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ON STEADY NON-NEWTONIAN FLUIDS WITH GROWTH CONDITIONS IN GENERALIZED ORLICZ SPACES

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Abstract. We are interested in the existence of weak solutions to steady non-Newtonian fluids with nonstandard growth conditions of the Cauchy stress tensor. Since the L^p framework is not suitable to capture the description of strongly inhomogeneous fluids, we formulate the problem in generalized Orlicz spaces. The existence proof consists in showing that for Galerkin approximations the sequence of symmetric gradients of the flow velocity converges modularly. As an example of motivation for considering non-Newtonian fluids in generalized Orlicz spaces we recall the smart fluids.

1. Introduction

Our considerations concentrate on the existence of weak solutions to steady flows of non-Newtonian incompressible fluids described by the equations

$$\begin{aligned} v\cdot\nabla v-\operatorname{div}T(x,Dv)+\nabla p&=f \quad \text{in } \Omega,\\ \operatorname{div} v&=0 \quad \text{in } \Omega,\\ v(x)&=0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $v\colon\Omega\to\mathbb{R}^d$ denotes the velocity field, $p\colon\Omega\to\mathbb{R}$ the pressure, T the stress tensor, f body forces, $\Omega\subset\mathbb{R}^d$ is a bounded domain with a Lipschitz bounary and we mean by $Dv=(\nabla v+\nabla^T v)/2$. Since we are interested in the fluids of

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strongly inhomogeneous behavior with the rapid increase in the viscosity, we assume non-standard growth conditions for the Cauchy stress tensor. An example of such fluids are the smart fluids (in particular electrorheological and magnetorheological fluids, e.g. [3], [14]). We allow the growth faster than polynomial and spatially inhomogeneous, hence the L^p-framework cannot capture the described situation. The growth conditions are formulated with help of a convex function, so-called *N-function*. Before stating the assumptions on T and the existence result we define an N-function and its complementary function.

Definition 1.1. We call a function $M:\Omega \times \mathbb{R}^{d\times d}_{\rm sym} \to \mathbb{R}$ an N-function if it satisfies the following conditions

(a) M is a Carathéodory function (i.e. measurable function of x for all $\xi \in \mathbb{R}^{d\times d}_{\rm sym}$ and continuous function of ξ for almost all $x \in \Omega$) such that $M(x,0) = 0$ and $M(x,\xi) = M(x,-\xi)$ almost everywhere in Ω and for all $\xi \in \mathbb{R}^{d\times d}_{\rm sym}$.
(b) $M(x,\xi)$ is a convex function of ξ a.e. in Ω.
(c) $\lim_{|\xi|\to 0} \sup_{x\in\Omega} M(x,\xi)/|\xi| = 0$.
(d) $\lim_{|\xi|\to \infty} \inf_{x\in\Omega} M(x,\xi)/|\xi| = \infty$.

Definition 1.2. The complementary function M^* to a function M is defined by

$$M^*(x,\eta) = \sup_{\xi\in\mathbb{R}^{d\times d}_{\rm sym}} (\xi\cdot\eta - M(x,\xi))$$

for $\eta \in \mathbb{R}^{d\times d}_{\rm sym}$ and almost all $x \in \Omega$.

Remark 1.3. The complementary function M^* is again an N-function.

Contrary to the usual consideration, e.g. [11], we consider an N-function M depending not only on $|\xi|$, but on the whole vector ξ. This recalls to the fact, that the stress tensor may differ in different directions.

Let us assume that the Cauchy stress tensor $T:\Omega \times \mathbb{R}^{d\times d}_{\rm sym} \to \mathbb{R}^{d\times d}_{\rm sym}$ satisfies the following conditions:

(T1) $T(x,\xi)$ is a Carathéodory function and $T(x,0) = 0$.
(T2) There exist a positive constant c_T and an N-function M such that for all $\xi \in \mathbb{R}^{d\times d}_{\rm sym}$ and almost all $x \in \Omega$ it holds

$$T(x,\xi)\cdot\xi \geq c_T[M(x,\xi) + M^*(x,T(x,\xi))]. \tag{1.2}$$

(T3) T is strictly monotone, i.e. for all $\xi_1,\xi_2 \in \mathbb{R}^{d\times d}_{\rm sym}$, $\xi_1 \neq \xi_2$ and almost all $x \in \Omega$

$$[T(x,\xi_1) - T(x,\xi_2)]\cdot[\xi_1 - \xi_2] > 0.$$

(T4) For all $\xi \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, almost all $x \in \Omega$, some $c > 0$ and $q > 3d/(d+2)$ the N-function M satisfies

$$M(x,\xi) \geq c|\xi|^q. \tag{1.3}$$

Before defining the solutions and stating the main result we need to introduce the proper spaces to capture the formulated problem. The generalized Orlicz class $\mathcal{L}_M(\Omega)$ is the set of all measurable functions $\xi\colon \Omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ such that

$$\int_\Omega M(x,\xi)\,dx < \infty.$$

By $L_M(\Omega)$ we denote the generalized Orlicz space which is the set of all measurable functions $\xi\colon \Omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ which satisfy

$$\int_\Omega M(x,\lambda\xi(x))\,dx \to 0 \quad \text{as } \lambda \to 0.$$

The Orlicz space is a Banach space with respect to the Luxemburg norm

$$\|u\| = \inf\left\{\lambda > 0\colon \int_\Omega M\left(x, \frac{u}{\lambda}\right) dx \leq 1\right\}.$$

By $E_M(\Omega)$ we denote the closure of all bounded functions in $L_M(\Omega)$. The space $L_{M^*}(\Omega)$ is the dual space of $E_M(\Omega)$. We are interested in the case of rapidly growing N-functions where the so-called Δ_2-*condition* is not satisfied. We say that an N-function M satisfies Δ_2-condition if for some nonnegative, integrable in Ω function h and a constant $k > 0$ it holds

$$M(x,2\xi) \leq kM(x,\xi) + h(x) \quad \text{for all } \xi \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \text{ and a.a. } x \in \Omega.$$

If this condition fails we lose numerous properties of the spaces, in particular we only know $E_M \subsetneq \mathcal{L}_M \subsetneq L_M$ and L_M is neither separable nor reflexive. These properties of Orlicz spaces with a vector-valued argument of an N-function are proved in [15]. The proofs essentially follow the same lines as the ones for classical Orlicz spaces, see e.g. [1, Chapter 8], [7, Chapter II, Theorem 10.2].

We are interested in the existence of weak solutions to problem (1.1). By $\mathcal{D}(\Omega)$ we understand the space of all C^∞−functions with compact support. Let $\mathcal{V}(\Omega)$ be the set of all functions which belong to $\mathcal{D}(\Omega)$ and are divergence-free. Moreover, by $L^q, W^{1,q}$, we mean the standard Lebesgue and Sobolev spaces, by L^2_{div} the closure of $\mathcal{V}$ w.r.t. the $\|\cdot\|_{L^2}$-norm. By q' we mean the conjugate exponent to q, namely $1/q + 1/q' = 1$. We mean by $\langle\cdot,\cdot\rangle_M$ the dual pair between $L_M(\Omega)$ and $L_{M^*}(\Omega)$ and by $(\cdot,\cdot)$ the scalar product in $L_2(\Omega)$. Finally, $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ stands for the space of symmetric $d \times d$-matrices.

Definition 1.4. Let f be in the form $f = \operatorname{div} F$ with $F \in \mathbb{R}^{d\times d}_{\rm sym}$ and $F \in L_{M^*}(\Omega)$. We call v a weak solution to (1.1) if $v \in L^2_{\rm div}(\Omega)$, $Dv \in L_M(\Omega)$ and the following is satisfied for all $\varphi \in \mathcal{V}(\Omega)$

$$\int_\Omega (v\cdot\nabla v\cdot\varphi + T(x,Dv)\cdot D\varphi)\,dx = -\langle F, D\varphi\rangle_M.$$

Theorem 1.5. *Let f be in the form $f = \operatorname{div} F$ with $F \in \mathbb{R}^{d\times d}_{\rm sym}$ and $F \in L_{M^*}(\Omega)$. Moreover, let T satisfy* (T1)–(T4). *Then there exists a weak solution to* (1.1).

Remark 1.6. For any N-functions M_1, M_2 with complementary functions M_1^* and M_2^* respectively, we have

$$M_1(x,\xi) \le cM_2(x,\xi) \Rightarrow M_2^*(x,\xi) \le \frac{1}{c} M_1^*(x,c\xi)$$

for some constant $c > 0$, almost all $x \in \Omega$ and all $\xi \in \mathbb{R}^{d\times d}_{\rm sym}$. Hence condition (1.3) provides

$$M^*(x,\xi) \le c^{(q'-1)}|\xi|^{q'} \quad \text{for a.a. } x\in\Omega \text{ and all } \xi \in \mathbb{R}^{d\times d}_{\rm sym}. \tag{1.4}$$

Consequently, $L_{M^*} = \mathcal{L}_{M^*} = E_{M^*}$ is a separable space.

Abstract elliptic equations with rapidly growing coefficients were widely studied, e.g. [4], [6]. Contrary to those results on elliptic equation we consider the system of equations containing the divergence-free condition. This imposes different definition of the spaces for existence of solutions and the characterization of the dual spaces to Orlicz and Orlicz–Sobolev spaces is not straightforward. Contrary to [4], [6] we do not use the generalization of the monotonicity methods to non reflexive and non separable Orlicz spaces.

The proof presented here provides that the mapping T is of class (S_m). The notion was introduced in the overview paper [12], nevertheless we recall it and adapt to the considered problem.

Definition 1.7. A mapping $T\colon \operatorname{Dom}(T) \subset L_M \to L_{M^*}$ is of class (S_m) if

$$\begin{cases} \{Du^n\} \subset \operatorname{Dom}(T), & \\ Du^n \overset{*}{\rightharpoonup} Du & \text{in } L_M,\\ T(Du^n) \overset{*}{\rightharpoonup} \chi & \text{in } L_{M^*},\\ \limsup \langle Du^n, T(Du^n)\rangle_M \le \langle Du,\chi\rangle_M, & \end{cases} \tag{1.5}$$

imply

$$\begin{cases} Du \in \operatorname{Dom}(T),\\ \chi = T(Du),\\ \langle Du^n, T(Du^n)\rangle_M \to \langle Du,\chi\rangle_M,\\ Du^n \to Du \text{ modularly in } L_M. \end{cases} \tag{1.6}$$

Mustonen and Tienari [12], using partial results of Landes [8] presented that the operator T satisfying certain set of conditions including the growth conditions formulated in Orlicz spaces, is of class (S_m). In the present paper we allow an N-function to be more general than in [8], [12] (namely, dependent on x and on the whole vector ξ) and using more advanced tools of Young measures, provide the proof much shorter to the one presented in [12].

The paper is organized as follows: we start in Section 2 with short preliminaries on generalized Orlicz spaces and Young measures, which are included in the paper for completeness. Section 3 is devoted to the proof of the main result Theorem 1.5. The proof essentially consists in showing that the mapping T is of class (S_m).

2. Preliminaries

2.1. Generalized Orlicz spaces. The functional

$$\xi \mapsto \int_\Omega M(x, \xi(x))\, dx$$

is a modular in $L_M(\Omega)$. A sequence $\{z^j\}$ converges modularly to z in $L_M(\Omega)$ if there exists $\lambda > 0$ such that

$$\int_\Omega M\bigg(x, \frac{z^j - z}{\lambda}\bigg)\, dx \to 0.$$

We will use the notation $z^j \xrightarrow{M} z$ for the modular convergence in $L_M(\Omega)$.

Next, we recall an analogue to the Vitali's lemma, however for the modular convergence instead of the strong convergence in L^q.

Lemma 2.1. *Let $z^j\colon \Omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ be a measurable sequence. Then $z^j \xrightarrow{M} z$ in $L_M(\Omega)$ modularly if and only if $z^j \to z$ in measure and there exist some $\lambda > 0$ such that the sequence $\{M(\,\cdot\,, \lambda z^j)\}$ is uniformly integrable, i.e.*

$$\lim_{R\to\infty} \bigg(\sup_{j\in\mathbb{N}} \int_{\{x : |M(x,\lambda z^j)| \geq R\}} M(x, \lambda z^j)\, dx\bigg) = 0.$$

Proof. Note that $z^j \to z$ in measure if and only if $M(\,\cdot\,, (z^j - z)/\lambda) \to 0$ in measure for all $\lambda > 0$. Moreover the convergence $z^j \to z$ in measure implies that for all measurable sets $A \subset \Omega$ it holds

$$\liminf_{j\to\infty} \int_A M(x, z^j)\, dx \geq \int_A M(x, z)\, dx.$$

Note also that the convexity of M implies

$$\int_A M\bigg(x, \frac{z^j - z}{\lambda}\bigg)\, dx \leq \int_A M\bigg(x, \frac{z^j}{2\lambda}\bigg)\, dx + \int_A M\bigg(x, \frac{z}{2\lambda}\bigg)\, dx.$$

Hence by the classical Vitali's lemma for $f^j(x) = M(x, (z^j - z)/\lambda)$ we obtain that $f^j \to 0$ strongly in $L^1(\Omega)$. □

For the proof of the following elementary estimate see e.g. [11].

PROPOSITION 2.2 (Fenchel–Young inequality). *Let M be an N-function and M^* a complementary to M, then the following inequality is satisfied*

$$|\xi \cdot \eta| \leq M(x, \xi) + M^*(x, \eta)$$

for all $\xi, \eta \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and a.a. $x \in \Omega$.

2.2. Young measures. In the last step of the existence proof we use the Young measures properties. For the reader not familiar with the theory of measure-valued solutions we recall the useful notions. The proofs of the following facts can be found in [13, Corollaries 3.2–3.4], [2], [10]. In the following[1] $C_0(\mathbb{R}^{d\times d}_{\mathrm{sym}})$ denotes the closure of the space of continuous functions on $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ with compact support with respect to the $\|\cdot\|_\infty$-norm. Its dual space can be identified with $\mathcal{M}(\mathbb{R}^{d\times d}_{\mathrm{sym}})$, the space of signed Radon measures with finite mass. The related duality pairing is given by

$$\langle \mu, f\rangle = \int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} f(\lambda)\, d\mu(\lambda).$$

DEFINITION 2.3. A map $\mu:\Omega \to \mathcal{M}(\mathbb{R}^{d\times d}_{\mathrm{sym}})$ is called weakly* measurable if the functions $x \mapsto \langle \mu(x), f\rangle$ are measurable for all $f \in C_0(\mathbb{R}^{d\times d}_{\mathrm{sym}})$.

THEOREM 2.4 (Fundamental theorem on Young measures). *Let $\Omega \subset \mathbb{R}^d$ be a measurable set of finite measure and let $z^j:\Omega \to \mathbb{R}^{d\times d}$ be a sequence of measurable functions. Then there exists a subsequence z^{j_k} and a weakly* measurable map $\nu:\Omega \to \mathcal{M}(\mathbb{R}^{d\times d}_{\mathrm{sym}})$ such that the following holds:*

(a) *$\nu_x \geq 0$, $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^{d\times d}_{\mathrm{sym}})} = \int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} d\nu_x \leq 1$ for almost all $x \in \Omega$.*

(b) *For all $g \in C_0(\mathbb{R}^{d\times d}_{\mathrm{sym}})$*

$$g(z^{j_k}) \overset{*}{\rightharpoonup} \overline{g} \quad \text{in } L^\infty(\Omega)$$

where $\overline{g}(x) = \langle \nu_x, g\rangle$.

(c) *Let $K \subset \mathbb{R}^{d\times d}_{\mathrm{sym}}$ be compact. Then $\operatorname{supp} \nu_x \subset K$ if $\operatorname{dist}(z^{j_k}, K) \to 0$ in measure.*

(d) *Additionally $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^{d\times d}_{\mathrm{sym}})} = 1$ for almost all $x \in \Omega$ if and only if the "tightness condition" is satisfied, i.e.*

$$\lim_{M\to\infty} \sup_k |\{|z^{j_k}| \geq M\}| = 0.$$

(1) We adapt the statement of most of the facts to the considered problem, however in general all the recalled theorems hold for any $\mathbb{R}^n$.

(e) *If the tightness condition is satisfied and moreover if $A \subset \Omega$ is measurable, $g \in C(\mathbb{R}^{d\times d}_{\mathrm{sym}})$ and $g(z^{j_k})$ is relatively weakly compact in $L^1(A)$, then*

$$g(z^{j_k}) \rightharpoonup \overline{g} \quad \text{in } L^1(A), \quad \overline{g}(x) = \langle \nu_x, g\rangle.$$

(f) *If the tightness condition is satisfied, then in* (c) *one can replace "if" by "if and only if".*

Remark 2.5. The map $\nu\colon \Omega \to \mathcal{M}(\mathbb{R}^{d\times d}_{\mathrm{sym}})$ is called the Young measure generated by the sequence $\{z^{j_k}\}$. Every (weakly* measurable map) $\nu\colon \Omega \to \mathcal{M}(\mathbb{R}^{d\times d}_{\mathrm{sym}})$ that satisfies (a) is generated by some sequence $\{z^k\}$.

Remark 2.6. If, for some $s > 0$ and all $j \in \mathbb{N}$ holds $\int_\Omega |z^j|^s \le k$, then the tightness condition is satisfied.

We recall the following lemmas

Lemma 2.7. *Suppose that the sequence of maps $z^j\colon \Omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ generates the Young measure ν. Let $F\colon \Omega \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}$ be a Carathéodory function. Let also assume that the negative part $F^-(x, z^j(x))$ is weakly relatively compact in $L^1(\Omega)$. Then*

$$\liminf_{j\to\infty} \int_\Omega F(x, z^j(x))\,dx \ge \int_\Omega \int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} F(x,\lambda)\,d\nu_x(\lambda)\,dx.$$

If, in addition, the sequence of functions $x \mapsto |F|(x, z^j(x))$ is weakly relatively compact in $L^1(\Omega)$ then

$$F(\,\cdot\,, z^j(\,\cdot\,)) \rightharpoonup \int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} F(x,\lambda)\,d\nu_x(\lambda) \quad \text{in } L^1(\Omega)$$

Lemma 2.8. *Suppose that a sequence $\{z^j\}$ of measurable functions from Ω to $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ generates the Young measure $\nu\colon \Omega \to \mathcal{M}(\mathbb{R}^{d\times d}_{\mathrm{sym}})$. Then*

$$z^j \to z \quad \text{in measure if and only if} \quad \nu_x = \delta_{z(x)} \quad \text{a.e.}$$

3. Existence of solutions. Proof of Theorem 1.5

We define an operator induced by T, which we denote in the same way, with a domain $\mathrm{Dom}(T) = \mathcal{L}_M(\Omega)$. For the proof of Theorem 1.5 we use the Galerkin method to construct approximate solutions. From the energy estimates we conclude the boundedness and weak precompactness of the sequence of approximate solutions. The appropriate compact embedding provides the weak convergence of the term $\{v^n \cdot \nabla v^n\}$. For details we refer to e.g. [5], [9]. The main difficulty resolves to characterizing the limit of the highest order nonlinear term $\{T(x, Dv^n)\}$. This follows once we show that T is of class (S_m). Thus we divide this section into two lemmas showing first that T satisfies (1.5), which in the second step leads to conditions (1.6).

Lemma 3.1. *Let T satisfy* (T1)–(T4). *Then T and the sequence $\{Dv^n\}$, where $\{v^n\}$ are the Galerkin approximations to* (1.1), *satisfy* (1.5).

Proof. We construct the Galerkin approximations to (1.1). Let $\{\omega_i\}_{i=1}^{\infty}$ be the set of eigenvectors of the Stokes operator. Then ω_i are smooth and divergence-free functions for all $i \in \mathbb{N}$. We define $v^n = \sum_{i=1}^n \alpha_i^n \omega_i$, $\alpha_i^n \in \mathbb{R}$

$$(v^n \cdot \nabla v^n, \omega_i) + (T(x, Dv^n), D\omega_i) = -\langle F, D\omega_i\rangle_M$$

Multiplying by α_i^n and summing over $i = 1, \dots, j$ with $j \le n$ yields

$$\int_\Omega v^n \cdot \nabla v^n \cdot v^j \, dx + \int_\Omega T(x, Dv^n) \cdot Dv^j \, dx = -\langle F, D\omega_i\rangle_M \tag{3.1}$$

hence for $j = n$ it holds

$$\int_\Omega T(x, Dv^n) \cdot Dv^n \, dx = -\langle F, D\omega_i\rangle_M. \tag{3.2}$$

We estimate the right-hand side of the above

$$\begin{aligned}
|\langle F, D\omega_i\rangle_M| &\le \int_\Omega \left| \frac{2}{c_T} F \cdot \frac{c_T}{2} Dv^n \right| dx \\
&\overset{F-Y}{\le} \int_\Omega M^*\left(x, \frac{2}{c_T}F\right) dx + \int_\Omega M\left(x, \frac{c_T}{2} Dv^n\right) dx \\
&\le \int_\Omega M^*\left(x, \frac{2}{c_T}F\right) dx + \frac{c_T}{2}\int_\Omega M\left(x, Dv^n\right) dx.
\end{aligned}$$

Using then coercivity condition (1.2) on T we obtain

$$\frac{c_T}{2}\int_\Omega M(x, Dv^n)\, dx + c_T \int_\Omega M^*(x, T(x, Dv^n))\, dx \le \int_\Omega M^*\left(x, \frac{2}{c_T}F\right) dx. \tag{3.3}$$

Since (1.4) holds, then $F \in \mathcal{L}_{M^*}(\Omega)$ and the right-hand side is finite. Condition (1.3) provides that $\{Dv^n\}$ is uniformly bounded in the space $L^q(\Omega)$ for $q > 3d/(d+2)$. Since $W^{1,q}(\Omega)$ is compactly embedded in $L^2(\Omega)$, we conclude the following

$$Dv^n \rightharpoonup Dv \quad \text{weakly in } L^q(\Omega)$$

and

$$v^n \to v \quad \text{strongly in } L^2(\Omega).$$

Moreover, there exists a $\chi \in L_{M^*}(\Omega)$ such that

$$T(\,\cdot\,, Dv^n) \overset{*}{\rightharpoonup} \chi \quad \text{weakly* in } L_{M^*}(\Omega).$$

Letting $n \to \infty$ in (3.1) provides

$$\int_\Omega v \cdot \nabla v \cdot v^j \, dx + \int_\Omega \chi \cdot Dv^j \, dx = -\langle F, Dv^j\rangle_M. \tag{3.4}$$

The first term vanishes, see e.g. [10, Lemma 2.44, p. 216]. Since (1.4) holds, then $L_{M^*} = \mathcal{L}_{M^*} = E_{M^*}$ is a separable space. Recalling that $(E_{M^*})^* = L_M$ we let $j \to \infty$ in (3.4) and using Banach–Alaoglu theorem obtain

$$\int_\Omega \chi \cdot Dv\, dx = -\langle F, Dv\rangle_M. \tag{3.5}$$

For later use we observe that letting $n \to \infty$ in (3.2) gives

$$\limsup_{n\to\infty} \int_\Omega T(x, Dv^n) \cdot Dv^n\, dx = -\langle F, Dv\rangle_M$$

which together with (3.5) provides

$$\limsup_{n\to\infty} \int_\Omega T(x, Dv^n) \cdot Dv^n\, dx = \int_\Omega \chi \cdot Dv\, dx. \tag{3.6}$$

□

Lemma 3.2. *Let T satisfy* (T1)–(T4) *and* $\mathrm{Dom}(T) = \mathcal{L}_M(\Omega)$. *Then T is of class (S_m).*

Proof. Since T is monotone and $T(x, 0) = 0$, then trivially the negative part is weakly relatively compact in $L^1(\Omega)$. Hence due to Lemma 2.7

$$\liminf_{n\to\infty} \int_\Omega T(x, Dv^n(x)) \cdot Dv^n\, dx \geq \int_\Omega \int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} T(x, \xi) \cdot \xi\, d\nu_x(\xi)\, dx$$

where ν_x is the Young measure generated by the sequence $\{Dv^n\}$. Combining it with (3.6) we obtain that

$$\int_\Omega \int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} T(x, \xi) \cdot \xi\, d\nu_x(\xi)\, dx \leq \int_\Omega \chi \cdot Dv\, dx. \tag{3.7}$$

The monotonicity of T provides that

$$\int_\Omega \int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} h(x, \xi)\, d\nu_x(\xi)\, dx \geq 0, \tag{3.8}$$

where h is defined by

$$h(x, \xi) := [T(x, \xi) - T(x, Dv)] \cdot [\xi - Dv].$$

Since $\{Dv^n\}$ and $\{T(\cdot, Dv^n)\}$ are weakly relatively compact in $L^1(\Omega)$ and T is a Carathéodory function, then

$$Dv = \int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} \xi\, d\nu_x(\xi) \quad \text{a.e. in } \Omega$$

and

$$\chi = \int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} T(x, \xi)\, d\nu_x(\xi) \quad \text{a.e. in } \Omega \tag{3.9}$$

by the second part of Lemma 2.7. Then

$$\int_\Omega \int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} h(x,\xi)\, d\nu_x(\xi)\, dx = \int_\Omega \int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} T(x,\xi)\cdot\xi\, d\nu_x(\xi)\, dx - \int_\Omega \chi\cdot Dv\, dx \tag{3.10}$$

which is nonpositive due to (3.7). Combining (3.7), (3.8) and (3.10) implies that $\int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} h(x,\xi)\, d\nu_x(\xi) = 0$ for almost all $x \in \Omega$. Moreover, since $\nu_x \geq 0$ is a probability measure and $T(x,\,\cdot\,)$ is strongly monotone, we conclude that

$$\operatorname{supp}\{\nu_x\} = \{Dv(x)\} \quad \text{a.e. in } \Omega.$$

Finally, we obtain that $\nu_x = \delta_{Dv(x)}$ almost everywhere, which inserted to (3.9) provides

$$\chi = T(x, Dv) \quad \text{a.e. in } \Omega.$$

To show that $Dv^n \xrightarrow{M} Dv$ in $L_M(\Omega)$, observe that the direct application of Lemma 2.8 implies that $Dv^n \to Dv$ in measure. To apply Lemma 2.1 we recall (1.2) and establish the convergence in $L^1(\Omega)$ of the term $T(x, Dv^n)\cdot Dv^n$.

We can set $a^n = T(x, Dv^n)\cdot Dv^n$, $a = T(x, Dv)\cdot Dv$ and claim that

$$a^n \geq 0, \quad a \in L^1(\Omega), \quad \int_\Omega a^n\, dx \to \int_\Omega a\, dx, \quad a^n \to a \quad \text{a.e. in } \Omega.$$

Noticing that

$$\int_\Omega |a^n - a|\, dx = \int_\Omega (a^n - a)\, dx + 2\int_{\{x: a^n \leq a\}} (a - a^n)\, dx$$

we conclude by Lebesgue's Dominated Convergence Theorem that

$$T(x, Dv^n)\cdot Dv^n \to T(x, Dv)\cdot Dv \quad \text{in } L^1(\Omega).$$

This implies the uniform integrability, which together with coercivity conditions (1.2) provides the uniform integrability of the sequence $\{M(x, Dv^n)\}$ and completes the proof.[2] □

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(2) Notice that (1.2) provides also the uniform integrability of $\{M^*(x, T(x, Dv^n))\}$, hence in the same way we obtain $T(\,\cdot\,, Dv^n) \xrightarrow{M^*} T(\,\cdot\,, Dv)$ in $L_{M^*}(\Omega)$.

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