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NONLINEAR BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL INCLUSIONS WITH CAPUTO FRACTIONAL DERIVATIVE

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Abstract. In this paper, we shall establish sufficient conditions for the existence of solutions for a class of boundary value problem for fractional differential inclusions involving the Caputo fractional derivative of order $\alpha \in (1,2]$. The both cases of convex valued and nonconvex valued right hand side are considered.

1. Introduction

This paper deals with the existence of solutions for boundary value problems (BVP for short), for fractional order differential inclusions. In Section 3 we consider the boundary value problem

$$ {}^cD^\alpha y(t) \in F(t,y), \quad \text{for almost each } t \in J = [0,T],\ 1 < \alpha \leq 2, \tag{1.1} $$

$$ y(0) = y_0, \quad y(T) = y_T, \tag{1.2} $$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $F\colon J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$), y_0, y_T are real constants.

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Section 4 is devoted to BVP for fractional order differential inclusions with nonlocal conditions

$$(1.3)\qquad {}^cD^\alpha y(t) \in F(t,y), \quad \text{for each } t \in J = [0,T],\ 1 < \alpha \leq 2,$$

$$(1.4)\qquad y(0) = g(y), \quad y(T) = y_T,$$

where F, y_T are as in problem (1.1)–(1.2), and $g: C(J,\mathbb{R}) \to \mathbb{R}$ a continuous function. Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [13], [21], [22], [24], [31], [32], [36], [38]). There has been a significant development in fractional differential equations in recent years; see the monographs of Kilbas *et al* [27], Miller and Ross [33], Podlubny [38], Samko *et al* [40] and the papers of Delbosco and Rodino [12], Diethelm *et al* [13], [14], [15], El-Sayed [17]–[19], Kilbas and Marzan [26], Mainardi [31], Momani and Hadid [34], Momani *et al* [35], Podlubny *et al* [39], Yu and Gao [41] and the references therein. Very recently, some basic theory for initial value problems for fractional differential equations involving the Riemann–Liouville differential operator of order $\alpha \in (0,1]$ was discussed by Lakshmikantham and Vatsala [28]–[30]. In [4], [5] the authors studied the existence and uniqueness of solutions of classes of functional differential equations with infinite delay and fractional order $\alpha \in (0,1]$ in Riemann–Liouville sense, and in [3] a class of perturbed functional differential equations involving the Caputo fractional derivative has been considered. El-Sayed and Ibrahim [20] initiated the study of fractional multivalued differential inclusions. In the case, where $\alpha \in (1,2]$, existence results for fractional boundary value problem and relaxation theorem, were studied by Ouahab [37].

Nonlocal conditions were initiated by Byszewski [6] for evolution equations when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As remarked by Byszewski [7], [8], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena. For example, $g(y)$ may be given by

$$g(y) = \sum_{i=1}^{p} c_i y(t_i),$$

where c_i, $i = 1,\dots,p$, are given constants and $0 < t_1 < \dots < t_p < T$.

In this paper we shall present two existence results for the problem (1.1)–(1.2) and (1.3)–(1.4) when the right hand side is convex as well as nonconvex valued. The first result relies on the nonlinear alternative of Leray–Schauder type, while the second one is based upon a fixed point theorem for contraction multivalued maps due to Covitz and Nadler. These results extend to the multivalued case some previous results in the literature.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper. Let $C(J,\mathbb{R})$ be the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : 0 \le t \le T\},$$

and we let $L^1(J,\mathbb{R})$ denote the Banach space of functions $y: J \to \mathbb{R}$ that are Lebesgue integrable with norm

$$\|y\|_{L^1} = \int_0^T |y(t)|\, dt.$$

$AC^1(J,\mathbb{R})$ is the space of functions $y: J \to \mathbb{R}$, which are absolutely continuous whose first derivative, y', is absolutely continuous. Let $(X, \|\cdot\|)$ be a Banach space. Let $P_{\rm cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$, $P_{\rm cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$ and $P_{{\rm cp},c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact and convex}\}$. A multivalued map $G: X \to P(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \bigcup_{x\in B} G(x)$ is bounded in X for all $B \in P_b(X)$ (i.e. $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subseteq N$. G is said to be completely continuous if $G(\mathcal{B})$ is relatively compact for every $\mathcal{B} \in P_b(X)$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $\operatorname{Fix} G$. A multivalued map $G: J \to P_{\rm cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \mapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable. For more details on multivalued maps see the books of Aubin and Cellina [1], Aubin and Frankowska [2], Deimling [11] and Hu and Papageorgiou [23].

DEFINITION 2.1. A multivalued map $F: J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be *Carathéodory* if

(a) $t \mapsto F(t,u)$ is measurable for each $u \in \mathbb{R}$;
(b) $u \mapsto F(t,u)$ is upper semicontinuous for almost all $t \in J$;

For each $y \in C(J,\mathbb{R})$, define the set of selections of F by

$$S_{F,y} = \{v \in L^1(J,\mathbb{R}) : v(t) \in F(t,y(t)) \text{ a.e. } t \in J\}.$$

Let (X,d) be a metric space induced from the normed space $(X,|\cdot|)$. Consider $H_d\colon \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A,B) = \max\Big\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\Big\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$, $d(a,B) = \inf_{b\in B} d(a,b)$. Then $(P_{b,\mathrm{cl}}(X), H_d)$ is a metric space and $(P_{\mathrm{cl}}(X), H_d)$ is a generalized metric space (see [25]).

Definition 2.2. A multivalued operator $N\colon X \to P_{\mathrm{cl}}(X)$ is called

(a) γ-*Lipschitz* if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \leq \gamma d(x,y), \quad \text{for each } x, y \in X,$$

(b) a *contraction* if and only if it is γ-Lipschitz with $\gamma < 1$.

The following lemma will be used in the sequel.

Lemma 2.3 ([10]). *Let (X,d) be a complete metric space. If $N\colon X \to P_{\mathrm{cl}}(X)$ is a contraction, then* $\operatorname{Fix} N \neq \emptyset$.

Definition 2.4 ([27], [38]). The fractional (arbitrary) order integral of the function $h \in L^1([a,b],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds,$$

where Γ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = h(t) * \varphi_\alpha(t)$, where $\varphi_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t \leq 0$, and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function.

Definition 2.5 ([27], [38]). For a function h given on the interval $[a,b]$, the αth Riemann–Liouville fractional-order derivative of h, is defined by

$$(D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} h(s)\, ds.$$

Here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.6 ([27]). For a function h given on the interval $[a,b]$, the Caputo fractional-order derivative of h of order α is defined by

$$({}^cD_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)\, ds,$$

where $n = [\alpha] + 1$.

3. Main results

In this section, we are concerned with the existence of solutions for the problem (1.1)–(1.2) when the right hand side has convex as well as nonconvex values. Initially, we assume that F is a compact and convex valued multivalued map.

DEFINITION 3.1. A function $y \in AC^1(J,\mathbb{R})$ is said to be a *solution* of (1.1)–(1.2), if there exists a function $v \in L^1(J,\mathbb{R})$ with $v(t) \in F(t,y(t))$, for a.e. $t \in J$, such that

$$^cD^\alpha y(t) = v(t), \quad \text{a.e. } t \in J,\ 1 < \alpha < 2,$$

and the function y satisfies condition (1.2).

For the existence of solutions for the problem (1.1)–(1.2), we need the following auxiliary lemmas:

LEMMA 3.2 ([42]). *Let $\alpha > 0$. Then the differential equation*

$$^cD^\alpha h(t) = 0$$

has solutions $h(t) = c_0+c_1t+c_2t^2+\ldots+c_nt^{n-1}$, $c_i \in \mathbb{R}$, $i = 0,\ldots,n$, $n = [\alpha]+1$.

LEMMA 3.3 ([42]). *Let $\alpha > 0$. Then*

$$I^{\alpha\, c}D^\alpha h(t) = c_0 + c_1t + c_2t^2 + \ldots + c_nt^{n-1} + h(t)$$

for some $c_i \in \mathbb{R}$, $i = 0,\ldots,n$, $n = [\alpha]+1$.

As a consequence of Lemmas 3.2 and 3.3 we have the following result which is useful in what follow:

LEMMA 3.4. *Let $1 < \alpha \le 2$ and let $h\colon J \to \mathbb{R}$ be continuous. A function y is a solution of the fractional integral equation*

$$y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds - \frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)\,ds - \left(\frac{t}{T}-1\right)y_0 + \frac{t}{T}y_T \tag{3.1}$$

if and only if y is a solution of the fractional BVP

$$^cD^\alpha y(t) = h(t), \quad t \in J, \tag{3.2}$$

$$y(0) = y_0, \qquad y(T) = y_T. \tag{3.3}$$

PROOF. Assume y satisfies (3.1), then Lemma 3.3 implies that

$$y(t) = c_0 + c_1t + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds.$$

From (3.3), a simple calculation gives $c_0 = y_0$ and

$$c_1 = \frac{-1}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s)\, ds - \frac{1}{T} y_0 + \frac{1}{T} y_T.$$

Hence we get equation (1.3). Inversely, it is clear that if y satisfies equation (1.3), then equations (3.2)–(3.3) hold. $\square$

Theorem 3.5. *Assume the following hypotheses hold:*

(H1) $F\colon J \times \mathbb{R} \to \mathcal{P}_{\mathrm{cp},c}(\mathbb{R})$ *is a Carathéodory multi-valued map;*

(H2) *there exist* $p \in C(J, \mathbb{R}^+)$ *and* $\psi\colon [0,\infty) \to (0,\infty)$ *continuous and nondecreasing such that*

$$\|F(t,u)\|_{\mathcal{P}} \leq p(t)\psi(|u|) \quad \textit{for } t \in J \textit{ and each } u \in \mathbb{R};$$

(H3) *there exists* $l \in L^1(J, \mathbb{R}^+)$, *with* $I^\alpha l < \infty$, *such that*

$$H_d(F(t,u), F(t,\overline{u})) \leq l(t)|u - \overline{u}| \quad \textit{for every } u, \overline{u} \in \mathbb{R},$$

and

$$d(0, F(t,0)) \leq l(t), \quad \textit{a.e. } t \in J;$$

(H4) *there exists an number* $M > 0$ *such that*

$$\frac{M}{\psi(M)\|I^\alpha p\|_\infty + \psi(M)(I^\alpha p)(T) + |y_0| + |y_T|} > 1. \tag{3.4}$$

Then the BVP (1.1)–(1.2) *has at least one solution on* J.

Proof. Transform the problem (1.1)–(1.2) into a fixed point problem. Consider the multivalued operator

$$\begin{aligned} N(y) = \bigg\{ h \in C(J,\mathbb{R}) : y(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)\, ds \\ &- \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v(s)\, ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{t}{T} y_T,\ v \in S_{F,y}. \bigg\} \end{aligned}$$

$\square$

Remark 3.6. Clearly, from Lemma 3.4, the fixed points of N are solutions to (1.1)–(1.2).

We shall show that N satisfies the assumptions of the nonlinear alternative of Leray–Schauder type [16]. The proof will be given in several steps.

Step 1. $N(y)$ is convex for each $y \in C(J,\mathbb{R})$.

Indeed, if h_1, h_2 belong to $N(y)$, then there exist $v_1, v_2 \in S_{F,y}$ such that for each $t \in J$ we have

$$\begin{aligned} h_i(t) = &\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_i(s)\, ds \\ &- \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v_i(s)\, ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{t}{T} y_T, \end{aligned}$$

for $i = 1, 2$. Let $0 \le d \le 1$. Then, for each $t \in J$, we have

$$\begin{aligned} (dh_1 + (1-d)h_2)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [dv_1(s) + (1-d)v_2(s)]\, ds \\ - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} [dv_1(s) + (1-d)v_2(s)]\, ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{t}{T} y_T. \end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values), we have

$$dh_1 + (1-d)h_2 \in N(y).$$

Step 2. N maps bounded sets into bounded sets in $C(J, \mathbb{R})$.

Let $B_{\eta^*} = \{y \in C(J, \mathbb{R}) : \|y\|_\infty \le \eta^*\}$ be bounded set in $C(J, \mathbb{R})$ and $y \in B_{\eta^*}$. Then for each $h \in N(y)$, there exists $v \in S_{F,y}$ such that

$$\begin{aligned} h(t) = &\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)\, ds \\ &- \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v(s)\, ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{t}{T} y_T. \end{aligned}$$

By (H2) we have, for each $t \in J$,

$$\begin{aligned} |h(t)| \le &\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |v(s)|\, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |v(s)|\, ds + |y_0| + |y_T| \\ \le &\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s)\psi(|y(s)|)\, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} p(s)\psi(|y(s)|)\, ds + |y_0| + |y_T| \\ \le &\psi(\eta^*) I^\alpha(p)(t) + \psi(\eta^*) I^\alpha(p)(T) + |y_0| + |y_T|. \end{aligned}$$

Thus

$$\|h\|_\infty \le \psi(\eta^*)\|I^\alpha(p)\|_\infty + \psi(\eta^*) I^\alpha(p)(T) + |y_0| + |y_T| := \ell.$$

Step 3. N maps bound ed sets into equicontinuous sets of $C(J, \mathbb{R})$.

Let $t_1, t_2 \in J$, $t_1 < t_2$, B_{η^*} be a bounded set of $C(J,\mathbb{R})$ as in Step 2, let $y \in B_{\eta^*}$ and $h \in N(y)$. Then

$$\begin{aligned}
|h(t_2) - h(t_1)| &= \bigg| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] v(s)\, ds \\
&\quad + \frac{(t_2 - t_1)}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} v(s)\, ds \bigg| + \frac{(t_2 - t_1)}{T}|y_0| + \frac{(t_2 - t_1)}{T}|y_T| \\
&\le \frac{\|p\|_\infty \psi(\eta^*)}{\Gamma(\alpha)} \int_0^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}]\, ds \\
&\quad + \frac{(t_2 - t_1)|p\|_\infty \psi(\eta^*)}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1}\, ds + \frac{(t_2 - t_1)}{T}|y_0| + \frac{(t_2 - t_1)}{T}|y_T| \\
&\le \frac{\|p\|_\infty \psi(\eta^*)}{\Gamma(\alpha+1)} [(t_2 - t_1)^\alpha + t_1^\alpha - t_2^\alpha] \\
&\quad + \frac{(t_2 - t_1)\|p\|_\infty \psi(\eta^*)}{\Gamma(\alpha+1)} (t_2 - t_1)^\alpha + \frac{(t_2 - t_1)}{T}|y_0| + \frac{(t_2 - t_1)}{T}|y_T|.
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzelá–Ascoli theorem, we can conclude that $N\colon C(J,\mathbb{R}) \to \mathcal{P}(C(J,\mathbb{R}))$ is completely continuous.

Step 4. N has a closed graph.

Let $y_n \to y_*$, $h_n \in N(y_n)$ and $h_n \to h_*$. We need to show that $h_* \in N(y_*)$. $h_n \in N(y_n)$ means that there exists $v_n \in S_{F,y_n}$ such that, for each $t \in J$,

$$\begin{aligned}
h_n(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} v_n(s)\, ds \\
&\quad - \frac{t}{T\Gamma(\alpha)} \int_0^T (T - s)^{\alpha-1} v_n(s)\, ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{t}{T} y_T.
\end{aligned}$$

We must show that there exists $v_* \in S_{F,y_*}$ such that, for each $t \in J$,

$$\begin{aligned}
h_*(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} v_*(s)\, ds \\
&\quad - \frac{t}{T\Gamma(\alpha)} \int_0^T (T - s)^{\alpha-1} v_*(s)\, ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{t}{T} y_T.
\end{aligned}$$

Since $F(t, \cdot\,)$ is upper semicontinuous, then for every $\varepsilon > 0$, there exist $n_0(\varepsilon) \ge 0$ such that, for every $n \ge n_0$, we have

$$v_n(t) \in F(t, y_n(t)) \subset F(t, y_*(t)) + \varepsilon B(0,1), \quad \text{a.e. } t \in J.$$

Since $F(\,\cdot\,, \cdot\,)$ has compact values, then there exists a subsequence $v_{n_m}(\,\cdot\,)$ such that

$$v_{n_m}(\,\cdot\,) \to v_*(\,\cdot\,) \quad \text{as } m \to \infty$$

and

$$v_*(t) \in F(t, y_*(t)), \quad \text{a.e. } t \in J.$$

For every $w \in F(t, y_*(t))$, we have

$$|v_{n_m}(t) - v_*(t)| \leq |v_{n_m}(t) - w| + |w - v_*(t)|.$$

Then

$$|v_{n_m}(t) - v_*(t)| \leq d(v_{n_m}(t), F(t, y_*(t)).$$

By an analogous relation, obtained by interchanging the roles of v_{n_m} and v_*, it follows that

$$|v_{n_m}(t) - v_*(t)| \leq H_d(F(t, y_n(t)), F(t, y_*(t))) \leq l(t)\|y_n - y_*\|_\infty.$$

Then

$$\begin{aligned} |h_n(t) - h_*(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |v_{n_m}(s) - v_*(s)|\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |v_{n_m}(s) - v_*(s)|\, ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} l(s)\, ds \|y_{n_m} - y_*\|_\infty \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} l(s)\, ds \|y_{n_m} - y_*\|_\infty. \end{aligned}$$

Hence

$$\begin{aligned} \|h_{n_m} - h_*\|_\infty &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} l(s)\, ds \|y_{n_m} - y_*\|_\infty \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} l(s)\, ds \|y_{n_m} - y_*\|_\infty \to 0 \end{aligned}$$

as $m \to \infty$.

Step 5. A priori bounds on solutions.

Let y be such that $y \in \lambda N(y)$ with $\lambda \in (0,1]$. Then there exists $v \in S_{F,y}$ such that, for each $t \in J$,

$$\begin{aligned} y(t) = \frac{\lambda}{\Gamma(\alpha)} &\int_0^t (t-s)^{\alpha-1} v(s)\, ds \\ &- \frac{\lambda t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v(s)\, ds - \lambda\left(\frac{t}{T} - 1\right) y_0 + \frac{\lambda t}{T} y_T. \end{aligned}$$

This implies by (H2) that, for each $t \in J$, we have

$$\begin{aligned} |y(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |v(s)|\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |v(s)| + |y_0| + |y_T| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s)\psi(|y(s)|)\, ds \end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} p(s)\psi(|y(s)|)\, ds + |y_0| + |y_T| \\
&\le \frac{\psi(\|y\|_\infty)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s)\, ds \\
&\quad + \frac{\psi(\|y\|_\infty)}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} p(s)\, ds + |y_0| + |y_T| \\
&\le \psi(\|y\|_\infty)(I^\alpha p)(t) + \psi(\|y\|_\infty)(I^\alpha p)(T) + |y_0| + |y_T|.
\end{aligned}
$$

Thus

$$
\frac{\|y\|_\infty}{\psi(\|y\|_\infty)\|I^\alpha p\|_\infty + \psi(\|y\|_\infty)(I^\alpha p)(T) + |y_0| + |y_T|} < 1.
$$

Then by condition (3.4), there exists $M > 0$ such that $\|y\|_\infty \neq M$.

Let $U = \{y \in C(J,\mathbb{R}) : \|y\|_\infty < M\}$. The operator $N\colon \overline{U} \to \mathcal{P}(C(J,\mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray–Schauder type [16], we deduce that N has a fixed point y in $\overline{U}$ which is a solution of the problem (1.1)–(1.2). This completes the proof. □

We present now a result for the problem (1.1)–(1.2) with a nonconvex valued right hand side. Our considerations are based on the fixed point theorem for contraction multivalued map given by Covitz and Nadler [10].

Theorem 3.7. *Assume* (H3) *and the following hypothesis holds:*

(H5) *$F\colon J \times \mathbb{R} \longrightarrow P_{\rm cp}(\mathbb{R})$ has the property that $F(\,\cdot\,,u)\colon J \to P_{\rm cp}(\mathbb{R})$ is measurable for each $u \in \mathbb{R}$;*

If

$$
(I^\alpha l)(T) < \frac{1}{2} \tag{3.5}
$$

then the BVP (1.1)–(1.2) *has at least one solution on J.*

Remark 3.8. For each $y \in C(J,\mathbb{R})$, the set $S_{F,y}$ is nonempty since by (H5), F has a measurable selection (see [9, Theorem III.6]).

Proof. We shall show that N satisfies the assumptions of Lemma 2.3. The proof will be given in two steps.

Step 1. $N(y) \in P_{\rm cl}(C(J,\mathbb{R}))$ for each $y \in C(J,\mathbb{R})$.

Indeed,let $(y_n)_{n\ge 0} \in N(y)$ such that $y_n \longrightarrow \widetilde{y}$ in $C(J,\mathbb{R})$. Then, $\widetilde{y} \in C(J,\mathbb{R})$ and there exists $v_n \in S_{F,y}$ such that, for each $t \in [0,T]$,

$$
\begin{aligned}
y_n(t) = \frac{1}{\Gamma(\alpha)} &\int_0^t (t-s)^{\alpha-1} v_n(s)\, ds \\
&- \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v_n(s)\, ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{t}{T} y_T.
\end{aligned}
$$

Using the fact that F has compact values and from (H3), we may pass to a subsequence if necessary to get that v_n converges weakly to v in $L^1_w(J,\mathbb{R})$ (the space endowed with the weak topology). An application of Mazur's theorem implies that v_n converges strongly to v and hence $v \in S_{F,y}$. Then, for each $t \in J$,

$$y_n(t) \to \widetilde{y}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)\,ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v(s)\,ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{t}{T} y_T.$$

So, $\widetilde{y} \in N(y)$.

Step 2. There exists $\gamma < 1$ such that

$$H_d(N(y), N(\overline{y})) \le \gamma \|y - \overline{y}\|_\infty \quad \text{for each } y, \overline{y} \in C(J,\mathbb{R}).$$

Let $y, \overline{y} \in C(J,\mathbb{R})$ and $h_1 \in N(y)$. Then, there exists $v_1(t) \in F(t, y(t))$ such that for each $t \in J$

$$h_1(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_1(s)\,ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v_1(s)\,ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{t}{T} y_T.$$

From (H3) it follows that

$$H_d(F(t, y(t)), F(t, \overline{y}(t))) \le l(t)|y(t) - \overline{y}(t)|.$$

Hence, there exists $w \in F(t, \overline{y}(t))$ such that

$$|v_1(t) - w| \le l(t)|y(t) - \overline{y}(t)|, \quad t \in J.$$

Consider $U\colon J \to \mathcal{P}(\mathbb{R})$ given by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \le l(t)|y(t) - \overline{y}(t)|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \overline{y}(t))$ is measurable (see Proposition III.4 in [9]), there exists a function $v_2(t)$ which is a measurable selection for V. So, $v_2(t) \in F(t, \overline{y}(t))$, and for each $t \in J$,

$$|v_1(t) - v_2(t)| \le l(t)|y(t) - \overline{y}(t)|.$$

Let us define for each $t \in J$

$$h_2(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_2(s)\,ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v_2(s)\,ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{t}{T} y_T.$$

Then for $t \in J$

$$\begin{aligned}|h_1(t) - h_2(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |v_1(s) - v_2(s)|\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |v_1(s) - v_2(s)|\, ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} l(s) |y(s) - \overline{y}(s)|\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} l(s) |y(s) - \overline{y}(s)|\, ds.\end{aligned}$$

Thus

$$\|h_1 - h_2\|_\infty \leq 2(I^\alpha l)(T) \|y - \overline{y}\|_\infty.$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H_d(N(y), N(\overline{y})) \leq 2(I^\alpha l)(T) \|y - \overline{y}\|_\infty.$$

So, N is a contraction by (3.5) and thus, by Lemma 2.3, N has a fixed point y which is solution to (1.1)–(1.2). The proof is complete. □

4. Nonlocal Boundary Value Problem

Let us start by defining what we mean by a solution of BVP (1.3)–(1.4).

Definition 4.1. A function $y \in AC^1(J, \mathbb{R})$ is said to be a solution of (1.3)–(1.4), if there exists a function $v \in L^1(J, \mathbb{R})$ with $v(t) \in F(t, y(t))$, for almost every $t \in J$, such that

$$^cD^\alpha y(t) = v(t), \quad \text{a.e. } t \in J,\ 1 < \alpha \leq 2,$$

and the function y satisfies condition (1.4).

Lemma 4.2. *Let $1 < \alpha \leq 2$ and let $h\colon [0,T] \to \mathbb{R}$ be continuous. A function y is a solution of the fractional integral equation*

$$\begin{aligned}y(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds \\ &\quad - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s)\, ds - \left(\frac{t}{T} - 1\right) g(y) + \frac{t}{T} y_T\end{aligned} \tag{4.1}$$

if and only if y is a solution of the fractional BVP

$$^cD^\alpha y(t) = h(t), \quad t \in J, \tag{4.2}$$

$$y(0) = g(y), \quad y(T) = y_T. \tag{4.3}$$

THEOREM 4.3. *Assume* (H1)–(H3), *and the following conditions hold:*

(H5) *there exists a constant* $M_1 > 0$ *with*

$$|g(y)| \le M_1 \quad \text{for all } y \in C(J, \mathbb{R}),$$

(H6) *there exists an number* $M_2 > 0$ *such that*

$$\frac{M_2}{\psi(M_2)\|I^\alpha p\|_\infty + \psi(M_2)(I^\alpha p)(T) + M_1 + |y_T|} > 1. \tag{4.4}$$

Then the BVP (1.3)–(1.4) *has at least one solution on* J.

PROOF. Transform the problem (1.3)–(1.4) into a fixed point problem. Consider the multivalued operator

$$\begin{aligned} N_1(y) = \bigg\{ h \in C(J, \mathbb{R}) : y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)\, ds \\ - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v(s)\, ds - \left(\frac{t}{T} - 1\right) g(y) + \frac{t}{T} y_T,\ v \in S_{F,y}. \bigg\}. \end{aligned}$$

REMARK 4.4. Clearly, from Lemma 4.2, the fixed points of N are solutions to (1.3)–(1.4).

As in Theorem 3.5 we can show that N_1 is completely continuous and “upper” semicontinuous.

Let y be such that $y \in \lambda N_1(y)$ with $\lambda \in (0, 1]$. Then, there exists $v \in S_{F,y}$ such that, for each $t \in J$,

$$\begin{aligned} y(t) = \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)\, ds \\ - \frac{\lambda t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v_*(s)\, ds - \lambda\left(\frac{t}{T} - 1\right) g(y) + \frac{\lambda t}{T} y_T. \end{aligned}$$

This implies by (H2)–(H5) that, for each $t \in J$, we have

$$\begin{aligned} |y(t)| &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |v(s)|\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |v(s)| + M_1 + |y_T| \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s)\psi(|y(s)|)\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} p(s)\psi(|y(s)|)\, ds + M_1 + |y_T| \\ &\le \frac{\psi(\|y\|_\infty)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s)\, ds \\ &\quad + \frac{\psi(\|y\|_\infty)}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} p(s)\, ds + M_1 + |y_T| \end{aligned}$$

$$\leq \psi(\|y\|_\infty)(I^\alpha p)(t) + \psi(\|y\|_\infty)(I^\alpha p)(T) + M_1 + |y_T|.$$

Thus

$$\frac{\|y\|_\infty}{\psi(\|y\|_\infty)\|I^\alpha p\|_\infty + \psi(\|y\|_\infty)(I^\alpha p)(T) + M_1 + |y_T|} < 1.$$

Then by condition (4.4), there exists M_2 such that $\|y\|_\infty \neq M_2$.

Let $U_1 = \{y \in C(J,\mathbb{R}) : \|y\|_\infty < M_2\}$. The operator $N_1 \colon \overline{U}_1 \to \mathcal{P}(C(J,\mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U_1, there is no $y \in \partial U_1$ such that $y \in \lambda N_1(y)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray–Schauder type [16], we deduce that N_1 has a fixed point y in $\overline{U}_1$ which is a solution of the problem (1.3)–(1.4). This completes the proof. □

Theorem 4.5. *Assume* (H3), (H5) *and the following hypothesis holds:*

(H7) *there exists* $k > 0$ *with such that*

$$|g(u) - g(\overline{u})| \leq k|u - \overline{u}|, \quad \text{for each } t \in J \text{ and all } u, \overline{u} \in C(J,\mathbb{R}).$$

If $2(I^\alpha l)(T) + k < 1$ *then the* BVP (1.3)–(1.4) *has at least one solution on* J.

Remark 4.6. For each $y \in C(J,\mathbb{R})$, the set $S_{F,y}$ is nonempty since, by (H5), F has a measurable selection (see [9, Theorem III.6]).

Proof. Using similar steps as in Theorem 3.7 we can show that N_1 satisfies the assumptions of Lemma 2.3. □

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