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ON ATTRACTIVITY AND ASYMPTOTIC STABILITY OF SOLUTIONS OF A QUADRATIC VOLTERRA INTEGRAL EQUATION OF FRACTIONAL ORDER

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Abstract. We study the existence of solutions of a nonlinear quadratic Volterra integral equation of fractional order. In our considerations we apply the technique of measures of noncompactness in conjunction with the classical Schauder fixed point principle. The mentioned equation is considered in the Banach space of real functions defined, continuous and bounded on an unbounded interval. We will show that solutions of the investigated integral equation are locally attractive.

1. Introduction

Differential and integral equations of fractional order play a very important role in describing some real world problems. For example, these equations can be used to describe some problems occurring in physics and mechanics (cf. [8], [11], [13], [14], [17], for example).

The theory of differential and integral equations of fractional order constitutes a significant branch of nonlinear analysis, which has been rapidly developed. In recent years numerous research papers and monographs devoted to differential and integral equations of fractional order have been published, which contain a lot of various type existence results (cf. [1], [3], [7], [9], [11]–[17]).

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In this paper we will study the existence of solutions of nonlinear quadratic Volterra integral equation of fractional order in the space of real functions defined, continuous and bounded on the interval $\mathbb{R}_+ = [0, \infty)$. Moreover, we will study locally attractivity and asymptotic stability of solutions of examine equation.

The result obtained in the paper create generalization of a lot of ones obtained previously (cf. [3]–[7], [9], [16], [17], for example).

2. Notation, definitions and auxiliary facts

This section contains some definitions and auxiliary facts which will be needed in further considerations.

At the beginning we recall some basic facts concerning measures of noncompactness [2].

Assume that E is an infinite dimensional Banach space with the norm $\|\cdot\|$ and zero element θ. Denote by $B(x, r)$ the closed ball centered at x and with radius r. The symbol B_r stands for the ball $B(\theta, r)$. If X is a subset of E we write $\overline{X}$, $\operatorname{Conv} X$ to denote the closure and the convex closure of X, respectively.

Further, denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets.

We accept the following definition of the notion of a measure of noncompactness [2].

Definition 2.1. A mapping $\mu\colon \mathfrak{M}_E \to \mathbb{R}_+ = [0, \infty)$ is said to be a *measure of noncompactness in E* if it satisfies the following conditions:

(a) The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$,
(b) $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$,
(c) $\mu(\overline{X}) = \mu(X)$,
(d) $\mu(\operatorname{Conv} X) = \mu(X)$
(e) $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1]$
(f) If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ $(n = 1, 2, \dots)$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the intersection $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker \mu$ described in (a) is said to be *the kernel of the measure of noncompactness* μ. Note that the intersection set X_∞ from (f) belongs to $\ker \mu$. Indeed, since $\mu(X_\infty) \le \mu(X_n)$ for any n then we conclude that $\mu(X_\infty) = 0$, so $X_\infty \in \ker \mu$. This simple observation will be essential in our further considerations.

Other facts concerning measures of noncompactness and their properties may be found in [2].

Our considerations will be placed in the Banach space $BC(\mathbb{R}_+)$ consisting of all real functions defined, continuous and bounded on the interval $\mathbb{R}_+ = [0, \infty)$ which is equipped with the standard norm

$$\|x\| = \sup\{|x(t)| : t \geq 0\}.$$

In our paper we will use a measure of noncompactness in the space $BC(\mathbb{R}_+)$ which was introduced in [2]. In order to recall the definition of this measure let us fix a nonempty, bounded subset X of the space $BC(\mathbb{R}_+)$ and a positive number $T > 0$. For $x \in X$ and $\varepsilon \geq 0$ denote by $\omega^T(x, \varepsilon)$ *the modulus of continuity* of the function x on the interval $[0, T]$, i.e.

$$\omega^T(x, \varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [0, T],\ |t - s| \leq \varepsilon\}.$$

Further, let us put

$$\begin{aligned} \omega^T(X, \varepsilon) &= \sup\{\omega^T(x, \varepsilon) : x \in X\}, \\ \omega_0^T(X) &= \lim_{\varepsilon \to 0} \omega^T(X, \varepsilon), \\ \omega_0(X) &= \lim_{T \to \infty} \omega_0^T(X). \end{aligned}$$

If $t \in \mathbb{R}_+$ is a fixed number, we denote

$$X(t) = \{x(t) : x \in X\}$$

and

$$\operatorname{diam} X(t) = \sup\{|x(t) - y(t)| : x, y \in X\}.$$

Finally, consider the mapping μ defined on the family $\mathfrak{M}_{BC(\mathbb{R}_+)}$ by the formula

$$\mu(X) = \omega_0(X) + \limsup_{t \to \infty} \operatorname{diam} X(t). \tag{2.1}$$

It can be shown that the mapping μ is a measure of noncompactness in the space $BC(\mathbb{R}_+)$ [2]. The kernel $\ker \mu$ of this measure consists of nonempty and bounded sets X such that functions from X are locally equicontinuous on $\mathbb{R}_+$ and the thickness of the bundle formed by functions belonging to X tends to zero at infinity. This property can be applied to characterize solutions of the integral equation considered in the next section.

Now we recall definitions of the concepts of global attractivity, local attractivity and asymptotic stability of solutions. Those definitions may be found in the papers [3]–[6], [12].

Assume that Ω is a nonempty subset of the space $BC(\mathbb{R}_+)$ and Q is an operator acting from Ω into $BC(\mathbb{R}_+)$. Let us consider the following operator equation

$$x(t) = (Qx)(t), \quad t \in \mathbb{R}_+. \tag{2.2}$$

Definition 2.2. The solution $x = x(t)$ of the equation (2.2) is said to be globally attractive if for each solution $y = y(t)$ of the equation (2.2) we have that

$$\lim_{t\to\infty} (x(t) - y(t)) = 0. \tag{2.3}$$

Other words we may say that solutions of the equation (2.2) are *globally attractive* if for arbitrary solutions $x(t)$ and $y(t)$ of this equation the condition (2.3) is satisfied.

Definition 2.3. We say that solutions of the equation (2.2) are *locally attractive* if there exists a ball $B(x_0, r)$ in the space $BC(\mathbb{R}_+)$ such that for arbitrary solutions $x(t)$ and $y(t)$ of (2.2) belonging to $B(x_0, r) \cap \Omega$ the condition (2.3) does hold.

In the case when the limit (2.3) is uniform with respect to the set $B(x_0, r) \cap \Omega$, i.e. when for each $\varepsilon > 0$ there exists $T > 0$ such that

$$|x(t) - y(t)| \le \varepsilon \tag{2.4}$$

for all solutions $x(t)$, $y(t)$ of the equation (2.2) from $B(x_0, r) \cap \Omega$ and for any $t \ge T$, we will say that solutions of the equation (2.2) are uniformly locally attractive.

Observe that the global attractivity of solutions imply local attractivity.

Definition 2.4. We say that solutions of the operator equation are asymptotic stable if there exists a ball $B(x_0, r)$ such that for each $\varepsilon > 0$ there exists $T > 0$ such that if $x = x(t)$, $y = y(t)$ are arbitrary solutions of considered equation and $x, y \in B(x_0, r)$, then (2.4) holds for any $t \ge T$.

The concept of uniform local attractivity of solutions is equivalent to the concept of asymptotic stability of solutions (introduced in the paper [4] (cf. also [5])).

3. Main result

In this section we will consider the following quadratic Volterra integral equation of fractional order

$$x(t) = p(t) + \frac{f(t, x(t))}{\Gamma(\alpha)} \int_0^t \frac{u(t, s, x(s))}{(t - s)^{1-\alpha}}\, ds, \tag{3.1}$$

where $t \in \mathbb{R}_+$ and α is a fixed number, $\alpha \in (0, 1)$. Here $\Gamma(\alpha)$ denotes the classical Euler gamma function. Let us mention that the term "quadratic" used above has a mainly historical meaning.

We will investigate the equation (3.1) under the following assumptions:

(i) The function $p: \mathbb{R}_+ \to \mathbb{R}$ is continuous and bounded on $\mathbb{R}_+$.

(ii) The function $f\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a continuous function $m\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t,x) - f(t,y)| \le m(t)|x-y|$$

for any $t \in \mathbb{R}_+$ and for all $x, y \in \mathbb{R}$.

(iii) The function $u(t,s,x) = u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous. Moreover, there exist a function $n\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ being continuous on $\mathbb{R}_+ \times \mathbb{R}_+$ and a function $\Phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ which is continuous and nondecreasing on $\mathbb{R}_+$ with $\Phi(0) = 0$ and such that

$$|u(t,s,x) - u(t,s,y)| \le n(t,s)\Phi(|x-y|)$$

for all $t, s \in \mathbb{R}_+$ such that $s \le t$ and for all $x, y \in \mathbb{R}$.

(iv) The functions $a, b, c, d\colon \mathbb{R}_+ \to \mathbb{R}_+$ defined by the formulas

$$\begin{aligned} a(t) &= m(t) \int_0^t \frac{n(t,s)}{(t-s)^{1-\alpha}}\, ds, & b(t) &= m(t) \int_0^t \frac{|u(t,s,0)|}{(t-s)^{1-\alpha}}\, ds, \\ c(t) &= |f(t,0)| \int_0^t \frac{n(t,s)}{(t-s)^{1-\alpha}}\, ds, & d(t) &= |f(t,0)| \int_0^t \frac{|u(t,s,0)|}{(t-s)^{1-\alpha}}\, ds \end{aligned}$$

are bounded on $\mathbb{R}_+$ and the functions $a(t)$ and $c(t)$ vanish at infinity i.e. $\lim_{t\to\infty} a(t) = \lim_{t\to\infty} c(t) = 0$.

Taking into account the above assumption we may define the following finite constants:

$$\begin{aligned} A &= \sup\{a(t) : t \in \mathbb{R}_+\}, & B &= \sup\{b(t) : t \in \mathbb{R}_+\}, \\ C &= \sup\{c(t) : t \in \mathbb{R}_+\}, & D &= \sup\{d(t) : t \in \mathbb{R}_+\}. \end{aligned}$$

Now we formulate our last assumption.

(v) There exists a positive solution r_0 of the inequality

$$\|p\| + \frac{1}{\Gamma(\alpha)}(Ar\Phi(r) + Br + C\Phi(r) + D) \le r$$

such that $A\Phi(r_0) + B < \Gamma(\alpha)$.

The main result of the paper is contained in the below given theorem.

Theorem 3.1. *Under the assumptions* (i)–(v) *the integral equation* (3.1) *has at least one solution* $x = x(t)$ *in the space* $BC(\mathbb{R}_+)$. *Moreover, solutions of the equation* (3.1) *are uniformly locally attractive.*

Proof. Consider the operators F, U, V defined on the space $BC(\mathbb{R}_+)$ by the formulas

$$\begin{aligned}(Fx)(t) &= f(t, x(t)),\\ (Ux)(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(t, s, x(s))}{(t-s)^{1-\alpha}}\, ds,\\ (Vx)(t) &= p(t) + (Fx)(t)(Ux)(t).\end{aligned}$$

Observe that on the basis of our assumptions, for any function $x \in BC(\mathbb{R}_+)$ the function Fx is continuous on $\mathbb{R}_+$.

Now we show that the same assertion holds also for the operator U. Let us take an arbitrary function $x \in BC(\mathbb{R}_+)$ and fix $T > 0$ and $\varepsilon > 0$. Assume that $t_1, t_2 \in [0, T]$ are such that $|t_2 - t_1| \leq \varepsilon$. Without loss of generality we can assume that $t_1 < t_2$. So, taking into account our assumptions, we get:

$$\begin{aligned}
|(Ux)(t_2) - (Ux)(t_1)| &\leq \frac{1}{\Gamma(\alpha)} \bigg| \int_0^{t_1} \frac{u(t_2, s, x(s))}{(t_2-s)^{1-\alpha}}\, ds \\
&\quad + \int_{t_1}^{t_2} \frac{u(t_2, s, x(s))}{(t_2-s)^{1-\alpha}}\, ds - \int_0^{t_1} \frac{u(t_1, s, x(s))}{(t_1-s)^{1-\alpha}}\, ds \bigg| \\
&\leq \frac{1}{\Gamma(\alpha)} \bigg| \int_0^{t_1} \frac{u(t_2, s, x(s))}{(t_2-s)^{1-\alpha}}\, ds - \int_0^{t_1} \frac{u(t_1, s, x(s))}{(t_2-s)^{1-\alpha}}\, ds \bigg| \\
&\quad + \frac{1}{\Gamma(\alpha)} \bigg| \int_0^{t_1} \frac{u(t_1, s, x(s))}{(t_2-s)^{1-\alpha}}\, ds - \int_0^{t_1} \frac{u(t_1, s, x(s))}{(t_1-s)^{1-\alpha}}\, ds \bigg| \\
&\quad + \frac{1}{\Gamma(\alpha)} \bigg| \int_{t_1}^{t_2} \frac{u(t_2, s, x(s))}{(t_2-s)^{1-\alpha}}\, ds \bigg| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{|u(t_2, s, x(s)) - u(t_1, s, x(s))|}{(t_2-s)^{1-\alpha}}\, ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} |u(t_1, s, x(s))| \left[\frac{1}{(t_1-s)^{1-\alpha}} - \frac{1}{(t_2-s)^{1-\alpha}} \right] ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{|u(t_2, s, x(s))|}{(t_2-s)^{1-\alpha}}\, ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \omega_1^T(u, \varepsilon, \|x\|) \frac{1}{(t_2-s)^{1-\alpha}}\, ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} [|u(t_1, s, x(s)) - u(t_1, s, 0)| + |u(t_1, s, 0)|] \\
&\quad \cdot \left[\frac{1}{(t_1-s)^{1-\alpha}} - \frac{1}{(t_2-s)^{1-\alpha}} \right] ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{|u(t_2, s, x(s)) - u(t_2, s, 0)| + |u(t_2, s, 0)|}{(t_2-s)^{1-\alpha}}\, ds \\
&\leq \frac{\omega_1^T(u, \varepsilon, \|x\|)}{\Gamma(\alpha)} \cdot \frac{t_2^\alpha - (t_2 - t_1)^\alpha}{\alpha}
\end{aligned} \tag{3.2}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}[n(t_1,s)\Phi(|x(s)|)+|u(t_1,s,0)|] \\
&\cdot\left[\frac{1}{(t_1-s)^{1-\alpha}}-\frac{1}{(t_2-s)^{1-\alpha}}\right]ds \\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{n(t_2,s)\Phi(|x(s)|)+|u(t_2,s,0)|}{(t_2-s)^{1-\alpha}}\,ds \\
\leq{}&\frac{\omega_1^T(u,\varepsilon,\|x\|)}{\Gamma(\alpha+1)}t_1^\alpha+\frac{\overline{n}(T)\Phi(\|x\|)+\overline{u}(T)}{\Gamma(\alpha+1)}[t_1^\alpha-t_2^\alpha+(t_2-t_1)^\alpha] \\
&+\frac{\overline{n}(T)\Phi(\|x\|)+\overline{u}(T)}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha \\
\leq{}&\frac{1}{\Gamma(\alpha+1)}\{t_1^\alpha\omega_1^T(u,\varepsilon,\|x\|)+2(t_2-t_1)^\alpha[\overline{n}(T)\Phi(\|x\|)+\overline{u}(T)] \\
&+(t_2-t_1)^\alpha[\overline{n}(T)\Phi(\|x\|)+\overline{u}(T)]\},
\end{aligned}
$$

where we denoted

$$
\begin{aligned}
\omega_1^T(u,\varepsilon,\|x\|)=\sup\{&|u(t_2,s,y)-u(t_1,s,y)|: \\
&s,t_1,t_2\in[0,T],\ s\leq t_1,\ s\leq t_2,\ |t_2-t_1|\leq\varepsilon,\ |y|\leq\|x\|\},
\end{aligned}
$$

$$
\begin{aligned}
\overline{n}(T)&=\sup\{n(t,s):t,s\in[0,T],\ s\leq t\}, \\
\overline{u}(T)&=\sup\{|u(t,s,0)|:t,s\in[0,T],\ s\leq t\}.
\end{aligned}
$$

Clearly, owing to the uniform continuity of the function $u(t,s,y)$ on the set $[0,T]\times[0,T]\times[-\|x\|,\|x\|]$ we have that $\omega_1^T(u,\varepsilon,\|x\|)\to 0$ as $\varepsilon\to 0$. Thus, in view of the estimate (3.2) we have

$$
\omega^T(Ux,\varepsilon)\leq\frac{1}{\Gamma(\alpha+1)}\{T^\alpha\omega_1^T(u,\varepsilon,\|x\|)+3\varepsilon^\alpha[\overline{n}(T)\Phi(\|x\|)+\overline{u}(T)]\}.
$$

From the above inequality we deduce that the function Ux is continuous on the inerval $[0,T]$ for any $T>0$. This yields the continuity of Ux on $\mathbb{R}_+$. Finally we infer that the function Vx is continuous on $\mathbb{R}_+$.

Now, let us take an arbitrary function $x\in BC(\mathbb{R}_+)$. Using our assumptions, for a fixed $t\in\mathbb{R}_+$ we obtain:

$$
\begin{aligned}
(3.3)\quad |(Vx)(t)|\leq{}&|p(t)|+\frac{1}{\Gamma(\alpha)}[|f(t,x(t))-f(t,0)|+|f(t,0)|] \\
&\cdot\int_0^t\frac{|u(t,s,x(s))-u(t,s,0)|+|u(t,s,0)|}{(t-s)^{1-\alpha}}\,ds \\
\leq{}&\|p\|+\frac{m(t)|x(t)|+|f(t,0)|}{\Gamma(\alpha)}\int_0^t\frac{n(t,s)\Phi(|x(s)|)+|u(t,s,0)|}{(t-s)^{1-\alpha}}\,ds \\
\leq{}&\|p\|+\frac{m(t)\|x\|\Phi(\|x\|)}{\Gamma(\alpha)}\int_0^t\frac{n(t,s)}{(t-s)^{1-\alpha}}\,ds \\
&+\frac{\Phi(\|x\|)|f(t,0)|}{\Gamma(\alpha)}\int_0^t\frac{n(t,s)}{(t-s)^{1-\alpha}}\,ds
\end{aligned}
$$

$$+\frac{m(t)\|x\|}{\Gamma(\alpha)}\int_0^t \frac{|u(t,s,0)|}{(t-s)^{1-\alpha}}\,ds+\frac{|f(t,0)|}{\Gamma(\alpha)}\int_0^t \frac{|u(t,s,0)|}{(t-s)^{1-\alpha}}\,ds$$
$$\le \|p\|+\frac{1}{\Gamma(\alpha)}[\|x\|\Phi(\|x\|)a(t)+\|x\|b(t)+\Phi(\|x\|)c(t)+d(t)].$$

Thus, in virtue of assumption (iv) we conclude that the function Vx is bounded on $\mathbb{R}_+$. This assertion in conjunction with the continuity of Vx on $\mathbb{R}_+$ allows us to infer that $Vx \in BC(\mathbb{R}_+)$. Besides, from estimate (3.3) we obtain

$$\|Vx\| \le \|p\|+\frac{1}{\Gamma(\alpha)}[A\|x\|\Phi(\|x\|)+B\|x\|+C\Phi(\|x\|)+D].$$

Joining this estimate with assumption (v) we deduce that there exists $r_0 > 0$ such that the operator V transforms the ball B_{r_0} into itself.

In what follows, let us take a nonempty set $X \subset B_{r_0}$. Then, for $x, y \in X$ and for an arbitrarily fixed $t \in \mathbb{R}_+$, in view of assumptions (ii)–(iv) we get:

$$\begin{aligned}
&|(Vx)(t) - (Vy)(t)|\\
&\quad\le \left|\frac{f(t,x(t))}{\Gamma(\alpha)}\int_0^t \frac{u(t,s,x(s))}{(t-s)^{1-\alpha}}\,ds-\frac{f(t,y(t))}{\Gamma(\alpha)}\int_0^t \frac{u(t,s,y(s))}{(t-s)^{1-\alpha}}\,ds\right|\\
&\quad\le \frac{1}{\Gamma(\alpha)}|f(t,x(t))-f(t,y(t))|\int_0^t \frac{|u(t,s,x(s))|}{(t-s)^{1-\alpha}}\,ds\\
&\qquad+\frac{|f(t,y(t))|}{\Gamma(\alpha)}\int_0^t \frac{|u(t,s,x(s))-u(t,s,y(s))|}{(t-s)^{1-\alpha}}\,ds\\
&\quad\le \frac{m(t)|x(t)-y(t)|}{\Gamma(\alpha)}\int_0^t \frac{|u(t,s,x(s))-u(t,s,0)|+|u(t,s,0)|}{(t-s)^{1-\alpha}}\,ds\\
&\qquad+\frac{|f(t,y(t))-f(t,0)|+|f(t,0)|}{\Gamma(\alpha)}\int_0^t \frac{n(t,s)\Phi(|x(s)-y(s)|)}{(t-s)^{1-\alpha}}\,ds\\
&\quad\le \frac{m(t)|x(t)-y(t)|}{\Gamma(\alpha)}\int_0^t \frac{n(t,s)\Phi(|x(s)|)+|u(t,s,0)|}{(t-s)^{1-\alpha}}\,ds\\
&\qquad+\frac{m(t)|y(t)|+|f(t,0)|}{\Gamma(\alpha)}\int_0^t \frac{n(t,s)\Phi(|x(s)|+|y(s)|)}{(t-s)^{1-\alpha}}\,ds\\
&\quad\le \frac{m(t)(|x(s)|+|y(s)|)\Phi(r_0)}{\Gamma(\alpha)}\int_0^t \frac{n(t,s)}{(t-s)^{1-\alpha}}\,ds\\
&\qquad+\frac{m(t)|x(t)-y(t)|}{\Gamma(\alpha)}\int_0^t \frac{|u(t,s,0)|}{(t-s)^{1-\alpha}}\,ds\\
&\qquad+\frac{m(t)r_0\Phi(2r_0)}{\Gamma(\alpha)}\int_0^t \frac{n(t,s)}{(t-s)^{1-\alpha}}\,ds+\frac{|f(t,0)|\Phi(2r_0)}{\Gamma(\alpha)}\int_0^t \frac{n(t,s)}{(t-s)^{1-\alpha}}\,ds\\
&\quad\le \frac{2r_0\Phi(r_0)}{\Gamma(\alpha)}\,a(t)+\frac{1}{\Gamma(\alpha)}\,b(t)\operatorname{diam}X(t)+\frac{r_0\Phi(2r_0)}{\Gamma(\alpha)}\,a(t)+\frac{\Phi(2r_0)}{\Gamma(\alpha)}\,c(t).
\end{aligned}$$

From the above estimate we derive the following inequality:

$$\operatorname{diam}(VX)(t)\le\frac{2r_0\Phi(r_0)}{\Gamma(\alpha)}\,a(t)+\frac{r_0\Phi(2r_0)}{\Gamma(\alpha)}\,a(t)+\frac{\Phi(2r_0)}{\Gamma(\alpha)}\,c(t)+\frac{1}{\Gamma(\alpha)}\,b(t)\operatorname{diam}X(t).$$

Hence, by assumption (iv) we have

$$\limsup_{t\to\infty} \operatorname{diam}(VX)(t) \le k \limsup_{t\to\infty} \operatorname{diam} X(t), \tag{3.4}$$

where we denoted $k = [A\Phi(r_0) + B]/\Gamma(\alpha)$. Evidently, on account of assumption (v) we have that $k < 1$.

Further, let us take arbitrary numbers $T > 0$ and $\varepsilon > 0$. Next, fix arbitrarily a function $x \in X$ and $t_1, t_2 \in [0, T]$ such that $|t_2 - t_1| \le \varepsilon$. Without loss of generality we may assume that $t_1 < t_2$. Then, in view of our assumptions and the preceding obtained estimate (3.2) we obtain:

$$\begin{aligned}
&|(Vx)(t_2) - (Vx)(t_1)| \\
&\le |p(t_2) - p(t_1)| + |(Fx)(t_2)(Ux)(t_2) - (Fx)(t_1)(Ux)(t_2)| \\
&\quad + |(Fx)(t_1)(Ux)(t_2) - (Fx)(t_1)(Ux)(t_1)| \\
&\le \omega^T(p,\varepsilon) + |f(t_2, x(t_2)) - f(t_1, x(t_1))| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{|u(t_2, s, x(s))|}{(t_2 - s)^{1-\alpha}}\, ds \\
&\quad + \frac{|f(t_1, x(t_1))|}{\Gamma(\alpha+1)} \{T^\alpha \omega_1^T(u, \varepsilon, r_0) + 3\varepsilon^\alpha [\overline{n}(T)\Phi(r_0) + \overline{u}(T)]\} \\
&\le \omega^T(p,\varepsilon) + \left[\frac{|f(t_2, x(t_2)) - f(t_2, x(t_1))|}{\Gamma(\alpha)} + \frac{|f(t_2, x(t_1)) - f(t_1, x(t_1))|}{\Gamma(\alpha)}\right] \\
&\quad \cdot \int_0^{t_2} \frac{|u(t_2, s, x(s)) - u(t_2, s, 0)| + |u(t_2, s, 0)|}{(t_2 - s)^{1-\alpha}}\, ds \\
&\quad + \frac{|f(t_1, x(t_1)) - f(t_1, 0)| + |f(t_1, 0)|}{\Gamma(\alpha+1)} \\
&\quad \cdot \{T^\alpha \omega_1^T(u, \varepsilon, r_0) + 3\varepsilon^\alpha [\overline{n}(T)\Phi(r_0) + \overline{u}(T)]\} \\
&\le \omega^T(p,\varepsilon) + \frac{m(t_2)|x(t_2) - x(t_1)| + \omega_1^T(f,\varepsilon)}{\Gamma(\alpha)} \\
&\quad \cdot \int_0^{t_2} \frac{n(t_2, s)\Phi(|x(s)|) + |u(t_2, s, 0)|}{(t_2 - s)^{1-\alpha}}\, ds \\
&\quad + \frac{m(t_1)|x(t_1)| + |f(t_1, 0)|}{\Gamma(\alpha+1)} \{T^\alpha \omega_1^T(u, \varepsilon, r_0) + 3\varepsilon^\alpha [\overline{n}(T)\Phi(r_0) + \overline{u}(T)]\} \\
&\le \omega^T(p,\varepsilon) + \frac{m(t_2)\omega^T(x,\varepsilon) + \omega_1^T(f,\varepsilon)}{\Gamma(\alpha)} \\
&\quad \cdot \left(\int_0^{t_2} \frac{n(t_2, s)\Phi(r_0)}{(t_2 - s)^{1-\alpha}}\, ds + \int_0^{t_2} \frac{|u(t_2, s, 0)|}{(t_2 - s)^{1-\alpha}}\, ds\right) \\
&\quad + \frac{\overline{m}(T) r_0 + \overline{f}(T)}{\Gamma(\alpha+1)} \{T^\alpha \omega_1^T(u, \varepsilon, r_0) + 3\varepsilon^\alpha [\overline{n}(T)\Phi(r_0) + \overline{u}(T)]\} \\
&\le \omega^T(p,\varepsilon) + \left(\frac{\Phi(r_0) m(t_2)}{\Gamma(\alpha)} \int_0^{t_2} \frac{n(t_2, s)}{(t_2 - s)^{1-\alpha}}\, ds \right. \\
&\quad \left. + \frac{m(t_2)}{\Gamma(\alpha)} \int_0^{t_2} \frac{|u(t_2, s, 0)|}{(t_2 - s)^{1-\alpha}}\, ds\right) \omega^T(x,\varepsilon)
\end{aligned} \tag{3.5}$$

$$
\begin{aligned}
&\quad + \frac{\omega_1^T(f,\varepsilon)}{\Gamma(\alpha+1)}(\overline{n}(T)\Phi(r_0)+\overline{u}(T))t_2^\alpha \\
&\quad + \frac{\overline{m}(T)r_0+\overline{f}(T)}{\Gamma(\alpha+1)}\{T^\alpha\omega_1^T(u,\varepsilon,r_0)+3\varepsilon^\alpha[\overline{n}(T)\Phi(r_0)+\overline{u}(T)]\} \\
&\le \omega^T(p,\varepsilon)+\left(\frac{\Phi(r_0)a(t_2)}{\Gamma(\alpha)}+\frac{b(t_2)}{\Gamma(\alpha)}\right)\omega^T(x,\varepsilon) \\
&\quad + \frac{\omega_1^T(f,\varepsilon)}{\Gamma(\alpha+1)}(\overline{n}(T)\Phi(r_0)+\overline{u}(T))t_2^\alpha \\
&\quad + \frac{\overline{m}(T)r_0+\overline{f}(T)}{\Gamma(\alpha+1)}\{T^\alpha\omega_1^T(u,\varepsilon,r_0)+3\varepsilon^\alpha[\overline{n}(T)\Phi(r_0)+\overline{u}(T)]\} \\
&\le \omega^T(p,\varepsilon)+\frac{A\Phi(r_0)+B}{\Gamma(\alpha)}\omega^T(x,\varepsilon)+\frac{\omega_1^T(f,\varepsilon)}{\Gamma(\alpha+1)}(\overline{n}(T)\Phi(r_0)+\overline{u}(T))t_2^\alpha \\
&\quad + \frac{\overline{m}(T)r_0+\overline{f}(T)}{\Gamma(\alpha+1)}\{T^\alpha\omega_1^T(u,\varepsilon,r_0)+3\varepsilon^\alpha[\overline{n}(T)\Phi(r_0)+\overline{u}(T)]\},
\end{aligned}
$$

where we denoted

$$
\begin{gathered}
\omega_1^T(f,\varepsilon)=\sup\{|f(t_2,x)-f(t_1,x)|:t_1,t_2\in[0,T],\ |t_2-t_1|\le\varepsilon,\ x\in[-r_0,r_0]\}, \\
\overline{m}(T)=\max\{m(t):t\in[0,T]\},\quad \overline{f}(T)=\max\{|f(t,0)|:t\in[0,T]\}.
\end{gathered}
$$

Now, keeping in mind the uniform continuity of the function $u=u(t,s,x)$ on the set $[0,T]\times[0,T]\times[-r_0,r_0]$ and the uniform continuity of the function $f=f(t,x)$ on the set $[0,T]\times[-r_0,r_0]$, from the estimate (3.5) we derive the following one:

$$
\omega_0^T(VX)\le\frac{A\Phi(r_0)+B}{\Gamma(\alpha)}\omega_0^T(X)=k\omega_0^T(X).
$$

Consequently, we obtain:

$$
\omega_0(VX)\le k\omega_0(X). \tag{3.6}
$$

Finally, linking (3.4) and (3.6) and the definition of the measure of noncompactness μ given by the formula (2.1), we derive the following inequality

$$
\mu(VX)\le k\mu(X). \tag{3.7}
$$

Further on, let us put $B_{r_0}^1=\operatorname{Conv}V(B_{r_0})$, $B_{r_0}^2=\operatorname{Conv}V(B_{r_0}^1)$ and so on. Consider the sequence of sets $(B_{r_0}^n)$. Observe that this sequence is decreasing i.e. $B_{r_0}^{n+1}\subset B_{r_0}^n$ for $n=1,2,\ldots$ Moreover, the sets of this sequence are nonempty, closed and convex. Hence, in view of (3.7) we obtain

$$
\mu(B_{r_0}^n)\le k^n\mu(B_{r_0})
$$

for $n=1,2,\ldots$ This yields that $\lim_{n\to\infty}\mu(B_{r_0}^n)=0$, because $k<1$ (cf. assumption (v)). Thus, keeping in mind Definition 2.1 we infer that the set

$Y = \bigcap_{n=1}^{\infty} B_{r_0}^n$ is nonempty, bounded, closed and convex. So, the set Y belongs to the kernel $\ker \mu$. In particular, we get

$$\limsup_{t\to\infty} \operatorname{diam} Y(t) = \lim_{t\to\infty} \operatorname{diam} Y(t) = 0. \tag{3.8}$$

Let us also note that the operator V transforms the set Y into itself.

Next we show that the operator V is continuous on the set Y. To prove this let us fix a number $\varepsilon > 0$ and take arbitrary functions $x, y \in Y$ such that $\|x - y\| \le \varepsilon$. Using (3.8) and the fact that $VY \subset Y$ we obtain that there exists $T > 0$ such that for an arbitrary $t \ge T$ we get

$$|(Vx)(t) - (Vy)(t)| \le \varepsilon. \tag{3.9}$$

Furthermore, let us assume that $t \in [0, T]$. Then, employing the imposed assumptions and estimating similarly as above, we have

$$\begin{aligned}
&|(Vx)(t) - (Vy)(t)| \\
&\le \frac{1}{\Gamma(\alpha)}\left| f(t,x(t)) \int_0^t \frac{u(t,s,x(s))}{(t-s)^{1-\alpha}}\,ds - f(t,y(t)) \int_0^t \frac{u(t,s,x(s))}{(t-s)^{1-\alpha}}\,ds \right. \\
&\quad \left. + f(t,y(t)) \int_0^t \frac{u(t,s,x(s))}{(t-s)^{1-\alpha}}\,ds - f(t,y(t)) \int_0^t \frac{u(t,s,y(s))}{(t-s)^{1-\alpha}}\,ds \right| \\
&\le \frac{1}{\Gamma(\alpha)} |f(t,x(t)) - f(t,y(t))| \int_0^t \frac{|u(t,s,x(s))|}{(t-s)^{1-\alpha}}\,ds \\
&\quad + \frac{|f(t,y(t))|}{\Gamma(\alpha)} \int_0^t \frac{|u(t,s,x(s)) - u(t,s,y(s))|}{(t-s)^{1-\alpha}}\,ds \\
&\le \frac{m(t)|x(t)-y(t)|}{\Gamma(\alpha)} \int_0^t \frac{|u(t,s,x(s)) - u(t,s,0)| + |u(t,s,0)|}{(t-s)^{1-\alpha}}\,ds \\
&\quad + \frac{|f(t,y(t)) - f(t,0)| + |f(t,0)|}{\Gamma(\alpha)} \int_0^t \frac{n(t,s)\Phi(|x(s)-y(s)|)}{(t-s)^{1-\alpha}}\,ds \\
&\le \frac{m(t)\varepsilon}{\Gamma(\alpha)} \int_0^t \frac{n(t,s)\Phi(|x(s)|) + |u(t,s,0)|}{(t-s)^{1-\alpha}}\,ds \\
&\quad + \frac{m(t)|y(t)| + |f(t,0)|}{\Gamma(\alpha)} \int_0^t \frac{n(t,s)\Phi(\varepsilon)}{(t-s)^{1-\alpha}}\,ds \\
&\le \frac{\varepsilon\Phi(r_0)}{\Gamma(\alpha)} a(t) + \frac{\varepsilon}{\Gamma(\alpha)} b(t) + \frac{r_0\Phi(\varepsilon)}{\Gamma(\alpha)} a(t) + \frac{\Phi(\varepsilon)}{\Gamma(\alpha)} c(t) \\
&\le \frac{A\Phi(r_0) + B}{\Gamma(\alpha)}\varepsilon + \frac{Ar_0 + C}{\Gamma(\alpha)}\Phi(\varepsilon).
\end{aligned} \tag{3.10}$$

Now, linking (3.9) and (3.10) and the assumption (iv) we deduce that the operator V transforms continuously the set Y into itself.

Finally, let us observe that taking into account all facts concerning the set Y and the operator $V\colon Y \to Y$ and applying the classical Schauder fixed point principle we infer that V has at least one fixed point x belonging to the set Y.

Obviously the function $x = x(t)$ is a solution of the integral equation (3.1). Since $Y \in \ker\mu$ we conclude that all solutions of the equation (3.1) are uniformly locally attractive in the sense of Definition 2.3. This completes the proof. $\square$

4. An example

In this section our aim is to illustrate main result contained in Theorem 3.1.

EXAMPLE4.1. Consider the following quadratic Volterra integral equation of fractional order

$$x(t) = t^2 e^{-4t} + \frac{L(x+1)}{(t^2+1)\Gamma(1/2)} \int_0^t \frac{\sqrt{t^3}\,\sqrt[3]{x^2(s)}/(\sqrt{t-s}+1) + t^2 e^{-\sqrt{t-s}}}{(t-s)^{1/2}}\, ds, \tag{4.1}$$

where $t \in \mathbb{R}_+$ and L is a positive constant.

Observe that the above equation is a special case of equation (3.1). Indeed, putting $\alpha = 1/2$ and

$$p(t) = t^2 e^{-4t}, \quad f(t,x) = \frac{L(x+1)}{t^2+1}, \quad u(t,s,x) = \frac{\sqrt{t^3}\,\sqrt[3]{x^2}}{\sqrt{t-s}+1} + t^2 e^{-\sqrt{t-s}},$$

we can easily see that there are satisfied the assumptions of Theorem 3.1. In fact, we have that p is continuous and bounded on $\mathbb{R}_+$ and $\|p\| = 0.03383382\ldots$ This shows that assumption (i) is satisfied.

Next, observe that the function $f(t,x)$ satisfies assumption (ii) with $m(t) = L/(t^2+1)$ and $f(t,0) = L/(t^2+1)$.

Further notice that the function $u(t,s,x)$ satisfies assumption (iii), where $n(t,s) = \sqrt{t^3}/(\sqrt{t-s}+1)$, $\Phi(r) = \sqrt[3]{r^2}$ and $u(t,s,0) = t^2 e^{-\sqrt{t-s}}$.

In order to check that assumption (iv) is satisfied let us notice that the functions a, b, c, d appearing in that assumption have form

$$\begin{aligned}
a(t) &= \frac{L\,t^{3/2}}{t^2+1} \int_0^t \frac{ds}{\sqrt{t-s}(\sqrt{t-s}+1)} = 2L\frac{t^{3/2}\ln(\sqrt{t}+1)}{t^2+1}, \\
b(t) &= \frac{L\,t^2}{t^2+1} \int_0^t \frac{e^{-\sqrt{t-s}}}{\sqrt{t-s}}\, ds = 2L\frac{t^2(1-e^{-\sqrt{t}})}{t^2+1}, \\
c(t) &= \frac{L\,t^{3/2}}{t^2+1} \int_0^t \frac{ds}{\sqrt{t-s}(\sqrt{t-s}+1)} = 2L\frac{t^{3/2}\ln(\sqrt{t}+1)}{t^2+1}, \\
d(t) &= \frac{L\,t^2}{t^2+1} \int_0^t \frac{e^{-\sqrt{t-s}}}{\sqrt{t-s}}\, ds = 2L\frac{t^2(1-e^{-\sqrt{t}})}{t^2+1}.
\end{aligned}$$

Now, let us observe that $a(t) \to 0$ as $t \to \infty$. Moreover, applying the standard inequality

$$\ln(\sqrt{t}+1) \le \sqrt{t}$$

for $t \ge 0$ we infer that $A \le 2L$.

Further on, we have that $b(t)$ is bounded on $\mathbb{R}_+$ and $B = 2L$. Moreover, it is easy to check that $c(t) \to 0$ as $t \to \infty$ and (similarly as above) we have that $C \leq 2L$. Finally, we can also verify that $d(t)$ is bounded on $\mathbb{R}_+$ and $D = 2L$.

Now, let us consider the inequality from assumption (v) which has the form

$$\|p\| + \frac{1}{\Gamma(1/2)}(Ar\sqrt[3]{r^2} + Br + C\sqrt[3]{r^2} + D) \leq r. \tag{4.2}$$

Observe that taking into account that $\Gamma(1/2) = \sqrt{\pi}$ (cf. [10]) and keeping in mind the above obtained estimates of the constants $\|p\|$, A, B, C, D we deduce that the number $r_0 = 1$ is a solution of the inequality (4.2) for $L \leq 0.214$. It is easily seen that $A\Phi(r_0)+B \leq 2L[\Phi(r_0)+1] = 2L[\Phi(1)+1] = 4L < \Gamma(1/2)$. Thus, in view of Theorem 3.1 we conclude that the equation (4.1) has a solution in the space $BC(\mathbb{R}_+)$ belonging to the ball B_1. Moreover, all solutions of equation (4.1) which belongs to B_1 are uniformly locally attractive in the sense of Definition 2.3. This means that for arbitrary solutions $x(t)$ and $y(t)$ of equation (4.1) belonging to B_1 we have that

$$\lim_{t\to\infty}(x(t) - y(t)) = 0$$

uniformly with respect to the ball B_1.

It is also worthwhile mentioning that the equation (3.1) is considered in [3], where in assumption (iii) the authors take into account the function n depending on t and continuous on $\mathbb{R}_+$. We assume in (iii) that exists the function $n = n(t, s)$ which depends on two variables t, s and is continuous on $\mathbb{R}_+ \times \mathbb{R}_+$. Thus the result contained in our Theorem 3.1 creates an essential generalization of that from [3], where the function n depends on the variable t only. It can be checked that functions involved in equation (4.1) do not satisfy assumptions of the existence result proved in [3].

References

[1] A. Babakhani and V. Daftardar-Gejii, *Existence of positive solutions of nonlinear fractional differential equations*, J. Math. Anal. Appl. **278** (2003), 434–442.

[2] J. Banaś and K. Goebel, *Measures of noncompactness in Banach spaces*, Lecture Notes in Pure and Applied Mathematics, vol. 60, Marcel Dekker, New York, 1980.

[3] J. Banaś and D. O'Regan, *On existence and local attractivity of solutions of a quadratic Volterra integral equation of fractional order*, preprint.

[4] J. Banaś and B. Rzepka, *An application of a measure of noncompactness in the study of asymptotic stability*, Appl. Math. Letters **16** (2003), 1–6.

[5] ———, *On existence and asymptotic stability of solutions of a nonlinear integral equation*, J. Math. Anal. Appl. **284** (2003), 165–173.

[6] T. A. Burton and B. Zhang, *Fixed points and stability of an integral equation: nonuniqueness*, Appl. Math. Letters **17** (2004), 839–846.

[7] M. Cichoń, A. M. A. El-Sayed and H. A. H. Salem, *Existence theorem for nonlinear integral equations of fractional orders*, Comment. Math. **41** (2001), 59–67.

[8] C. Corduneanu, *Integral Equations and Applications*, Cambridge Univ. Press, Cambridge, 1991.

[9] A. M. A. El-Sayed, *Nonlinear functional differential equations of arbitrary order*, Nonlinear Anal. **33** (1998), 181–186.

[10] G. H. Fichtenholz, *Differential and Integral Calculus* II, PWN, Warsaw, 1980. (Polish)

[11] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[12] X. Hu, J. Yan, *The global attractivity and asymptotic stability of solution of a nonlinear integral equation*, J. Math. Anal. Appl. **321** (2006), 147–156.

[13] A. A. Kilbas and J. J. Trujillo, *Differential equations of fractional order: Methods, results, problems* I, Appl. Anal. **78** (2001), 153–192.

[14] K. S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.

[15] D. O'Regan and M. Meehan, *Existence Theory for Nonlinear Integral and Integrodifferential Equations*, Kluwer Academic Publishers, Dordrecht, 1998.

[16] I. Podlubny, *Fractional Differencial Equations*, Academic Press, San Diego, 1999.

[17] S. Samko, A. A. Kilbas and O. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Amsterdam, 1993.

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