^n)}. \tag{2.4}$$

Proof. Applying (2.2) we get

$$\begin{aligned}\|q\|^p_{L^p(\mathbb{R},\mathbb{R}^n)} &= \int_{-\infty}^{\infty} |q(t)|^p\,dt \le \|q\|^{p-2}_{L^\infty(\mathbb{R},\mathbb{R}^n)} \int_{-\infty}^{\infty} |q(t)|^2\,dt \\ &\le (\sqrt{2}\|q\|_E)^{p-2}\|q\|^2_{L^2(\mathbb{R},\mathbb{R}^n)} \le 2^{(p-2)/2}\|q\|^p_E,\end{aligned}$$

which completes the proof. □

Remark that $E \subset L^p(\mathbb{R},\mathbb{R}^n)$ for $2 \le p \le \infty$ and the embedding is continuous.

Let $I: E \to \mathbb{R}$ be given by

$$I(q) := \int_{-\infty}^{\infty} \left[\frac{1}{2}|\dot q(t)|^2 - V(t,q(t)) + (f(t),q(t))\right] dt.$$

Then, by (H_3) and (H_4), $I \in C^1(E,\mathbb{R})$ and it is easy to verify that

$$I'(q)w = \int_{-\infty}^{\infty} [(\dot q(t),\dot w(t)) - (V_q(t,q(t)),w(t)) + (f(t),w(t))]\,dt$$

and any critical point of I on E is a classical solution of (1.1) with $q(\pm\infty) = 0$.

From the pinching condition (H_2), for every $q \in E$ we have

$$I(q) \ge \frac{1}{2}\bar b_1\|q\|^2_E - \int_{-\infty}^{\infty} W(t,q(t))\,dt - \|f\|_{L^2(\mathbb{R},\mathbb{R}^n)}\|q\|_E. \tag{2.5}$$

Assume that $\|q\|_E \le (\sqrt{2}/2)r$. It follows from (2.2) that $\|q\|_{L^\infty(\mathbb{R},\mathbb{R}^n)} \le r$.

Using (H_5) and (2.4), we get

$$\int_{-\infty}^{\infty} W(t,q(t))\,dt \le \int_{-\infty}^{\infty} M|q(t)|^\mu\,dt = M\|q\|^\mu_{L^\mu(\mathbb{R},\mathbb{R}^n)} \le M\|q\|^2_E \tag{2.6}$$

and therefore by (2.5) and (2.6), we receive

$$I(q) \ge \frac{1}{2}\|q\|_E[(\bar b_1 - 2M)\|q\|_E - 2\|f\|_{L^2(\mathbb{R},\mathbb{R}^n)}]. \tag{2.7}$$

Hence I is bounded from below on a disc $\{q \in E: \|q\|_E \le (\sqrt{2}/2)r\}$.

Let

$$c := \inf\left\{I(q): \|q\|_E \le \frac{\sqrt{2}}{2}r\right\} \le I(0) = 0.$$

Furthermore, by (2.7) and (H_6),

$$I(q) \ge \frac{\sqrt{2}}{4}r\left[\frac{\sqrt{2}}{2}r\left(\bar b_1 - 2M\right) - 2\|f\|_{L^2(\mathbb{R},\mathbb{R}^n)}\right] \equiv \alpha > 0,$$

if $\|q\|_E = (\sqrt{2}/2)r$. Hence, by Ekeland's variational principle (see Theorem 4.2 in [10]) there exists a sequence $\{q_k\}_{k\in\mathbb{N}} \subset \{q \in E\colon \|q\|_E \leq (\sqrt{2}/2)r\}$ such that

$$I(q_k) \to c \quad \text{and} \quad I'(q_k) \to 0, \tag{2.8}$$

as $k \to \infty$. Since $\{q_k\}_{k\in\mathbb{N}}$ is a bounded sequence in a reflexive Banach space E, it possesses a weakly convergent subsequence.

Let q_0 denote a weak limit of a weakly convergent subsequence of $\{q_k\}_{k\in\mathbb{N}}$. Without loss of generality, we will write

$$q_k \rightharpoonup q_0 \quad \text{in } E, \tag{2.9}$$

as $k \to \infty$. This implies that $q_k \to q_0$ in $L^\infty_{\text{loc}}(\mathbb{R}, \mathbb{R}^n)$, as $k \to \infty$.

Lemma 2.4. *q_0 given by* (2.9) *is an almost homoclinic solution of* (1.1).

Proof. Since $q_0 \in E$, by Fact 2.1, $q_0(t) \to 0$, as $t \to \pm\infty$. Therefore, it is sufficient to show that $I'(q_0) = 0$. Fix $w \in C^\infty_0(\mathbb{R}, \mathbb{R}^n)$ and assume that for some $A > 0$, $\text{supp}(w) \subset [-A, A]$. We have

$$I'(q_k)w = \int_{-A}^{A} [(\dot{q}_k(t), \dot{w}(t)) - (V_q(t, q_k(t)), w(t)) + (f(t), w(t))]\, dt$$

for each $k \in \mathbb{N}$. From (2.8), it follows that $I'(q_k)w \to 0$, as $k \to \infty$. On the other hand,

$$\int_{-A}^{A} (\dot{q}_k(t), \dot{w}(t)) dt \to \int_{-A}^{A} (\dot{q}_0(t), \dot{w}(t))\, dt,$$

as $k \to \infty$, by (2.9), and

$$\int_{-A}^{A} (V_q(t, q_k(t)), w(t))\, dt \to \int_{-A}^{A} (V_q(t, q_0(t)), w(t))\, dt,$$

as $k \to \infty$, because $q_k \to q_0$ uniformly on $[-A, A]$. Thus $I'(q_k)w \to I'(q_0)w$, as $k \to \infty$, and, in consequence, $I'(q_0)w = 0$. Since $C^\infty_0(\mathbb{R}, \mathbb{R}^n)$ is dense in E, we get $I'(q_0) = 0$. □

Lemma 2.5. *Let q_0 be given by* (2.9). *Then $\dot{q}_0(t) \to 0$, as $t \to \pm\infty$.*

Proof. From Fact 2.1, we obtain

$$|\dot{q}_0(t)|^2 \leq 2 \int_{t-1/2}^{t+1/2} |\ddot{q}_0(s)|^2\, ds + 2 \int_{t-1/2}^{t+1/2} (|q_0(s)|^2 + |\dot{q}_0(s)|^2)\, ds.$$

For this reason, it suffices to notice that

$$\int_{r}^{r+1} |\ddot{q}_0(s)|^2\, ds \to 0,$$

as $r \to \pm\infty$. Since q_0 satisfies (1.1), we have

$$\int_r^{r+1} |\ddot{q}_0(s)|^2\,ds = \int_r^{r+1} (|V_q(s,q_0(s))|^2 + |f(s)|^2)\,ds - 2\int_r^{r+1} (V_q(s,q_0(s)), f(s))\,ds.$$

From this, we get

$$\int_r^{r+1} |\ddot{q}_0(s)|^2\,ds \le 2\int_r^{r+1} (|V_q(s,q_0(s))|^2 + |f(s)|^2)\,ds.$$

(H_6) implies,

$$\int_r^{r+1} |f(s)|^2 ds \to 0, \quad \text{as } r \to \pm\infty.$$

Take $\varepsilon > 0$. By (H_3) and (H_4), there is $\eta > 0$ such that for each $s \in \mathbb{R}$, if $|q| < \eta$, then $|V_q(s,q)| < \varepsilon$. Moreover, there is $\delta > 0$ such that, if $|s| > \delta$, then $|q_0(s)| < \eta$. Hence, if $|r| > \delta + 1$, then

$$\int_r^{r+1} |V_q(s,q_0(s))|^2\,ds < \varepsilon^2,$$

which completes the proof. □

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