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NODAL SOLUTIONS OF A PERTURBED ELLIPTIC PROBLEM

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Abstract. Multiple nodal solutions are obtained for the elliptic problem

$$\begin{aligned} -\Delta u &= f(x,\ u) + \varepsilon g(x,\ u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where ε is a parameter, Ω is a smooth bounded domain in $\mathbb{R}^N$, $f \in C(\overline{\Omega}\times\mathbb{R})$, and $g \in C(\overline{\Omega} \times \mathbb{R})$. For a superlinear C^1 function f which is odd in u and for any C^1 function g, we prove that for any $j \in \mathbb{N}$ there exists $\varepsilon_j > 0$ such that if $|\varepsilon| \leq \varepsilon_j$ then the above problem possesses at least j distinct nodal solutions. Except C^1 continuity no further condition is needed for g. We also prove a similar result for a continuous sublinear function f and for any continuous function g. Results obtained here refine earlier results of S. J. Li and Z. L. Liu in which the nodal property of the solutions was not considered.

1. Introduction

In this paper, we consider the perturbed elliptic boundary value problem

$$(1.1)_\varepsilon \qquad \begin{aligned} -\Delta u &= f(x,\ u) + \varepsilon g(x,u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, ε is a parameter, $f, g \in C(\overline{\Omega} \times \mathbb{R})$, and $f(x,t)$ is odd in t. Based on the idea of essential values

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developed in [14] and [15], existence of multiple solutions was studied in [19]. However, the nodal property of the solutions was not considered there. In the present paper, we shall refine the results in [19] and show that the solutions obtained there could be nodal solutions. A solution $u \in C(\overline{\Omega})$ is said to be a nodal solution if the set $\{x \in \Omega : u(x) \neq 0\}$ has at least two connected components.

For various elliptic problems, symmetry of the nonlinearity usually results in existence of multiple solutions. In some cases, problems with perturbations from symmetry also have multiple solutions; see [3]–[5], [22] and [24] for instance. But only for problems which are invariant under various symmetry groups, multiple nodal solutions were obtained in the past. (Please refer to [2], [6], [9], [10], [17], [20] and references therein.) In this paper the perturbation problem $(1.1)_\varepsilon$ does not have any symmetry and no known results on existence of multiple nodal solutions can be applied to it.

We shall present two kinds of results on existence of multiple nodal solutions for $(1.1)_\varepsilon$. To state our first result, we need the following assumptions.

- (g_0) $g \in C^1(\overline{\Omega} \times \mathbb{R})$.
- (f_0) $f \in C^1(\overline{\Omega} \times \mathbb{R})$.
- (f_1) $f(x,-t) = -f(x,t)$ for all $x \in \Omega$ and $t \in \mathbb{R}$.
- (f_2) $\lim_{|t|\to+\infty} |f'_t(x,t)||t|^{-p} = 0$ uniformly in x if $N \geq 3$, where $p \in (0, 4/(N-2))$;
  $\lim_{|t|\to+\infty} \ln(|f'_t(x,t)|+1)|t|^{-2} = 0$ uniformly in x if $N = 2$;
  and no assumption if $N = 1$.
- (f_3) There exist constants $M > 0$ and $\mu > 2$ such that

$$0 < F(x,t) := \int_0^t f(x,s)\,ds \leq \frac{1}{\mu} t f(x,t), \quad \text{for all } x \in \overline{\Omega},\ |t| \geq M.$$

THEOREM 1.1. *Suppose that* (g_0) *and* (f_0)–(f_3) *are satisfied. Then for any* $j \in \mathbb{N}$, *there exists* $\varepsilon_j > 0$ *such that if* $|\varepsilon| \leq \varepsilon_j$ *problem* $(1.1)_\varepsilon$ *possesses at least* j *distinct nodal solutions corresponding to positive critical values. That is, for such a solution* u,

$$\frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(x,u) - \varepsilon \int_\Omega G(x,u) > 0$$

where $G(x,t) = \int_0^t g(x,s)\,ds$.

REMARK 1.2. The solutions obtained in Theorem 1.1 are classical solutions and are in $C^{2,\alpha}$ for any $0 < \alpha < 1$.

The case we deal with in Theorem 1.1 is called the superlinear case because of condition (f_3). Next we consider the sublinear case. For the sublinear case, we just need to impose conditions on f and g for t near 0. We assume that there exists $\delta > 0$ such that the following assumptions hold.

(g_0') $g \in C(\overline{\Omega} \times (-\delta, \delta))$.
(f_0') $f \in C(\overline{\Omega} \times (-\delta, \delta))$.
(f_1') $f(x, -t) = -f(x, t)$ for all $x \in \Omega$ and $t \in (-\delta, \delta)$.
(f_2') $\lim_{|t| \to 0} F(x, t) t^{-2} = +\infty$.
(f_3') $2F(x, t) > t f(x, t) > 0$ for all $x \in \Omega$ and $0 < |t| < \delta$.

THEOREM 1.3. *Suppose that* (g_0') *and* (f_0')–(f_3') *are satisfied. Then for any* $j \in \mathbb{N}$*, there exists* $\varepsilon_j > 0$ *such that if* $|\varepsilon| \leq \varepsilon_j$ *then problem* $(1.1)_\varepsilon$ *possesses at least* j *distinct nodal solutions corresponding to negative critical values. That is,*

$$\frac{1}{2} \int_\Omega |\nabla u|^2 - \int_\Omega F(x, u) - \varepsilon \int_\Omega G(x, u) < 0$$

for such a solution u.

REMARK 1.4. (a) The solutions obtained in Theorem 1.3 are weak solutions and are in $C^{1,\alpha}$ for any $0 < \alpha < 1$. Because of (f_2') and (f_3'), f can not be a C^1 function. But if f and g are C^α functions for some $0 < \alpha < 1$ then the solutions in Theorem 1.3 are classical solutions in $C^{2,\alpha}$.

(b) To make the arguments more transparent, we do not seek maximum generality of the assumptions in Theorem 1.1 or Theorem 1.3. The conditions listed above are a little bit stronger than those in [19, Theorem 1].

REMARK 1.5. (a) For the unperturbed problem (that is, $\varepsilon = 0$), the existence of infinitely many nodal solutions has been studied by several authors. If f satisfies (f_0)–(f_3), infinitely many nodal solutions for Dirichlet boundary value problems were first proved to exist by Bartsch in [6]. The result in [6] was improved by Li and Wang in [20], where results on nodal solutions were presented in several interesting cases. Infinitely many nodal solutions for equations on the whole space $\mathbb{R}^N$ were obtained in [7]. Again for the unperturbed problem (that is, $\varepsilon = 0$), if f satisfies (f_0')–(f_3'), infinitely many nodal solutions of Dirichlet boundary value problems were also obtained in [20] and [25]. It seems that the approaches developed in [6], [7], [20] and [25] for the unperturbed problem do not suit the perturbed case.

(b) Multiple solutions of the perturbed problem have been obtained by many authors, without nodal properties of the solutions being considered. Results in this direction are mainly obtained through variational approaches. Similar problems to the one considered here were recently studied by Chambers and Ghoussoub in [11] where more references on related results can be found. Degiovanni and Radulescu studied a perturbation eigenvalue problem in [15] and obtained multiple solutions by using the idea of essential value, which was initiated and developed by Degiovanni and Lancelotti in [14]. The method devised in [14] has also been successfully used in [19]. In all the references mentioned here on perturbation problems, the authors were mainly interested in obtaining multiple solutions, leaving nodal properties of the solutions unconsidered.

(c) We would like to emphasize that, except for being of class C^1 or C, no further condition is imposed on g in Theorem 1.1 or Theorem 1.3. This brings three major difficulties. First, since we do not assume $g(x,0)=0$, at least one of the two cones $P_E=\{u\in H_0^1(\Omega):u\geq 0\}$ and $-P_E$ is not invariant for any descending flow associated with problem $(1.1)_\varepsilon$. But the methods for finding nodal solutions developed in the literature were essentially based on invariance of P and $-P$. Thus we must adjust the reliance on invariance of P and $-P$. Second, the functional associated with problem $(1.1)_\varepsilon$ is not even and genus can not be used in a direct way; instead, we shall construct essential values via genus for the unperturbed problem. Third, problem $(1.1)_\varepsilon$ itself does not have a variational setting and it must be modified in order for a variational argument to be effective. For example $\int_\Omega G(x,u)$ is not well-defined in $H_0^1(\Omega)$.

In the following, we use C_i to stand for a constant.

2. Preliminaries

In their study of existence of multiple solutions for a perturbed elliptic eigenvalue problem in [14], Degiovanni and Lancelotti introduced the notion of essential value for a functional of class C . It turns out that an essential value c is a critical value if the functional is in C^1 and satisfies the $(\mathrm{PS})_c$ condition. The most important property of this notion is that if c is an essential value of I then in any neighbourhood of c any small perturbation of I must also have an essential value. This approach has been successfully applied in obtaining multiple solutions in several interesting cases. In this paper, we shall follow the idea of Degiovanni and Lancelotti, but we shall define essential value in quite a different way, so that it can be applied to problem $(1.1)_\varepsilon$.

Assume that E is a Banach space, $\mathcal{D}$ is a closed subset of E, and $I\in C^1(E,\mathbb{R})$. For $b\in\mathbb{R}$, we set $I^b=\{u\in E:I(u)\leq b\}$ and call I^b a level set of I.

Definition 2.1. Let $a,b\in\mathbb{R}$ with $a<b$. The pair (I^b,I^a) is said to be *trivial with respect to* $\mathcal{D}$ if for any $\varepsilon>0$, any compact topological space Y, and any $h\in C(Y,I^b\cup\mathcal{D})$, there exists $\widetilde{h}\in C(Y,I^a\cup\mathcal{D})$ such that $\widetilde{h}(x)=h(x)$ for any $x\in Y$ with $h(x)\in I^{a-\varepsilon}\cup\mathcal{D}$.

Definition 2.2. A real number c is said to be an *essential value of I with respect to* $\mathcal{D}$ if for any $\varepsilon>0$ there exist $a,b\in(c-\varepsilon,c+\varepsilon)$ with $a<b$ such that the pair (I^b,I^a) is not trivial with respect to $\mathcal{D}$.

The following theorem is similar to [14, Theorem 2.5] and we include a proof here for completeness.

Theorem 2.3. *Let $a, b \in \mathbb{R}$ with $a < b$. Let us assume that I has no essential value in $[a, b]$ with respect to $\mathcal{D}$. Then the pair (I^b, I^a) is trivial with respect to $\mathcal{D}$.*

Proof. Since I has no essential value with respect to $\mathcal{D}$ in $[a, b]$, there exist a finite number of open intervals $\{(a_i, b_i)\}_{i=1}^k$ such that $[a, b] \subset \bigcup_{i=1}^k (a_i, b_i)$ and (I^d, I^c) is trivial with respect to $\mathcal{D}$ for any $a_i \le c < d \le b_i$ and $i = 1, \ldots, k$. Without loss of generality, we assume that

$$a_i < b_{i-1} < a_{i+1} < b_i \quad \text{for } i = 2, \ldots, k-1, \tag{2.1}$$

$$a_1 < a < a_2, \tag{2.2}$$

$$b_{k-1} < b < b_k. \tag{2.3}$$

Assume $\varepsilon > 0$, Y a compact topological space, and $h \in C(Y, I^b \cup \mathcal{D})$. Then increasing a_1 if necessary, we have, in addition to (2.2), $a - \varepsilon < a_1$. Thus we can choose $\varepsilon^* > 0$ such that

$$a - a_1 < \varepsilon - \varepsilon^*. \tag{2.4}$$

By (2.3), $h \in C(Y, I^{b_k} \cup \mathcal{D})$. Since (I^{b_k}, I^{a_k}) is trivial with respect to $\mathcal{D}$, there exists $h_1 \in C(Y, I^{a_k} \cup \mathcal{D})$ such that $h_1(x) = h(x)$ for $x \in Y$ with $h(x) \in I^{a_k - \varepsilon^*} \cup D$. Now by (2.1), $h_1 \in C(Y, I^{b_{k-1}} \cup \mathcal{D})$. Since $(I^{b_{k-1}}, I^{a_{k-1}})$ is trivial with respect to $\mathcal{D}$, there exists $h_2 \in C(Y, I^{a_{k-1}} \cup \mathcal{D})$ such that $h_2(x) = h_1(x) = h(x)$ for $x \in Y$ with $h_1(x) = h(x) \in I^{a_{k-1} - \varepsilon^*} \cup D$. After k steps in this way, we find $h_k \in C(Y, I^{a_1} \cup \mathcal{D})$ such that $h_k(x) = \ldots = h_1(x) = h(x)$ for $x \in Y$ with $h_{k-1}(x) = \ldots = h_1(x) = h(x) \in I^{a_1 - \varepsilon^*} \cup D$. But (2.2) implies $h_k \in C(Y, I^a \cup \mathcal{D})$ while (2.4) yields $h_k(x) = h(x)$ for $x \in Y$ with $h(x) \in I^{a - \varepsilon} \cup D$. Therefore, the pair (I^b, I^a) is trivial with respect to $\mathcal{D}$. $\square$

3. Proof of Theorem 1.1

By (f_0)–(f_3), there exists a constant $L^* > 0$ such that

$$t(f(x, t) + L^* t) > 0 \quad \text{for} \quad t \neq 0. \tag{3.1}$$

Denote $E = H_0^1(\Omega)$. Except the usual norm $\|u\|$ of E induced by the inner product $(u, v) = \int_\Omega (\nabla u \cdot \nabla v)$ we shall make use of the equivalent norm $\|u\|_*$ induced by the inner product

$$(u, v)_* = \int_\Omega (\nabla u \cdot \nabla v + L^* uv). \tag{3.2}$$

And apart from E, the space $X := C_0^1(\overline{\Omega})$ will also be used. Weak solutions of the unperturbed problem $(1.1)_0$ are precisely critical points of

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(x,u), \quad u \in E,$$

which is a C^2 functional, by (f_0) and (f_2). Since there is no growth condition imposed on g near $|t| = \infty$, in order for the perturbed problem to have a variational structure the function g should be cut off. For any $k \in \mathbb{N}$, choose a smooth function $\beta_k(t)$ such that $\beta_k(t) = 1$ if $|t| \le k$, that $\beta_k(t) = 0$ if $|t| \ge k+1$, and that $0 < \beta_k(t) < 1$ if $k < |t| < k+1$. Let $g_k(x,t) = \beta_k(t)g(x,t)$ and $G_k(x,t) = \int_0^t g_k(x,s)\,ds$. For any $k \in \mathbb{N}$, choose $\varepsilon_1(k) > 0$ such that for all $x \in \Omega$ and $t \in \mathbb{R}$,

$$\varepsilon_1(k)|g_k(x,t)| < 1, \quad \varepsilon_1(k)|G_k(x,t)| < 1, \quad \varepsilon_1(k)|tg_k(x,t)| < 1. \tag{3.3}$$

For $k \in \mathbb{N}$ and $|\varepsilon| \le \varepsilon_1(k)$, set

$$I_{\varepsilon,k}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega [F(x,u) + \varepsilon G_k(x,u)], \quad u \in E.$$

Note that $I_{\varepsilon,k}$ is a C^2 functional and critical points of $I_{\varepsilon,k}$ are solutions of

$$\begin{aligned} -\Delta u &= f(x,u) + \varepsilon g_k(x,u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.4}$$

Any solution of (3.4) with L^∞ norm less than k is a solution of problem $(1.1)_\varepsilon$.

The positive cone P and the negative cone $-P$ in the space X are very important in obtaining nodal solutions. They are defined by

$$\pm P = \{u \in X : \pm u \ge 0\}.$$

Denote $\mathcal{D} = P \cup (-P)$. Then any solution of $(1.1)_\varepsilon$ in $X \setminus \mathcal{D}$ is a nodal solution. Clearly, the interior $\mathrm{int}(\mathcal{D})$ of $\mathcal{D}$ in X is nonempty. This property plays an important role in obtaining solutions of (3.4) in $X \setminus \mathcal{D}$, and this is why we have taken the space X into consideration. The positive cone P_E and the negative cone $-P_E$ in the space E do not have this property.

Now we plan to prove the existence of an unbounded increasing sequence of essential values of I with respect of $\mathcal{D}$. We follow an argument in [19]. Denote by $\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$ all the eigenvalues of $-\Delta$ with Dirichlet boundary condition and by $e_1, e_2, e_3, \ldots$ the corresponding eigenfunctions, with the explicit meaning that each λ_i is counted as many times as its multiplicity. Denote $E_k = \mathrm{span}\{e_1, \ldots, e_k\}$ and use $E_k^\perp$ to represent the orthogonal complement of E_k in E. By (f_2) and (f_3), there exists an increasing sequence of positive numbers $\{R_k\}$ (see [1], [23]) such that $R_k \to \infty$ and

$$I(u) \le 0, \quad \text{for all } u \in E_k,\ \|u\| \ge R_k.$$

Let $B_k = \{u : u \in E_k,\ \|u\| \le R_k\}$ and ∂B_k be the boundary of B_k in E_k. Define a sequence $\{\Phi_k\}$ of sets of functions inductively as

$$\Phi_1 = \{h : h \in C(B_1, X),\ h \text{ is odd, and } h|_{\partial B_1} = \mathrm{id}\}$$

and, for $k = 1, 2, \ldots,$

$$\Phi_{k+1} = \{h : h \in C(B_{k+1}, X),\ h \text{ is odd},\ h|_{\partial B_{k+1}} = \mathrm{id}, \text{ and } h|_{B_k} \in \Phi_k\}.$$

Note that unlike [19], here we use the space X instead of E in constructing Φ_k. This is important for obtaining nodal solutions. Define, for $k = 1, 2, \ldots,$

$$b_k = \inf_{h \in \Phi_k} \sup_{u \in h(B_k) \setminus \mathcal{D}} I(u).$$

It is obvious that $b_1 \le b_2 \le b_3 \le \ldots$

We are going to prove $b_k \to \infty$. Here we adopt a direct argument; the argument in [19] is indirect. For this we first have

LEMMA 3.2. *For any $R > 0$ and $r > 0$, there exists $n_0 \in \mathbb{N}$ such that if $n \ge n_0$ then*

$$\sup_{u \in E_n^{\perp}, \|u\| \le R} \int_\Omega |F(x, u)| \le r.$$

PROOF. We first consider the case $N \ge 3$. For any $\delta > 0$ the condition (f_2) implies the existence of two constants K_δ and C_δ depending on δ such that

$$\int_\Omega |F(x, u)| \le \delta \int_{|u| \ge K_\delta} |u|^{2^*} + C_\delta \int_{|u| < K_\delta} |u|^2 \le C_1 \delta \|u\|^{2^*} + C_\delta \|u\|^2_{L^2(\Omega)},$$

where $2^* = 2N/(N-2)$. Thus if $u \in E_n^{\perp}$ and $\|u\| \le R$, then

$$\int_\Omega |F(x, u)| \le C_1 \delta \|u\|^{2^*} + \frac{C_\delta}{\lambda_{n+1}} \|u\|^2 \le C_1 \delta R^{2^*} + \frac{C_\delta R^2}{\lambda_{n+1}}.$$

First choosing $\delta > 0$ sufficient small and then n_0 sufficiently large, we obtain the result. Now we consider the case $N = 2$. By [16, Theorem 7.15], there are positive constants C_2 and C_3 depending on R such that for $u \in E$ and $\|u\| \le R$,

$$\int_\Omega \exp(C_2 u^2) \le C_3.$$

For C_2 in the last inequality, the condition (f_2) yields $K > 0$ such that

$$\int_\Omega |F(x, u)| \le \int_{|u| \ge K} \exp\left(\frac{1}{2} C_2 u^2\right) |u| + C_4 \int_{|u| < K} |u|.$$

Therefore, Hölder inequality yields for $u \in E_n^{\perp}$ and $\|u\| \le R$,

$$\int_\Omega |F(x, u)| \le C_5 \|u\|_{L^2(\Omega)} \le C_5 R \lambda_{n+1}^{-1/2}.$$

The constants K, C_4, and C_5 may depend on R. The last inequality implies the result. The case $N = 1$ is trivial. $\square$

Now we have

Lemma 3.3. *$b_k \to \infty$.*

Proof. For any $G > 0$, by Lemma 3.2 there exists $n \in \mathbb{N}$ such that $R_{n+1} > 2\sqrt{G}$ and

$$\sup_{u \in E_n^\perp, \|u\| \le 2\sqrt{G}} \int_\Omega |F(x,u)| \le G.$$

Then for $u \in E_n^\perp$ and $\|u\| = 2\sqrt{G}$,

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(x,u) \ge G.$$

We claim that for any $h \in \Phi_{n+1}$,

$$(h(B_{n+1}) \setminus \mathcal{D}) \cap E_n^\perp \cap S_{2\sqrt{G}} \ne \emptyset. \tag{3.5}$$

Here $S_{2\sqrt{G}}$ is the sphere in $(E, \|\cdot\|)$ centered at 0 and with radius $2\sqrt{G}$. To prove (3.5) it is sufficient to show

$$(h(B_{n+1})) \cap E_n^\perp \cap S_{2\sqrt{G}} \ne \emptyset, \tag{3.6}$$

since $\mathcal{D} \cap E_n^\perp \cap S_{2\sqrt{G}} = \emptyset$. To prove (3.6), we define $\mathcal{O}_1 = \{u \in B_{n+1} : \|h(u)\| < 2\sqrt{G}\}$, which is a bounded open neighbourhood of 0 in E_{n+1}. Let $\mathcal{O}$ be the connected component of $\mathcal{O}_1$ containing 0. Then the genus of $\partial\mathcal{O}$ is $\gamma(\partial\mathcal{O}) = n+1$; see [23] for the definition and properties of genus. If (3.6) is not true, then $h(\partial\mathcal{O}) \subset S_{2\sqrt{G}} \backslash E_n^\perp$. Let $P_n : E \to E_n$ be the orthogonal projection operator. The operator $Q_n : \partial\mathcal{O} \to E_n \setminus \{0\}$ defined by $Q_n(u) = P_n h(u)$ is odd and continuous, and then $\gamma(\partial\mathcal{O}) \le n$. But this is a contradiction. Equation (3.5) implies for any $h \in \Phi_{n+1}$,

$$\sup_{u \in h(B_{n+1}) \backslash \mathcal{D}} I(u) \ge \inf_{u \in E_n^\perp \cap S_{2\sqrt{G}}} I(u) \ge G.$$

This implies $b_k \ge G$ for $k \ge n+1$. Then the result follows. □

We are ready to prove the existence of a sequence of essential values of I with respect to $\mathcal{D}$ so that the sequence converges to ∞.

Lemma 3.4. *Define $\Lambda = \{ c \in \mathbb{R} : c$ is an essential value of I with respect to $\mathcal{D}\}$. Then $\Lambda \ne \emptyset$ and $\sup \Lambda = \infty$.*

Proof. If this statement were false then by Lemma 3.3 there would exist $k \in \mathbb{N}$ such that $0 < b_k < b_{k+1}$ and $[b_k, \infty) \cap \Lambda = \emptyset$. Choose real numbers d and a such that

$$b_k < d < a < b_{k+1}. \tag{3.7}$$

Let $h \in \Phi_k$ be such that

$$\sup_{u \in h(B_k) \backslash \mathcal{D}} I(u) < d.$$

For $k \in \mathbb{N}$, define $B_{k+1}^{+} = \{u : u = v + te_{k+1},\ v \in E_k,\ t \geq 0,\ \|u\| \leq R_{k+1}\}$ and let ∂B_{k+1}^{+} be the boundary of B_{k+1}^{+} in E_{k+1}. Extend h to a function $h_1 \in C(\partial B_{k+1}^{+}, X)$ as

$$h_1(u) = \begin{cases} h(u) & \text{if } u \in B_k, \\ u & \text{if } u \in \partial B_{k+1}^{+} \setminus B_k. \end{cases}$$

Clearly, h_1 is well defined and continuous and $h_1(\partial B_{k+1}^{+}) \subset \widehat{I}^d \cup \mathcal{D}$. Here and in what follows, for any $c \in \mathbb{R}$, $\widehat{I}^c$ is the level set of $\widehat{I} = I|_X$. That is $\widehat{I}^c = \{u \in X : I(u) \leq c\}$. Extend h_1 to a function $h_2 \in C(B_{k+1}^{+}, X)$ and let $b = \sup\{I(h_2(u)) : u \in B_{k+1}^{+}\}$. Then $h_2 \in C(B_{k+1}^{+}, \widehat{I}^b \cup \mathcal{D})$. As a consequence of the density of X in E, $\widehat{I}$ and I have the same essential values. By Theorem 2.3, the pair $(\widehat{I}^b, \widehat{I}^a)$ is trivial with respect to $\mathcal{D}$. So there exists $h_3 \in C(B_{k+1}^{+}, \widehat{I}^a \cup \mathcal{D})$ such that $h_3(x) = h_2(x)$ for all $x \in B_{k+1}^{+}$ with $h_2(x) \in \widehat{I}^d \cup \mathcal{D}$. Then h_3 satisfies

$$h_3(B_{k+1}^{+}) \subset \widehat{I}^a \cup \mathcal{D}, \tag{3.8}$$

$$h_3 \text{ is odd on } B_{k+1}^{+} \cap E_k, \tag{3.9}$$

$$h_3 = \mathrm{id} \text{ on } \partial B_{k+1}^{+} \cap \partial B_{k+1}, \tag{3.10}$$

$$h_3|_{B_k} = h. \tag{3.11}$$

Define $h_4 \in C(B_{k+1}, X)$ as

$$h_4(u) = \begin{cases} h_3(u) & \text{if } u \in B_{k+1}^{+}, \\ -h_3(-u) & \text{if } u \in B_{k+1} \setminus B_{k+1}^{+}. \end{cases}$$

Then, (3.9) implies that h_4 is odd, (3.10) implies $h_4|_{\partial B_{k+1}} = \mathrm{id}$, and (3.11) implies $h_4|_{B_k} \in \Phi_k$. So $h_4 \in \Phi_{k+1}$, which is a contradiction since by (3.7) and (3.8) we have

$$b_{k+1} \leq \sup_{u \in h_4(B_{k+1}) \setminus \mathcal{D}} I(u) = \sup_{u \in h_3(B_{k+1}^{+}) \setminus \mathcal{D}} I(u) \leq a < b_{k+1}.$$

Therefore $\Lambda \neq \emptyset$ and $\sup \Lambda = \infty$. □

According to Lemma 3.4, we can choose a strictly increasing sequence of positive numbers $\{d_k\} \subset \Lambda$ such that $d_1 > 2$ and $d_{k+1} > d_k + 3$ for all $k = 1, 2, \ldots$ Now we fix $j \in \mathbb{N}$ and want to find j nodal solutions for $(1.1)_\varepsilon$ for $|\varepsilon|$ small enough. For this we are going to prove that for any k if $|\varepsilon|$ is small enough then $I_{\varepsilon,k}$ has a critical point in $X \setminus \mathcal{D}$ with critical value in $(d_i - 1, d_i + 1)$ for all $i = 1, \ldots, j$ and then prove that these solutions have an L^∞ bound independent of k.

We need a series of lemmas.

Lemma 3.5. *For $\varepsilon_1(k)$ defined in* (3.3)*, there exists $\varepsilon_2(k) \in (0, \varepsilon_1(k))$ such that if $|\varepsilon| \leq \varepsilon_2(k)$, $u \in \partial\mathcal{D}$, and $I_{\varepsilon,k}(u) \in [d_1 - 1, d_j + 1]$, then $I'_{\varepsilon,k}(u) \neq 0$. Here $\partial\mathcal{D}$ is the boundary of $\mathcal{D}$ in X.*

Proof. If the result were not true, then there would exist ε_n for $n = 1, 2, \ldots$ with $|\varepsilon_n| \le \varepsilon_1(k)$ and $\varepsilon_n \to 0$ and $u_n \in \partial\mathcal{D}$ such that $I_{\varepsilon_n,k}(u_n) \in [d_1 - 1, d_j + 1]$ and $I'_{\varepsilon_n,k}(u_n) = 0$ for $n = 1, 2, \ldots$. Then u_n satisfy

$$d_1 - 1 \le \frac{1}{2}\int_\Omega |\nabla u_n|^2 - \int_\Omega [F(x, u_n) + \varepsilon_n G_k(x, u_n)] \le d_j + 1 \tag{3.12}$$

and

$$\begin{aligned} -\Delta u_n &= f(x,\ u_n) + \varepsilon_n g_k(x,\ u_n) && \text{in } \Omega, \\ u_n &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.13}$$

By (f_3), (3.3), (3.12) and (3.13), a standard calculation shows that $\{u_n\}$ is bounded in E. Then a result of Brézis and Kato [8] implies that $\{u_n\}$ is bounded in L^∞ and hence in $C^2(\overline{\Omega})$. Passing to a subsequence if necessary, we can assume that $u_n \to u$ in X. Taking the limit in (3.12) and (3.13) as $n \to \infty$ gives

$$d_1 - 1 \le \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(x, u) \le d_j + 1 \tag{3.14}$$

and

$$-\Delta u = f(x, u) \text{ in } \Omega \quad \text{and} \quad u = 0 \text{ on } \partial\Omega. \tag{3.15}$$

Since $u_n \in \partial\mathcal{D}$ and $u_n \to u$ in X, $u \in \partial\mathcal{D}$. The bound (3.14) with $d_1 > 2$ implies $u \ne 0$. Then using (3.1) and (3.12), the strong maximum principle yields $u \in \operatorname{int}(\mathcal{D})$, a contradiction. □

With respect to the inner product defined in (3.2), the gradient of $I_{\varepsilon,k}$ takes the form $I'_{\varepsilon,k}(u) = u - A_{\varepsilon,k}(u)$, where $A_{\varepsilon,k}\colon E \to E$ is defined by

$$A_{\varepsilon,k}(u) = (-\Delta + L^*)^{-1}(f(x, u) + \varepsilon g_k(x, u) + L^* u).$$

Clearly, $A_{\varepsilon,k}(X) \subset X$ and $A_{\varepsilon,k}$ is a C^1 operator from X to X. We also denote $A = A_{0,k}$. The strong maximum principle together with (3.1) implies that $A(\pm P \setminus \{0\}) \subset \operatorname{int}(\pm P)$.

Let $u \in E$ and consider the initial value problem in E

$$\begin{cases} \phi'(t) = -\phi(t) + A_{\varepsilon,k}(\phi(t)) & \text{for } t \ge 0, \\ \phi(0) = u. \end{cases} \tag{3.16}$$

This defines a descending flow of $I_{\varepsilon,k}$. The solution is denoted by $\phi_{\varepsilon,k}(t, u)$ with $[0, \eta_{\varepsilon,k}(u))$ the maximal interval of existence. It is well known that if $u \in X$ then $\phi_{\varepsilon,k}(t, u)$ stays in X for all $t \in [0, \eta_{\varepsilon,k}(u))$ and $\phi_{\varepsilon,k}(t, u)$ is continuously dependent on t and u with respect to the X norm. That is, for any $u \in X$, any $T \in [0, \eta_{\varepsilon,k}(u))$, and any $\delta > 0$, there exists $\delta_1 > 0$ such that if $v \in X$ and $\|v - u\|_X < \delta_1$ then $\phi_{\varepsilon,k}(t, v)$ exists for all $t \in [0, T]$ and in this interval

$\|\phi_{\varepsilon,k}(t,v)-\phi_{\varepsilon,k}(t,u)\|_X<\delta$. For $a<b$ two real numbers, G a subset of E, $k\in\mathbb{N}$, and $|\varepsilon|\le\varepsilon_2(k)$, we define a set

$$\mathcal{M}(a,b,G,k,\varepsilon)=\{\phi_{\varepsilon,k}(t,u): t\in[0,\eta_{\varepsilon,k}(u)), u\in G,\ \phi_{\varepsilon,k}(t,u)\in I^b_{\varepsilon,k}\setminus \operatorname{int}(I^a_{\varepsilon,k})\}.$$

We also define

$$\mathcal{M}(a,b,G,k)=\bigcup_{|\varepsilon|\le\varepsilon_2(k)}\mathcal{M}(a,b,G,k,\varepsilon).$$

Lemma 3.6. *If $a<b$ are two real numbers, $k\in\mathbb{N}$, and G is a bounded subset of E, then $\mathcal{M}(a,b,G,k)$ is a bounded subset of E.*

Proof. Since G is a bounded subset of E, we may assume that $G\subset I^b_{\varepsilon,k}$ for all $|\varepsilon|\le\varepsilon_2(k)$ by enlarging b if necessary. If the result were false, for each $n\in\mathbb{N}$ there would exist $|\varepsilon_n|\le\varepsilon_2(k)$, $u_n\in G$, and $t_n\in[0,\eta_{\varepsilon_n,k}(u_n))$ such that $\phi_{\varepsilon_n,k}(t_n,u_n)\in I^b_{\varepsilon_n,k}\setminus\operatorname{int}(I^a_{\varepsilon_n,k})$ and $\|\phi_{\varepsilon_n,k}(t_n,u_n)\|_*\ge 2n$. Since G is a bounded subset of $I^b_{\varepsilon_n,k}$, for each n large enough there exists $t'_n\in(0,t_n)$ such that $\|\phi_{\varepsilon_n,k}(t'_n,u_n)\|_*=n$ and

$$\|\phi_{\varepsilon_n,k}(t,u_n)\|_*>n \text{ for } t'_n<t<t_n. \tag{3.17}$$

Using (3.16), we estimate as follows:

$$\begin{aligned} n&\le\|\phi_{\varepsilon_n,k}(t_n,u_n)-\phi_{\varepsilon_n,k}(t'_n,u_n)\|_*\\ &\le\int_{t'_n}^{t_n}\|I'_{\varepsilon_n,k}(\phi_{\varepsilon_n,k}(s,u_n))\|_*\,ds\\ &\le\bigg(\int_{t'_n}^{t_n}\|I'_{\varepsilon_n,k}(\phi_{\varepsilon_n,k}(s,u_n))\|_*^2\,ds\bigg)^{1/2}(t_n-t'_n)^{1/2}\\ &\le(b-a)^{1/2}(t_n-t'_n)^{1/2}. \end{aligned}\tag{3.18}$$

Also from (3.16), there exists $t''_n\in(t'_n,t_n)$ such that

$$\begin{aligned}\frac{a-b}{t_n-t'_n}&\le\frac{I_{\varepsilon_n,k}(\phi_{\varepsilon_n,k}(t_n,u_n))-I_{\varepsilon_n,k}(\phi_{\varepsilon_n,k}(t'_n,u_n))}{t_n-t'_n}\\ &=-\|I'_{\varepsilon_n,k}(\phi_{\varepsilon_n,k}(t''_n,u_n))\|_*^2.\end{aligned}\tag{3.19}$$

As a consequence of (3.18) and (3.19) $I'_{\varepsilon_n,k}(\phi_{\varepsilon_n,k}(t''_n,u_n))\to0$ as $n\to\infty$. This together with $a\le I_{\varepsilon_n,k}(\phi_{\varepsilon_n,k}(t''_n,u_n))\le b$ implies that $\phi_{\varepsilon_n,k}(t''_n,u_n)$ is bounded in E. But this contradicts (3.17). □

Lemma 3.7. *If $a<b$ are two real numbers, $k\in\mathbb{N}$, and G is a compact subset of X, then $\operatorname{clos}_X\mathcal{M}(a,b,G,k)$ is a compact subset of X.*

Proof. As in [21] (cf. [6], [12], [13], [18], [20]), we can choose a finite sequence of Banach spaces $\{Z_i\}_{i=0}^n$ such that the embeddings $E\hookrightarrow Z_0$ and

$$X=Z_n\hookrightarrow Z_{n-1}\hookrightarrow\cdots\hookrightarrow Z_1\hookrightarrow Z_0$$

are continuous, $A_{\varepsilon,k}\colon Z_{i-1} \to Z_i$ is continuous and compact, and for any bounded subset M of Z_{i-1}, $A_{\varepsilon,k}(M)$ has a bound in Z_i independent of ε. Starting from the result of Lemma 3.6 and using the above fact and the expression

$$\phi_{\varepsilon,k}(t,u) = e^{-t}u + e^{-t}\int_0^t e^s A_{\varepsilon,k}(\phi_{\varepsilon,k}(s,u))\,ds$$

n times, it is easy to see that the set

$$\bigg\{ e^{-t}\int_0^t e^s A_{\varepsilon,k}(\phi_{\varepsilon,k}(s,u))\,ds : t \in [0, \eta_{\varepsilon,k}(u)),\ u \in G,\\ |\varepsilon| \le \varepsilon_2(k),\ \phi_{\varepsilon,k}(t,u) \in I^b_{\varepsilon,k} \setminus \operatorname{int}(I^a_{\varepsilon,k}) \bigg\}$$

is bounded in Z_n and hence is precompact in X. Thus $\operatorname{clos}_X \mathcal{M}(a,b,G,k)$ is a compact subset of X. □

Lemma 3.8. *Let $0 < a < b$ be two real numbers, $k \in \mathbb{N}$, and G a compact subset of X. There exists $\varepsilon_3(k) \in (0, \varepsilon_2(k))$ such that if $0 < t_1 < t_2$, $u \in G$, $|\varepsilon| \le \varepsilon_3(k)$, $\phi_{\varepsilon,k}(t_1,u) \in \partial\mathcal{D}$, and $\phi_{\varepsilon,k}(t_2,u) \in I^b_{\varepsilon,k} \setminus I^a_{\varepsilon,k}$, then $\phi_{\varepsilon,k}(t_2,u) \in \operatorname{int}(\mathcal{D})$.*

Proof. By Lemma 3.7, $\operatorname{clos}_X \mathcal{M}(a,b,G,k)$ is a compact subset of X. Since $A\big(\operatorname{clos}_X \mathcal{M}(a,b,G,k) \cap \partial(\pm P)\big)$ is a compact subset of $\operatorname{int}(\pm P)$, there exists $\varepsilon_3(k) \in (0, \varepsilon_2(k))$ such that if $|\varepsilon| \le \varepsilon_3(k)$ then

$$A_{\varepsilon,k}(\operatorname{clos}_X \mathcal{M}(a,b,G,k) \cap \partial(\pm P)) \subset \operatorname{int}(\pm P).$$

Thus if $0 < t_1$, $u \in G$, $|\varepsilon| \le \varepsilon_3(k)$, and $\phi_{\varepsilon,k}(t_1,u) \in (I^b_{\varepsilon,k} \setminus I^a_{\varepsilon,k}) \cap \partial P$, then there exists $\delta > 0$ such that $A_{\varepsilon,k}(\phi_{\varepsilon,k}(t,u)) \in \operatorname{int}(P)$ for $t_1 \le t \le t_1 + \delta$ and thus by the convexity of $\operatorname{int}(P)$,

$$\xi_{\varepsilon,k}(t,u) := \frac{1}{e^t - e^{t_1}} \int_{t_1}^t e^s A_{\varepsilon,k}(\phi_{\varepsilon,k}(s,u))\,ds \in \operatorname{int}(P)$$

for $t_1 \le t \le t_1 + \delta$. This together with

$$\phi_{\varepsilon,k}(t,u) = e^{-(t-t_1)}\phi_{\varepsilon,k}(t_1,u) + (1 - e^{-(t-t_1)})\xi_{\varepsilon,k}(t,u)$$

implies $\phi_{\varepsilon,k}(t,u) \in \operatorname{int}(P)$ for $t_1 < t \le t_1 + \delta$. The same argument is valid if P is replaced with $-P$.

Now assume that $0 < t_1 < t_2$, $u \in G$, $|\varepsilon| \le \varepsilon_3(k)$, $\phi_{\varepsilon,k}(t_1,u) \in \partial\mathcal{D}$, and $\phi_{\varepsilon,k}(t_2,u) \in I^b_{\varepsilon,k} \setminus I^a_{\varepsilon,k}$. If $\phi_{\varepsilon,k}(t_2,u) \notin \operatorname{int}(\mathcal{D})$, then according to the above discussion we may assume that $\phi_{\varepsilon,k}(t_2,u) \in \partial\mathcal{D}$ and $\phi_{\varepsilon,k}(t,u) \in \big(I^b_{\varepsilon,k} \setminus I^a_{\varepsilon,k}\big) \cap \operatorname{int}(\mathcal{D})$ for $t \in (t_1,t_2)$. But then $A_{\varepsilon,k}(\phi_{\varepsilon,k}(t,u)) \in \operatorname{int}(\mathcal{D})$ and thus $\xi_{\varepsilon,k}(t,u) \in \operatorname{int}(\mathcal{D})$ for $t \in [t_1,t_2]$. This implies $\phi_{\varepsilon,k}(t,u) \in \operatorname{int}(\mathcal{D})$ for $t \in (t_1,t_2]$, a contradiction. □

LEMMA 3.9. *Let a and b be two real numbers with $d_1 - 1 < a < b < d_j + 1$, $k \in \mathbb{N}$, $|\varepsilon| \le \varepsilon_3(k)$, and G a compact subset of X. If $I_{\varepsilon,k}$ has no critical point in $\widehat{I}^b_{\varepsilon,k} \setminus (\mathrm{int}(\widehat{I}^a_{\varepsilon,k}) \cup \mathcal{D})$ then there exists $\sigma = \sigma(a,b,G,k,\varepsilon)$ such that $\|I'_{\varepsilon,k}(\phi_{\varepsilon,k}(t,u))\|_* \ge \sigma$ for all $\phi_{\varepsilon,k}(t,u) \in \mathcal{M}(a,b,G,k,\varepsilon)$.*

PROOF. First recall that $\widehat{I} = I|_X$. If not, then for each $n \in \mathbb{N}$ there would exist $u_n \in G$ and $t_n \in [0, \eta_{\varepsilon,k}(u_n))$ with $\phi_{\varepsilon,k}(t_n,u_n) \in \widehat{I}^b_{\varepsilon,k} \setminus \big(\mathrm{int}(\widehat{I}^a_{\varepsilon,k}) \cup \mathcal{D}\big)$ such that $I'_{\varepsilon,k}(\phi_{\varepsilon,k}(t_n,u_n)) \to 0$ as $n \to \infty$. By Lemma 3.7, we can assume that $\phi_{\varepsilon,k}(t_n,u_n) \to u^*$ in E as $n \to \infty$ for some $u^* \in X$ which satisfies $I'_{\varepsilon,k}(u^*) = 0$. Since $\phi_{\varepsilon,k}(t_n,u_n) \notin \mathcal{D}$ the convergence of $\phi_{\varepsilon,k}(t_n,u_n)$ in X implies $u^* \notin \mathrm{int}(\mathcal{D})$. Then by Lemma 3.5, $u^* \notin \mathcal{D}$. Thus u^* is a critical point of $I_{\varepsilon,k}$ in $\widehat{I}^b_{\varepsilon,k} \setminus (\mathrm{int}(\widehat{I}^a_{\varepsilon,k}) \cup \mathcal{D})$, which contradicts the condition of the lemma. □

LEMMA 3.10. *For any $k \in \mathbb{N}$, there exists $\varepsilon_4(k) \in (0, \varepsilon_3(k))$ such that if $|\varepsilon| \le \varepsilon_4(k)$ then for any $i \in \{1,\ldots,j\}$, $I_{\varepsilon,k}$ has at least one critical point in $\widehat{I}^{d_i+1}_{\varepsilon,k} \setminus (\mathrm{int}(\widehat{I}^{d_i-1}_{\varepsilon,k}) \cup \mathcal{D})$.*

PROOF. If the result were not true, then for any $n \in \mathbb{N}$ there would exist ε_n with $\varepsilon_n \to 0$ such that $I_{\varepsilon_n,k}$ has no critical value in $\widehat{I}^{d_i+1}_{\varepsilon,k} \setminus (\mathrm{int}(\widehat{I}^{d_i-1}_{\varepsilon,k}) \cup \mathcal{D})$ for some $i \in \{1,\ldots,j\}$. Fix such an i. In order to obtain a contradiction, it suffices to show that d_i is not an essential value of I. For this we assume that a, b are any numbers satisfying $d_i - 1 < a < b < d_i + 1$, δ is a number with $\delta \in (0, a - d_i + 1)$, Y is a compact topological space, and $h \in C(Y, \widehat{I}^b \cup \mathcal{D})$. Choose a_1, b_1 such that

$$d_i - 1 < a - \delta < a_1 < a < b < b_1 < d_i + 1.$$

We fix an ε_n such that $|\varepsilon_n| \le \varepsilon_3(k)$ and

$$\widehat{I}^{a-\delta} \subset \widehat{I}^{a_1}_{\varepsilon_n,k} \subset \widehat{I}^a \subset \widehat{I}^b \subset \widehat{I}^{b_1}_{\varepsilon_n,k} \subset \widehat{I}^{d_i+1}.$$

For any $u \in \widehat{I}^{b_1}_{\varepsilon_n,k} \cap h(Y) \setminus (\widehat{I}^{a_1}_{\varepsilon_n,k} \cup \mathcal{D})$, consider $\phi_{\varepsilon_n,k}(t,u)$. Let $0 < t_1 < t_2 < \eta_{\varepsilon_n,k}(u)$. If $\phi_{\varepsilon_n,k}(t,u) \in \widehat{I}^{b_1}_{\varepsilon_n,k} \setminus (\widehat{I}^{a_1}_{\varepsilon_n,k} \cup \mathcal{D})$ for $t \in [t_1, t_2]$, then by (3.16) and Lemma 3.9, a simple computation shows that

$$\|\phi_{\varepsilon_n,k}(t_2,u) - \phi_{\varepsilon_n,k}(t_1,u)\|_* \le \frac{1}{\sigma}(I_{\varepsilon_n,k}(\phi_{\varepsilon_n,k}(t_1,u)) - I_{\varepsilon_n,k}(\phi_{\varepsilon_n,k}(t_2,u)))$$

and

$$t_2 - t_1 \le \frac{1}{\sigma^2}(I_{\varepsilon_n,k}(\phi_{\varepsilon_n,k}(t_1,u)) - I_{\varepsilon_n,k}(\phi_{\varepsilon_n,k}(t_2,u))).$$

From these inequalities we see that if $\phi_{\varepsilon_n,k}(t,u) \in \widehat{I}^{b_1}_{\varepsilon_n,k} \setminus (\widehat{I}^{a_1}_{\varepsilon_n,k} \cup \mathcal{D})$ for all $t \in [0, \eta_{\varepsilon_n,k}(u))$ then the limit $\lim_{t \to \eta_{\varepsilon_n,k}(u)^-} \phi_{\varepsilon_n,k}(t,u)$ exists in E and $\phi_{\varepsilon_n,k}(t,u)$ can be extended beyond $\eta_{\varepsilon_n,k}(u)$, which is a contradiction to the maximality of $[0, \eta_{\varepsilon_n,k}(u))$. Thus $\phi_{\varepsilon_n,k}(t,u)$ must reach $\widehat{I}^{a_1}_{\varepsilon_n,k} \cup \mathcal{D}$ at some $t \in [0, \eta_{\varepsilon_n,k}(u))$. Then as a consequence of Lemma 3.8, for any $u \in \widehat{I}^{b_1}_{\varepsilon_n,k} \cap h(Y) \setminus (\widehat{I}^{a_1}_{\varepsilon_n,k} \cup \mathcal{D})$, there exists $\tau_{\varepsilon_n,k}(u) > 0$ such that $\phi_{\varepsilon_n,k}(t,u) \in \widehat{I}^{b_1}_{\varepsilon_n,k} \setminus (\widehat{I}^{a_1}_{\varepsilon_n,k} \cup \mathcal{D})$ for $0 \le t < \tau_{\varepsilon_n,k}(u)$,

$\phi_{\varepsilon_n,k}(t,u) \in \partial(\widehat{I}^{a_1}_{\varepsilon_n,k} \cup \mathcal{D})$ for $t = \tau_{\varepsilon_n,k}(u)$, and $\phi_{\varepsilon_n,k}(t,u) \in \operatorname{int}(\widehat{I}^{a_1}_{\varepsilon_n,k} \cup \mathcal{D})$ for $\tau_{\varepsilon_n,k}(u) < t < \eta_{\varepsilon_n,k}(u)$. Here ∂ means the boundary in X. Since $\phi_{\varepsilon_n,k}(t,u)$ is continuously dependent on t and u, $\tau_{\varepsilon_n,k}(u)$ is continuous in u. Define $\widehat{h}\colon Y \to \widehat{I}^{a_1}_{\varepsilon_n,k} \cup \mathcal{D}$ by

$$\widehat{h}(x) = \begin{cases} h(x) & \text{if } x \in Y \text{ and } h(x) \in \widehat{I}^{a_1}_{\varepsilon_n,k} \cup \mathcal{D}, \\ \phi_{\varepsilon_n,k}(\tau_{\varepsilon_n,k}(u),u) & \text{if } x \in Y \text{ and } u := h(x) \in \widehat{I}^{b_1}_{\varepsilon_n,k} \setminus (\widehat{I}^{a_1}_{\varepsilon_n,k} \cup \mathcal{D}). \end{cases}$$

Then $\widehat{h} \in C(Y, \widehat{I}^a \cup \mathcal{D})$ and $\widehat{h}(x) = h(x)$ for any $x \in Y$ with $h(x) \in \widehat{I}^{a-\delta} \cup \mathcal{D}$. Thus d_i is not an essential value of I, a contradiction. $\square$

Since the positive cone and the negative cone in E have no interior, it can not be expected to construct a similar deformation which is continuous with respect to the E norm. It is natural to use the space X here.

COMPLETING THE PROOF OF THEOREM 1.1. For each $k \in \mathbb{N}$ and $|\varepsilon| \leq \varepsilon_4(k)$, according to Lemma 3.10, there are j critical points $u_{\varepsilon,k,i}$ $(i = 1, ldots, j)$ of $I_{\varepsilon,k}$ in $X \setminus \mathcal{D}$ such that

$$\begin{gathered} -\Delta u_{\varepsilon,k,i} = f(x, u_{\varepsilon,k,i}) + \varepsilon g_k(x, u_{\varepsilon,k,i}) \quad \text{in } \Omega, \quad u_{\varepsilon,k,i}|_{\partial\Omega} = 0, \\ d_i - 1 \leq \frac{1}{2}\int_\Omega |\nabla u_{\varepsilon,k,i}|^2\,dx - \int_\Omega [F(x, u_{\varepsilon,k,i}) + \varepsilon G_k(x, u_{\varepsilon,k,i})]\,dx \leq d_i + 1. \end{gathered}$$

Then it is easy to see that $\int_\Omega |\nabla u_{\varepsilon,k,i}|^2\,dx \leq C'_j$, where C'_j is a constant independent of ε and k. A result of Brézis and Kato [8] implies that there exists a constant C''_j independent of ε and k such that $\|u_{\varepsilon,k,i}\|_{C(\overline{\Omega})} \leq C''_j$, for every $k \in \mathbb{N}$, $|\varepsilon| \leq \varepsilon_4(k)$, and $i = 1, \ldots, j$. So if $k > C''_j$ then for any ε with $|\varepsilon| \leq \varepsilon_4(k)$, $(1.1)_\varepsilon$ possesses j distinct nodal solutions $u_{\varepsilon,k,1},\ u_{\varepsilon,k,2}, \ldots, u_{\varepsilon,k,j}$. The proof is finished. $\square$

4. Proof of Theorem 1.3

Now we assume that $f(x,t)$ satisfies (f'_0)–(f'_3). Choose numbers $a > 0$, $0 < q < 1$, and $\beta > 0$ such that $q < 1/(N-2)$ if $N \geq 4$, $2a < \delta$, and $F(x,t) > \beta|t|^{q+1}$ for $a \leq |t| \leq 2a$. Choose an even function $\rho \in C^1(\mathbb{R}, \mathbb{R})$ such that $\rho(t) = 1$ for $|t| \leq a$, $\rho(t) = 0$ for $|t| \geq 2a$, and $t\rho'(t) < 0$ for $a < |t| < 2a$. Define

$$\widetilde{F}(x,t) = \rho(t)F(x,t) + (1 - \rho(t))\beta|t|^{q+1}$$

and $\widetilde{f}(x,t) = (\partial/\partial t)\widetilde{F}(x,t)$. As in [19] we have the following lemma which is a slight modification of [25, Lemma 2.3].

LEMMA 4.1. *Assume that* $f(x,t)$ *satisfies* (f_0')–(f_3'). *Then* $\widetilde{f} \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ *satisfies*

$$\widetilde{f}(x,-t) = -\widetilde{f}(x,t) \quad \text{for all } x \in \Omega \text{ and } t \in \mathbb{R}, \tag{4.1}$$

$$\widetilde{f}(x,t) = f(x,t) \quad \text{for all } x \in \Omega \text{ and } |t| \le a, \tag{4.2}$$

$$2\widetilde{F}(x,t) > t\widetilde{f}(x,t) > 0 \quad \text{for all } x \in \Omega \text{ and } t \neq 0, \tag{4.3}$$

where $\widetilde{F}(x,t) = \int_0^t \widetilde{f}(x,s)\,ds$.

Set $\widetilde{G}(x,t) = \rho(t)G(x,t)$ and $\widetilde{g}(x,t) = (\partial/\partial t)\widetilde{G}(x,t)$, and define

$$\widetilde{I}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega \widetilde{F}(x,u), \quad u \in E$$

and

$$\widetilde{I}_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega [\widetilde{F}(x,u) + \varepsilon\widetilde{G}(x,u)], \quad u \in E.$$

By (4.1), $\widetilde{I}$ is an even functional. As a well known result, we have

LEMMA 4.2. *The functionals* $\widetilde{I}$ *and* $\widetilde{I}_\varepsilon$ *are in* C^1, $\widetilde{I}_\varepsilon$ *with* $|\varepsilon| \le 1$ *have a uniform lower bound, and* $\lim_{\|u\|\to\infty} \widetilde{I}_\varepsilon(u) = \infty$ *uniformly in* $|\varepsilon| \le 1$.

Denote $O = \{u \in X : \widetilde{I}(u) < 0\}$. As in [19], from Lemma 4.1 we can deduce that O is contractible. In [19], O is defined to be the set of points in E at which $\widetilde{I} < 0$. But the same argument can be used here. For $k \in \mathbb{N}$, let S^{k-1} be the unit sphere in $\mathbb{R}^k$ and define $\Phi_k = \{h \in C(S^{k-1}, O) : h \text{ is odd}\}$ and

$$b_k = \inf_{h\in\Phi_k} \sup_{u \in h(S^{k-1})\setminus\mathcal{D}} \widetilde{I}(u).$$

Then (f_2') and (4.2) yield $b_1 \le b_2 \le \ldots \le b_k \le \ldots < 0$.

LEMMA 4.3. $b_k \to 0$.

PROOF. Denote by Σ the class of closed symmetric subsets of $X\setminus\{0\}$. Define $\gamma_k = \{A \in \Sigma : \gamma(A) \ge k\}$ and

$$c_k = \inf_{A\in\gamma_k} \sup_{u\in A} \widetilde{I}(u).$$

To have a comparison between b_k and c_k, we observe that $\gamma(S^{k-1} \cap h^{-1}(\mathcal{D})) = 1$ for any $h \in \Phi_k$; this genus is easily computed through the odd and continuous map $\nu\colon S^{k-1} \cap h^{-1}(\mathcal{D}) \to S^0$ defined by

$$\nu(x) = \begin{cases} 1 & \text{if } x \in S^{k-1} \cap h^{-1}(P), \\ -1 & \text{if } x \in S^{k-1} \cap h^{-1}(-P). \end{cases}$$

Then for any $h \in \Phi_k$,

$$\gamma(h(S^{k-1}) \setminus \mathcal{D}) = \gamma(h(S^{k-1} \setminus h^{-1}(\mathcal{D}))) \ge \gamma(S^{k-1}) - \gamma(S^{k-1} \cap h^{-1}(\mathcal{D})) = k-1.$$

Then $\{h(S^{k-1}) \setminus \mathcal{D} : h \in \Phi_k\} \subset \gamma_{k-1}$ and $b_k \geq c_{k-1}$. By the proof of [25, Lemma 2.4], $c_k \to 0$ as $k \to \infty$. So $b_k \to 0$ as $k \to \infty$. □

The sequence $\{b_k\}$ is used to assist in proving the existence of a sequence of essential values of I with respect to $\mathcal{D}$ which is increasing and converges to 0.

Lemma 4.4. *Define $\Lambda = \{c < 0 : c$ is an essential value of $\widetilde{I}$ with respect to $\mathcal{D}\}$. Then $\Lambda \neq \emptyset$ and $\sup \Lambda = 0$.*

Proof. If the result were false then by Lemma 4.3 there would exist $k \in \mathbb{N}$ such that $b_k < b_{k+1}$ and $[b_k, 0) \cap \Lambda = \emptyset$. Choose real numbers d and a such that

$$b_k < d < a < b_{k+1}.$$

Let $h \in \Phi_k$ be such that

$$\sup_{u \in h(S^{k-1}) \setminus \mathcal{D}} \widetilde{I}(u) < d.$$

For $k \in \mathbb{N}$, define $S^k_+ = \{x : x = (x', x_{k+1}),\ x' \in \mathbb{R}^k,\ x_{k+1} \geq 0,\ |x| = 1\}$. Since O is contractible, we can extend h to $h_1 \in C(S^k_+, O)$. Define $b = \sup\{\widetilde{I}(h_1(x)) : x \in S^k_+\}$. Then $b < 0$ and $h_1 \in C(S^k_+, \widetilde{I}^b \cup \mathcal{D})$. By Theorem 2.3, the pair $(\widetilde{I}^b, \widetilde{I}^a)$ is trivial with respect to $\mathcal{D}$. Then there exists $h_2 \in C(S^k_+, \widetilde{I}^a \cup \mathcal{D})$ such that $h_2(x) = h_1(x)$ for all $x \in S^k_+$ with $h_1(x) \in \widetilde{I}^d \cup \mathcal{D}$. Then $h_2(x)$ satisfies

$$h_2(S^k_+) \subset (\widetilde{I}^a \cup \mathcal{D}) \cap O, \quad h_2|_{S^{k-1}} \text{ is odd.}$$

So we can extend h_2 to an odd map $h_3 \in C(S^k, O)$ satisfying $h_3(S^k) \subset \widetilde{I}^a \cup \mathcal{D}$, which leads to

$$b_{k+1} \leq \sup_{u \in h_3(S^k) \setminus \mathcal{D}} \widetilde{I}(u) = \sup_{u \in h_2(S^k_+) \setminus \mathcal{D}} \widetilde{I}(u) \leq a < b_{k+1},$$

a contradiction. Therefore $\Lambda \neq \emptyset$ and $\sup \Lambda = 0$. □

By Lemma 4.4, we can choose a strictly increasing sequence $\{d_k\} \subset \Lambda$ such that $d_k \to 0$. For any k, define $\delta_k = (1/3) \min_{1 \leq i \leq k}(d_{i+1} - d_i)$. We are going to prove that for any k if $|\varepsilon|$ is small enough then I_ε has a critical point $u_{\varepsilon,i}$ in $X \setminus \mathcal{D}$ with critical value in $(d_i - \delta_k, d_i + \delta_k)$ for all $i = 1, \ldots, k$ and then prove that for k large, the j critical points $u_{\varepsilon,k-j+1}$, $u_{\varepsilon,k-j+2}$, $\ldots$, $u_{\varepsilon,k}$ of $\widetilde{I}_\varepsilon$ have L^∞ norms less than a and thus are nodal solutions of $(1.1)_\varepsilon$. For this we need to study a descending flow of $\widetilde{I}_\varepsilon$.

With respect to the usual norm $\|\cdot\|$, the gradient of $\widetilde{I}_\varepsilon$ takes the form $\widetilde{I}'_\varepsilon(u) = u - A_\varepsilon(u)$, where $A_\varepsilon \colon E \to E$ is defined by

$$A_\varepsilon(u) = (-\Delta)^{-1}(\widetilde{f}(x,u) + \varepsilon \widetilde{g}(x,u)).$$

By the construction of $\widetilde{f}$ and $\widetilde{g}$, there exists $C > 0$ such that for all $x \in \Omega$, $t \in \mathbb{R}$, and $|\varepsilon| \leq 1$,

$$|\widetilde{f}(x,t) + \varepsilon \widetilde{g}(x,t)| \leq C(1 + |t|^q).$$

Since $q < 1/(N-2)$ if $N \geq 4$, it is easy to see that $A_\varepsilon(E) \subset X$ and A_ε is continuous and compact from E to X. For any real number c, Lemma 4.2 implies that $\bigcup_{|\varepsilon|\leq 1} \widetilde{I}_\varepsilon^c$ is a bounded subset of E and thus $\mathrm{clos}_X(\bigcup_{|\varepsilon|\leq 1} A_\varepsilon(\widetilde{I}_\varepsilon^c))$ is a compact subset of X. Denote $A = A_0$. The strong maximum principle together with (4.3) implies that $A(\pm P\backslash\{0\}) \subset \mathrm{int}(\pm P)$. Define $K_\varepsilon = \{u \in E : \widetilde{I}'_\varepsilon(u) = 0\}$. Then $K_\varepsilon \subset X$. Since in this case $\widetilde{I}_\varepsilon$ is only C^1, to study a descending flow of $\widetilde{I}_\varepsilon$ we have to construct a pseudo-gradient vector field.

LEMMA 4.5. *For any $d < 0$, there exists $\varepsilon_1 = \varepsilon_1(d) \in (0,1)$ such that for each ε with $|\varepsilon| \leq \varepsilon_1$, there exists an operator $B_\varepsilon : E \to X$ such that*

(a) $B_\varepsilon|_{E\setminus K_\varepsilon}$ *is locally Lipschitz continuous from $E \setminus K_\varepsilon$ to X;*
(b) $B_\varepsilon(\pm P \cap \widetilde{I}_\varepsilon^d) \subset \mathrm{int}(\pm P)$;
(c) $\mathrm{clos}_X\big(\bigcup_{|\varepsilon|\leq\varepsilon_1} B_\varepsilon(\widetilde{I}_\varepsilon^0)\big)$ *is compact in X;*
(d) $(\widetilde{I}'_\varepsilon(u), u - B_\varepsilon(u)) \geq (1/2)\|\widetilde{I}'_\varepsilon(u)\|^2$ *for any $u \in E$;*
(e) $\|u - B_\varepsilon(u)\| \leq 2\|\widetilde{I}'_\varepsilon(u)\|$ *for any $u \in E$.*

PROOF. For any $d < 0$, choose $\varepsilon^* = \varepsilon^*(d) \in (0,1)$ such that for the set M defined by $M = \bigcup_{|\varepsilon|\leq\varepsilon^*} \widetilde{I}_\varepsilon^d$,

$$2\delta := \mathrm{dist}_E(0, M) > 0.$$

For any $u \in E \setminus K_\varepsilon$, let $r = r(u) \in (0,\delta)$ be such that

$$\|A_\varepsilon(v) - A_\varepsilon(w)\| < \frac{1}{2}\min\{\|v - A_\varepsilon(v)\|, \|w - A_\varepsilon(w)\|\} \tag{4.4}$$

for all v and w in $B_r(u)$ and that $B_r(u)$ does not intersect P and $-P$ simultaneously, where $B_r(u)$ is the ball in $(E, \|\cdot\|)$ centered at u and with radius r. Let $\{U_\lambda : \lambda \in \Lambda\}$ be a locally finite open refinement of the covering $\{B_r(u) : u \in E \setminus K_\varepsilon\}$ of $E \setminus K_\varepsilon$. For any $\lambda \in \Lambda$, choose $u_\lambda \in U_\lambda$ such that $u_\lambda \in U_\lambda \cap P$ if $U_\lambda \cap P \neq \emptyset$ and $u_\lambda \in U_\lambda \cap (-P)$ if $U_\lambda \cap (-P) \neq \emptyset$. This is possible since for any $\lambda \in \Lambda$ there exists $u_\lambda \in E \setminus K_\varepsilon$ such that $U_\lambda \subset B_r(u_\lambda)$ and $B_r(u_\lambda)$ does not intersect P and $-P$ simultaneously. Let π_λ, $\lambda \in \Lambda$, be a locally finite partition of unity subordinated to $\{U_\lambda : \lambda \in \Lambda\}$ such that the $\pi_\lambda : E \setminus K_\varepsilon \to \mathbb{R}$ are Lipschitz continuous. Define $B_\varepsilon : E \to X$ by

$$B_\varepsilon(u) = \begin{cases} \sum_{\lambda\in\Lambda} \pi_\lambda(u) A_\varepsilon(u_\lambda) & \text{if } u \in E \setminus K_\varepsilon, \\ u & \text{if } u \in K_\varepsilon. \end{cases}$$

Since $2\delta = \mathrm{dist}_E(0, M)$, $A(\mathrm{clos}_E(N_\delta(M) \cap (\pm P)))$ is a compact subset of $\mathrm{int}(\pm P)$. Here $N_\delta(M)$ is the δ-neighbourhood of M in E. Then there exists $\varepsilon_1 = \varepsilon_1(d) \in (0, \varepsilon^*(d))$ such that if $|\varepsilon| \leq \varepsilon_1$ then $A_\varepsilon(\mathrm{clos}_E(N_\delta(M) \cap (\pm P))) \subset \mathrm{int}(\pm P)$. In what follows fix an ε with $|\varepsilon| \leq \varepsilon_1$. For $u \in (P \cap \widetilde{I}_\varepsilon^d) \setminus K_\varepsilon$, if $\pi_\lambda(u) \neq 0$ then

$u \in U_\lambda \cap P$, which implies $U_\lambda \cap P \neq \emptyset$ and $u_\lambda \in U_\lambda \cap P$. Since $u \in M$ and $\|u_\lambda - u\| < \delta$, $u_\lambda \in N_\delta(M) \cap P$. Therefore,

$$B_\varepsilon(u) = \sum_{\lambda \in \Lambda} \pi_\lambda(u) A_\varepsilon(u_\lambda) \in \operatorname{int}(P).$$

For $u \in P \cap \widetilde{I}_\varepsilon^d \cap K_\varepsilon$, $B_\varepsilon(u) = u \in \operatorname{int}(P)$. Thus $B_\varepsilon(u) \in \operatorname{int}(P)$ for $u \in P \cap \widetilde{I}_\varepsilon^d$. Similarly, $B_\varepsilon(u) \in \operatorname{int}(-P)$ for $u \in -P \cap \widetilde{I}_\varepsilon^d$, finishing the proof of (b). Result (c) follows from the same property associated with A_ε. For any $u \in E$, (4.4) implies that

$$\|A_\varepsilon(u) - B_\varepsilon(u)\| \leq \frac{1}{2} \|\widetilde{I}'_\varepsilon(u)\|,$$

which leads to (d) and (e). □

Let $u \in E \setminus K_\varepsilon$ and consider the initial value problem in $E \setminus K_\varepsilon$

$$\begin{cases} \phi'(t) = -\phi(t) + B_\varepsilon(\phi(t)) & \text{for } t \geq 0, \\ \phi(0) = u. \end{cases}$$

The solution, which exists uniquely as a consequence of Lemma 4.5(a), is denoted by $\phi_\varepsilon(t, u)$ with $[0, \eta_\varepsilon(u))$ the maximal interval of existence. Note that if $u \in X$ then $\phi_\varepsilon(t, u)$ stays in X for all $t \in [0, \eta_\varepsilon(u))$ and $\phi_\varepsilon(t, u)$ is continuously dependent on t and u with respect to the X norm. By Lemma 4.5(b), as in the proof of Lemma 3.8, we can prove the following lemma.

LEMMA 4.6. *Assume $d < 0$ and $|\varepsilon| \leq \varepsilon_1(d)$. If $0 < t_1 < t_2 < \eta_\varepsilon(u)$ and $\phi_\varepsilon(t_1, u) \in \partial\mathcal{D} \cap \widetilde{I}_\varepsilon^d$ then $\phi_\varepsilon(t_2, u) \in \operatorname{int}(\mathcal{D})$.*

By Lemma 4.5(b) and (c), as in the proof of Lemma 3.9, we can prove the following lemma.

LEMMA 4.7. *Let a, b be two real numbers with $a < b < 0$ and $|\varepsilon| \leq \varepsilon_1(d)$. If $\widetilde{I}_\varepsilon$ has no critical point in $\widetilde{I}_\varepsilon^b \setminus (\operatorname{int}(\widetilde{I}_\varepsilon^a) \cup \mathcal{D})$ then there exists $\sigma = \sigma(a, b, \varepsilon)$ such that $\|\widetilde{I}'_\varepsilon(u)\| \geq \sigma$ for all $u \in \widetilde{I}_\varepsilon^b \setminus (\operatorname{int}(\widetilde{I}_\varepsilon^a) \cup \mathcal{D})$.*

With the help of Lemma 4.5(d) and (e), Lemmas 4.6 and Lemma 4.7, in a similar way as in the proof of Lemma 3.10, we obtain the next lemma.

LEMMA 4.8. *For any $k \in \mathbb{N}$, there exists $\varepsilon_2(k) \in (0, \min\{\varepsilon_1(d_k), 1/k\})$ such that if $|\varepsilon| \leq \varepsilon_2(k)$ then for any $i \in \{1, \ldots, k\}$, $\widetilde{I}_\varepsilon$ has at least one critical point in $\widetilde{I}_\varepsilon^{d_i + \delta_k} \setminus (\operatorname{int}(\widetilde{I}_\varepsilon^{d_i - \delta_k}) \cup \mathcal{D})$.*

Now we are ready to complete the proof of Theorem 1.3.

COMPLETING THE PROOF OF THEOREM 1.3. For $k \in \mathbb{N}$ and $|\varepsilon| \leq \varepsilon_2(k)$, by Lemma 4.8, there exists k critical points $u_{\varepsilon,1}, \ldots, u_{\varepsilon,k}$ of $\widetilde{I}_\varepsilon$ in $X \setminus \mathcal{D}$. These critical points satisfy

$$d_i - \delta_k \leq \widetilde{I}_\varepsilon(u_{\varepsilon,i}) \leq d_i + \delta_k.$$

By the same argument as [19, Theorem 2], there exists $k \in \mathbb{N}$ such that $u_{\varepsilon,k-j+1}$, $u_{\varepsilon,k-j+2}, \dots, u_{\varepsilon,k}$ have L^∞ norms less than a for any $|\varepsilon| \le \varepsilon_2(k)$. So they are j distinct nodal solutions of $(1.1)_\varepsilon$ with negative critical values. □

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