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SPECTRAL PROPERTIES AND NODAL SOLUTIONS FOR SECOND-ORDER, m-POINT, p-LAPLACIAN BOUNDARY VALUE PROBLEMS

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Abstract. We consider the m-point boundary value problem consisting of the equation

$$-\phi_p(u')' = f(u), \quad \text{on } (0,1), \tag{1}$$

together with the boundary conditions

$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \tag{2}$$

where $p > 1$, $\phi_p(s) := |s|^{p-1}\operatorname{sgn} s$, $s \in \mathbb{R}$, $m \geq 3$, $\alpha_i, \eta_i \in (0,1)$, for $i = 1, \ldots, m-2$, and $\sum_{i=1}^{m-2} \alpha_i < 1$. We assume that the function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous, satisfies $sf(s) > 0$ for $s \in \mathbb{R} \setminus \{0\}$, and that $f_0 := \lim_{\xi\to 0} f(\xi)/\phi_p(\xi) > 0$.

Closely related to the problem (1), (2), is the spectral problem consisting of the equation

$$-\phi_p(u')' = \lambda\phi_p(u), \tag{3}$$

together with the boundary conditions (2). It will be shown that the spectral properties of (2), (3), are similar to those of the standard Sturm–Liouville problem with separated (2-point) boundary conditions (with a minor modification to deal with the multi-point boundary condition). The topological degree of a related operator is also obtained. These spectral and degree theoretic results are then used to prove a Rabinowitz-type global bifurcation theorem for a bifurcation problem related to the problem (1), (2). Finally, we use the global bifurcation theorem to obtain nodal solutions of (1), (2), under various conditions on the asymptotic behaviour of f.

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1. Introduction

We consider the m-point boundary value problem consisting of the equation

$$-\phi_p(u')' = f(u), \quad \text{on } (0,1), \tag{1.1}$$

together with the boundary conditions

$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \tag{1.2}$$

where $p > 1$, $\phi_p(s) := |s|^{p-1}\operatorname{sgn} s$, $s \in \mathbb{R}$, $m \geq 3$, $\alpha_i, \eta_i \in (0,1)$, $i = 1, \ldots, m-2$, and

$$\sum_{i=1}^{m-2} \alpha_i < 1. \tag{1.3}$$

We assume that the function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous, satisfies $sf(s) > 0$ for $s \in \mathbb{R} \setminus \{0\}$, and that

$$f_0 := \lim_{\xi \to 0} \frac{f(\xi)}{\phi_p(\xi)} > 0 \tag{1.4}$$

(we assume that the limit exists and is finite).

When $p = 2$ the problem (1.1)–(1.2) has been considered in many recent papers. For example, positive solutions are obtained in [17], and in the references therein. On the other hand, 'nodal solutions' (that is, sign-changing solutions having a given number of zeros) have been obtained in [11], [16], and in the references therein. When $1 < p \neq 2$, positive solutions have been obtained in [1], while the existence of solutions in general (ignoring nodal structure) has been studied in [7], and the references therein. However, in this case the existence of nodal solutions has not previously been studied, and it is this that we will consider in this paper.

Previous results on the existence of nodal solutions in the case $p = 2$ rely heavily on spectral properties of the linearization of the problem. Here the related nonlinear eigenvalue problem

$$-\phi_p(u')' = \lambda \phi_p(u), \tag{1.5}$$

together with the boundary conditions (1.2), is relevant and will be central to many arguments below. The spectral properties of the p-Laplacian problem (1.5), together with standard separated (2-point) boundary conditions, are known, see [2]. However, in the case of the multi-point boundary condition (1.2) these spectral properties have not been obtained previously. Thus, in Section 3 we set up a suitable operator formulation of the m-point, p-Laplacian, we show that this operator is invertible, and we derive the required spectral results for this operator. The results we obtain are similar to the standard spectral theory of the linear, separated Sturm–Liouville problem, with a slight difference in the nodal counting method used, to deal with the multi-point boundary conditions. The topological degree of a related operator is also obtained.

In Section 4 we consider a bifurcation problem related to (1.1)–(1.2), and prove a Rabinowitz-type global bifurcation theorem for this problem. The proof uses the spectral and degree theoretic results obtained in Section 3. Finally, in Section 5, we use the global bifurcation theorem from Section 4 to obtain nodal solutions of (1.1)–(1.2), under various hypotheses on the asymptotic behaviour of f.

2. Preliminary definitions and results

In this section we describe some notation, and some preliminary results, which will be required in later sections to prove our main results.

2.1. Function spaces. For any integer $n \geq 0$, let $C^n[0,1]$ denote the usual Banach space of n-times continuously differentiable functions on $[0,1]$, with the usual sup-type norm, denoted by $|\cdot|_n$. A suitable space in which to search for solutions of (1.1), and which incorporates the boundary conditions (1.2), is the space

$$X := \{u \in C^1[0,1] : \phi_p(u') \in C^1[0,1] \text{ and } u \text{ satisfies (1.2) }\},$$
$$\|u\|_X := |u|_1 + |\phi_p(u')|_1, \quad u \in X.$$

We also let $Y := C^0[0,1]$, with the norm $\|\cdot\|_Y := |\cdot|_0$.

For any continuous function $f\colon \mathbb{R} \to \mathbb{R}$, we use the notation $f\colon Y \to Y$ to denote the corresponding Nemitskii operator defined by $f(u)(x) := f(u(x))$, $x \in [0,1]$, for $u \in Y$ (using the same notation for the function and the operator should not cause any confusion). The operator $f\colon Y \to Y$ is bounded (in the sense that bounded sets are mapped to bounded sets) and continuous. In addition, it can easily be seen that the operator $\phi_p\colon Y \to Y$ is invertible, with inverse $\phi_p^{-1} = \phi_{p^*}$, where $p^* := p/(p-1) > 1$.

2.2. Nodal properties. We now introduce some notation to describe the nodal properties of solutions of the various differential equations that we consider. For any C^1 function u, if $u(x_0) = 0$ then x_0 is a *simple* zero of u if $u'(x_0) \neq 0$. Now, for any integer $k \geq 1$ and any $\nu \in \{\pm\}$, we define $T_k^\nu \subset X$ to be the set of functions $u \in X$ satisfying the following conditions:

(i) $u(0) = 0$, $\nu u'(0) > 0$ and $u'(1) \neq 0$;
(ii) $\phi_p(u')$ has only simple zeros in $(0,1)$, and has exactly k such zeros;
(iii) u has a zero strictly between each consecutive zero of u'.

We also define $T_k := T_k^+ \cup T_k^-$.

Remarks 2.1. (a) The zeros of u' and $\phi_p(u')$ clearly coincide, but simple zeros of u' need not be simple zeros of $\phi_p(u')$, and vice-versa.

(b) If $u \in T_k^\nu$ then u has exactly one zero between each consecutive zero of u', and all zeros of u are simple. Thus, u has at least $k-1$ zeros in $(0,1)$, and at most k zeros in $(0,1]$.

(c) The sets T_k^ν are open in X and disjoint.

Remark 2.2. The sets T_k^ν above, in the case $p \neq 2$, are analogous to the sets defined in [16] in the case $p = 2$. One could regard these sets as counting 'bumps' of u. The nodal properties of solutions of nonlinear Sturm–Liouville problems with separated boundary conditions are usually described in terms of sets similar to T_k^ν which count zeros of u (with an additional condition at $x = 1$ to incorporate the boundary condition there), see, for example, [12, Section 2] (for the case $p = 2$). However, it was shown in [16], in the case $p = 2$, that when considering the multi-point boundary condition (1.2) the sets T_k^ν are in fact the appropriate sets to describe the nodal properties.

2.3. Existence and uniqueness for initial value problems. We will need an existence and uniqueness result for the initial value problem

$$
\begin{aligned}
-\phi_p(u')' &= f(u), \quad \text{on } \mathbb{R}, \\
u(x_0) = a, &\quad u'(x_0) = b,
\end{aligned}
\tag{2.1}
$$

for arbitrary $x_0, a, b \in \mathbb{R}$. The following result extends [8, Theorem 3] on existence and uniqueness of solutions of (2.1), and also gives some information on the general structure of solutions which will be required below. In this result we will use the following terminology regarding a solution u: the portion of u between two consecutive zeros will be termed a *bump*; if u has infinitely many zeros it is *oscillatory*.

Theorem 2.3. *Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $sf(s) > 0$ for $s \in \mathbb{R} \setminus \{0\}$. Then for any $x_0, a, b \in \mathbb{R}$, the problem* (2.1) *has a unique solution u. This solution is defined on the whole of $\mathbb{R}$ and $u, \phi_p(u') \in C^1(\mathbb{R})$. If $u \not\equiv 0$ then u and $\phi_p(u')$ have only simple zeros. In addition, u satisfies one of the following alternatives.*

(a) *u is bounded. Then u is periodic and oscillatory* (*and so consists of a sequence of positive and negative bumps*).
All the positive bumps of u have the same shape, and any such bump B has the following properties:
(a1) *B is symmetric about its mid-point m_B;*
(a2) *u' is strictly decreasing on B, so B contains exactly one zero of u', at m_B.*

All the negative bumps of u have the same shape, and have similar properties to those of the positive bumps.
The positive and negative bumps need not have the same shape, although they will if f is odd.

(b) *u is unbounded. Then $\lim_{x\to\pm\infty}|u(x)| = \infty$ and u satisfies one of the following alternatives.*

(b1) *u has one zero and u' has no zeros on $\mathbb{R}$. If f is odd, u is anti-symmetric about its zero.*

(b2) *u has two zeros, and hence has a single bump B, with properties similar to* (a1) *and* (a2) *above.*

Proof. Define $F\colon\mathbb{R}\to\mathbb{R}$ by

$$F(s) := \int_0^s f(t)\,dt, \quad s\in\mathbb{R}.$$

Then F is a C^1 function, and is strictly increasing on $[0,\infty)$ and strictly decreasing on $(-\infty,0]$. In particular, $F\ge 0$, and $F(s)=0$ if and only if $s=0$. Suppose that u is an arbitrary solution of (2.1) on a maximal interval E. Multiplying the differential equation by $u'(x)$ and integrating from x_0 to $x\in E$ yields

$$\frac{p}{p-1}F(u(x)) + |u'(x)|^p = c(a,b) := \frac{p}{p-1}F(a) + |b|^p. \tag{2.2}$$

Hence, $|u'(x)|\le c(a,b)^{1/p}$, $x\in E$, and so u is bounded on any bounded interval. Standard existence theory for ordinary differential equations (for example, Theorem 1.4, Chapter 1 of [3]) now shows that $E=\mathbb{R}$, that is, there exists a solution u defined on $\mathbb{R}$.

It now follows from (2.2) that if $a=b=0$ then

$$\frac{p}{p-1}F(u(x)) + |u'(x)|^p = 0, \quad x\in\mathbb{R},$$

which implies that $u\equiv 0$ on $\mathbb{R}$. Hence, if $u\not\equiv 0$ then any zeros of u are simple. Furthermore, if $\phi_p(u'(x))=0$ then $u(x)\ne 0$, so by our hypothesis on f and the differential equation (2.1), $\phi_p(u'(x))'\ne 0$, that is, any zero of $\phi_p(u')$ is simple. From now on we suppose that $|a|+|b|>0$.

Next, we note that, for any $a\in\mathbb{R}$, if $b\ne 0$ then the local uniqueness part of the proof of [8, Theorem 3] is valid under our hypotheses. Hence, in this case the above solution u is uniquely defined between x_0 and any zero of u'.

Now suppose that $u>0$ on a maximal interval P containing x_0 (similar arguments apply to maximal intervals N on which $u<0$). Then from (2.1),

$$\phi_p(u'(x)) = \phi_p(b) - \int_{x_0}^x f(u(s))\,ds, \quad x\in P,$$

so that u' is strictly decreasing on P. Hence, P contains at most one zero of u', and it can easily be seen, using the differential equation in (2.1), that P is bounded if and only if P contains exactly one zero of u'. In addition, if P is unbounded then either P is bounded below and $\lim_{x\to\infty} u(x) = \infty$, or P is bounded above and $\lim_{x\to-\infty} u(x) = \infty$.

Now suppose that P is bounded, and write $P = (x_l, x_r)$, $m_B = (x_l + x_r)/2$. Then, by (2.2),

$$u(x_l) = u(x_r) = 0, \quad u'(x_l) = -u'(x_r) > 0. \tag{2.3}$$

Hence, by the above uniqueness result, the solution curves on the intervals $[x_l, m_B]$, $[m_B, x_r]$ are symmetric, and $u'(m_B) = 0$. These results show that any positive (and negative) bumps have the properties (a1) and (a2) described in the theorem.

Now suppose that there exist two adjacent bumps P and N. Then, by (2.3) the magnitude of u' is the same at each end point of P and N, and so, by the above uniqueness result, the solution u must now consist entirely of a sequence of copies of these bumps, that is, u must satisfy alternative (a). In addition, the uniqueness result also shows that the positive and negative bumps must have the same shape when f is odd.

The only other options for u are:

(i) adjacent unbounded positive and negative intervals;

(ii) a single bump between two unbounded positive or negative intervals.

These options correspond to (b1) and (b2) and this completes the proof. □

Corollary. *If F is as in the proof of Theorem* 2.3, *and* $\lim_{s\to\pm\infty} F(s) = \infty$, *then alternative* (b) *in Theorem* 2.3 *cannot hold.*

Proof. It follows from (2.2) and the assumption on F that u must be bounded in this case. □

2.4. The function $\sin_p$**.** Fundamental to the proofs below will be certain properties of the solution of the specific initial value problem

$$-\phi_p(u')' = (p-1)\phi_p(u), \quad \text{on } \mathbb{R}, \tag{2.4}$$

$$u(0) = 0, \quad u'(0) = 1. \tag{2.5}$$

We let $\sin_p$ denote the (unique) maximal solution of this problem. A construction of this function is described in [8], and it is shown there that $\sin_p$ is a $2\pi_p$-periodic, C^1 function on $\mathbb{R}$, where $\pi_p := 2(\pi/p)/\sin(\pi/p)$. Moreover, for any $x \in \mathbb{R}$,

$$\sin_p(x + \pi_p) = -\sin_p(x), \tag{2.6}$$

$$|\sin_p x|^p + |\sin_p' x|^p = 1, \tag{2.7}$$

and $\sin_p(n\pi_p) = 0$, $\sin_p'((n+1/2)\pi_p) = 0$, $n \in \mathbb{Z}$. Thus the graph of $\sin_p$ resembles a sine wave, and indeed, $\sin_2$ is the usual sin function, with $\pi_2 = \pi$.

Remark 2.5. The notations $\sin_p$, π_p have been used in various senses. The one used here is taken from [8]. Another one (used in, for example, [6], [10] or [13]) omits the factor $p-1$ on the right hand side of the equation (2.4), but this adds an additional factor to the formula (2.7), and it will be crucial for the results below to have (2.7) in this simple form.

3. Properties of the m-point, p-Laplacian operator

In this section we define an operator realization of the m-point, p-Laplacian boundary value problem, and investigate some of the basic spectral and degree theoretic properties of this operator.

We define $\Delta_p : X \to Y$ by

$$\Delta_p u := \phi_p(u')', \quad u \in X.$$

By the definition of the spaces X, Y, the operator Δ_p is well-defined and continuous. The following result shows that Δ_p has a continuous inverse. We say that an operator is *completely continuous* if it is continuous and maps bounded sets into relatively compact sets.

Theorem 3.1. *The operator $\Delta_p : X \to Y$ is bijective, and the inverse operator $\Delta_p^{-1} : Y \to X$ is continuous. In addition, $\Delta_p^{-1} : Y \to C^1[0,1]$ is completely continuous.*

Proof. For arbitrary $h \in Y$ we will construct a unique solution $u \in X$ of the equation

$$\Delta_p u = h. \tag{3.1}$$

Define a linear operator $\mathcal{I} : Y \to C^1[0,1]$ by

$$(\mathcal{I}h)(x) := \int_0^x h(s)\,ds, \quad h \in Y.$$

Clearly, $\mathcal{I} : Y \to C^1[0,1]$ is bounded, and $\mathcal{I} : Y \to Y$ is completely continuous. By integrating (3.1) twice, and using the boundary condition at $x = 0$, we see that any solution of (3.1) must have the form

$$u_\gamma := \mathcal{I}(\phi_{p^*}(\gamma + \mathcal{I}h)), \tag{3.2}$$

for some $\gamma \in \mathbb{R}$. It now follows from the multi-point boundary condition that u_γ is in fact a solution of (3.1) if and only if

$$\begin{aligned} 0 = \Phi(\gamma; h) : &= \int_0^1 \phi_{p^*}(\gamma + \mathcal{I}h) - \sum_{i=1}^{m-2} \alpha_i \int_0^{\eta_i} \phi_{p^*}(\gamma + \mathcal{I}h) \\ &= \biggl(1 - \sum_{i=1}^{m-2} \alpha_i\biggr) \int_0^1 \phi_{p^*}(\gamma + \mathcal{I}h) + \sum_{i=1}^{m-2} \alpha_i \int_{\eta_i}^1 \phi_{p^*}(\gamma + \mathcal{I}h). \end{aligned}$$

Clearly, the function $\Phi: \mathbb{R} \times Y \to \mathbb{R}$ is continuous and, for fixed $h \in Y$, it follows from (1.3) that $\Phi(\,\cdot\,; h)$ is strictly increasing, with $\lim_{\gamma \to \pm\infty} \Phi(\gamma; h) = \pm\infty$. Thus, there exists a unique $\gamma(h) \in \mathbb{R}$ such that $\Phi(\gamma(h); h) = 0$, and the functional $\gamma: Y \to \mathbb{R}$ is continuous. Hence, $u_{\gamma(h)}$ is the unique solution of (3.1).

Now, it follows from (3.2) and the properties of $\mathcal{I}$ and ϕ_{p^*}, that the mappings

$$\begin{aligned} &h \to u_{\gamma(h)}: Y \to C^1[0,1], \\ &h \to \phi_p(u'_{\gamma(h)}) = \gamma(h) + \mathcal{I}h: Y \to C^1[0,1], \end{aligned}$$

are continuous, and hence $u_{\gamma(h)} \in X$ and the mapping $h \to u_{\gamma(h)}: Y \to X$ is continuous. Thus, Δ_p is bijective, with continuous inverse $\Delta_p^{-1}(h) = u_{\gamma(h)}$.

Next, it follows directly from (1.3) and the definition of Φ that $\Phi(\gamma; h) > 0$ for $\gamma > |h|_0$ and $\Phi(\gamma; h) < 0$ for $\gamma < -|h|_0$, so that

$$|\gamma(h)| \le |h|_0, \quad h \in Y.$$

Hence, the mapping $h \to \phi_{p^*}(\gamma(h) + \mathcal{I}h): Y \to Y$ is completely continuous, so by (3.2), $\Delta_p^{-1}: Y \to C^1[0,1]$ is completely continuous. $\square$

Remark 3.2. We used the spaces $C^n[0,1]$, $n = 0, 1$, to define the above multi-point, p-Laplace operator Δ_p, and Theorem 3.1 showed that the resulting operator is invertible. This is the function space setting that we will mainly use here. However, one could also use a Sobolev space setting to define a similar operator — we outline this briefly here. Let $\|\cdot\|_1$ and $\|\cdot\|_{1,1}$ denote the standard norms on $L^1(0,1)$ and the Sobolev space $W^{1,1}(0,1)$, respectively. Let $\widetilde{Y} := L^1(0,1)$, and let

$$\begin{gathered} \widetilde{X} := \{u \in C^1[0,1] : \phi_p(u') \in W^{1,1}[0,1] \text{ and } u \text{ satisfies (1.2)}\}, \\ \|u\|_{\widetilde{X}} := |u|_1 + \|\phi_p(u')\|_{1,1}, \quad u \in \widetilde{X}. \end{gathered}$$

Then we can define $\widetilde{\Delta}_p: \widetilde{X} \to \widetilde{Y}$ in the obvious manner, and the analogue of Theorem 3.1 holds for $\widetilde{\Delta}_p$, with a similar proof, using a functional $\widetilde{\gamma}: \widetilde{Y} \to \mathbb{R}$.

The following result will be required below. The notation $\rightharpoonup$ will denote weak convergence.

Lemma 3.3. *Suppose that $q > 1$, (h_n) is a sequence in $L^q(0,1)$, with $h_n \rightharpoonup h_\infty$ in $L^q(0,1)$, and (p_n) is a sequence in $(1,\infty)$, with $p_n \to p_\infty \in (1,\infty)$. Then $\widetilde{\Delta}_{p_n}^{-1}(h_n) \to \widetilde{\Delta}_{p_\infty}^{-1}(h_\infty)$ in $C^1[0,1]$.*

Proof. In this proof, to indicate the dependence on p, we write $\widetilde{\gamma}_p$ for the functional $\widetilde{\gamma}$ mentioned in Remark 3.2. By weak convergence, $(\mathcal{I}h_n)(x) \to (\mathcal{I}h_\infty)(x)$ for all $x \in [0,1]$. Also, the sequence (h_n) is bounded in $L^q(0,1)$ (by weak convergence), so the set of functions $\{\mathcal{I}h_n\}$ is uniformly bounded and equicontinuous (by Hölder's inequality). Thus, by the proof of the Arzela–Ascoli theorem, $\mathcal{I}h_n \to \mathcal{I}h_\infty$ in $C^0[0,1]$. It follows from this, and the definition of $\widetilde{\gamma}_p$, that $\widetilde{\gamma}_{p_n}(h_n) \to \widetilde{\gamma}_{p_\infty}(h_\infty)$, and hence, by the properties of the operators $\mathcal{I}$ and ϕ_p, $p > 1$,

$$\widetilde{\Delta}_{p_n}^{-1}(h_n) = \mathcal{I}(\phi_{p_n^*}(\widetilde{\gamma}_{p_n}(h_n) + \mathcal{I}h_n)) \to \mathcal{I}(\phi_{p_\infty^*}(\widetilde{\gamma}_{p_\infty}(h_\infty) + \mathcal{I}h_\infty)) = \widetilde{\Delta}_{p_\infty}^{-1}(h_\infty),$$

in $C^1[0,1]$. □

Remark 3.4. Lemma 3.3 also holds if $q = 1$ and the set of functions $\{h_n\}$ is *equi-integrable*, that is, there exists $h \in L^1(0,1)$ such that $|h_n(x)| \le h(x)$ for a.e. $x \in [0,1]$ and any $n \ge 1$ (this condition implies equicontinuity of the set $\{\mathcal{I}h_n\}$ in this case).

Next, we consider the eigenvalue problem

$$-\Delta_p(u) = \lambda\phi_p(u), \quad u \in X. \tag{3.3}$$

Of course, a number $\lambda \in \mathbb{R}$ is said to be an *eigenvalue* of (3.3) if this problem has a non-trivial solution u, which is then an *eigenfunction* corresponding to λ. Clearly, if u is an eigenfunction then tu is an eigenfunction for all non-zero $t \in \mathbb{R}$ — the eigenvalue λ is said to be *simple* if every eigenfunction corresponding to λ is of the form tu, for some $t \in \mathbb{R}$ (for linear problems, 'simple' eigenvalues usually have some further properties, but here we will use the term in the above sense for all $p > 1$, even in the linear case $p = 2$).

Theorem 3.5. *The set of eigenvalues of* (3.3) *consists of a strictly increasing sequence of simple eigenvalues $\lambda_k > 0$, $k = 1, 2, \ldots$, with corresponding eigenfunctions $u_k(x) = \sin_p((\lambda_k/(p-1))^{1/p}x)$. In addition,*

(a) $\lim_{k\to\infty} \lambda_k = \infty$;
(b) *$u_k \in T_k^+$, for each $k \ge 1$, and u_1 is strictly positive on* $(0,1)$.

Proof. Combining the boundary condition (1.2) with Theorem 2.3 ensures that any eigenvalue λ is simple. Now let (λ, u) be a non-trivial solution of (3.3), with $u'(0) > 0$ (without loss of generality). Suppose that $\lambda < 0$. Then for x between 0 and the first positive zero of u we have

$$\phi_p(u'(x)) = \phi_p(u'(0)) - \lambda \int_0^x \phi_p(u(s))\, ds,$$

from which it follows that u is strictly increasing on $[0,1]$. Hence, by (1.2) and (1.3),

$$u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i) \le u(1) \sum_{i=1}^{m-2} \alpha_i < u(1),$$

and this contradiction shows that this case cannot occur. Also, by Theorem 3.1, the case $\lambda = 0$ cannot occur. Thus, from now on we may assume that $\lambda > 0$.

It now follows from the construction of the function $\sin_p$ and the uniqueness of the solution of the initial value problem that there exists $s > 0$ and $C \in \mathbb{R}$ such that $\lambda = s^p(p-1)$ and $u(x) = C\sin_p(sx)$, $x \in [0,1]$. Hence, if we define a function $\Gamma\colon (0,\infty) \times (1,\infty) \to \mathbb{R}$ by

$$\Gamma(s,p) := \sin_p s - \sum_{i=1}^{m-2} \alpha_i \sin_p(\eta_i s), \quad s \in (0,\infty),$$

then $\Gamma(s,p) = 0$ if and only if $\lambda = s^p(p-1)$ is an eigenvalue of (3.3), with eigenfunction $w_s(x) := \sin_p(sx)$. Thus, to prove the theorem it suffices to search for zeros of Γ.

For fixed $p > 1$, it follows from (1.3) and the properties of $\sin_p$ that $\Gamma(s,p) > 0$, $s \in (0,\pi_p/2]$, and $\Gamma(s,p) < 0$, $s \in [\pi_p, 3\pi_p/2]$, while $\Gamma((i+1/2)\pi_p, p) \ne 0$, for any integer $i \ge 1$. Hence all the zeros of $\Gamma(s,p)$ lie in $\cup_{k\ge 1} I_k(p)$, where $I_k(p)$, $k = 1,2,\ldots$, are the open intervals

$$\begin{aligned} I_1(p) &:= \left(\frac{1}{2}\pi_p, \pi_p\right), \\ I_k(p) &:= \left(\left(k - \frac{1}{2}\right)\pi_p, \left(k + \frac{1}{2}\right)\pi_p\right), \quad k = 2,3,\ldots \end{aligned}$$

In addition, it is straightforward to show that if $s \in I_k(p)$ then $w_s \in T_k^+$, and if $s \in I_1(p)$ then w_s is strictly positive on $(0,1)$. Thus, to prove the theorem it suffices to show that for each $p > 1$ and each $k \ge 1$, the function $\Gamma(\,\cdot\,,p)$ has exactly one zero in the interval $I_k(p)$. To do this we require the following lemma.

Lemma 3.6. *The function Γ and the partial derivative Γ_s are continuou on $(0,\infty)\times(1,\infty)$. For fixed $p > 1$, all the zeros of $\Gamma(\,\cdot\,,p)$ are simple.*

Proof. The first result is a consequence of equation (2.5) in [10]. Now suppose that s is a double zero of $\Gamma(\,\cdot\,,p)$, that is,

$$\sin_p s = \sum_{i=1}^{m-2} \alpha_i \sin_p(\eta_i s), \quad \sin_p' s = \sum_{i=1}^{m-2} \alpha_i \eta_i \sin_p'(\eta_i s). \tag{3.4}$$

Then, by (1.3) and (2.7),

$$\begin{aligned}
1 &= \Big(\sum_{i=1}^{m-2} \alpha_i \sin_p(\eta_i s)\Big)\phi_p(\sin_p s) + \Big(\sum_{i=1}^{m-2} \alpha_i \eta_i \sin_p'(\eta_i s)\Big)\phi_p(\sin_p' s) \\
&\le \sum_{i=1}^{m-2} \alpha_i (|\sin_p(\eta_i s)| |\sin_p s|^{p-1} + |\sin_p'(\eta_i s)| |\sin_p' s|^{p-1}) \\
&\le \sum_{i=1}^{m-2} \alpha_i (|\sin_p(\eta_i s)|^p + |\sin_p'(\eta_i s)|^p)^{1/p} (|\sin_p s|^p + |\sin_p' s|^p)^{(p-1)/p} < 1,
\end{aligned}$$

which shows that (3.4) cannot hold, and so $\Gamma(\,\cdot\,, p)$ has only simple zeros. □

Now, letting p vary over $(1,\infty)$, it follows from Lemma 3.6 and the form of the implicit function theorem in Theorem 15.1 of [4] that the zeros of $\Gamma(\,\cdot\,,p)$ depend continuously upon p, and the number of zeros in each interval $I_k(p)$ remains constant. By Lemma 3.2 in [16], $\Gamma(\,\cdot\,,2)$ has exactly one zero in $I_k(2)$, for each $k \ge 1$, so this remains true for all $p > 1$. This concludes the proof of Theorem 3.5. □

The above proof of Theorem 3.5 also proves the following result.

Corollary 3.7. *The eigenvalues of* (3.3) *depend continuously on* $p \in (1,\infty)$.

Now, the eigenvalue problem (3.3) is equivalent to the equation

$$u + K_\lambda(u) = 0, \quad u \in Y, \tag{3.5}$$

where $K_\lambda := \Delta_p^{-1} \circ (\lambda\phi_p) \colon Y \to Y$. In particular, (3.5) has a non-trivial solution u if and only if λ is an eigenvalue of the operator $-\Delta_p$. Furthermore, the operator K_λ is completely continuous (by Theorem 3.1), and homogeneous (in the sense that $K_\lambda(tu) = tK_\lambda(u)$, for any $t \in \mathbb{R}$ and $u \in Y$). Thus, if λ is not an eigenvalue of $-\Delta_p$ then the Leray-Schauder degree $\deg(I + K_\lambda, B_r, 0)$ is well defined for any $r > 0$, where B_r denotes the open ball in Y, centered at 0 with radius r.

Theorem 3.8. *For any* $r > 0$,

$$\deg(I + K_\lambda, B_r, 0) = \begin{cases} 1 & \text{if } \lambda < \lambda_1, \\ (-1)^k & \text{if } \lambda \in (\lambda_k, \lambda_{k+1}),\ k \ge 1. \end{cases}$$

Proof. We again prove the result by continuation with respect to p, varying p from the known, linear case $p = 2$ to general $p \neq 2$. Hence, to explicitly indicate the dependence of the operator K_λ on p we write $K_{\lambda,p}$. We also denote the eigenvalues of (3.3) by $\lambda_k(p)$, $k \ge 1$. By Corollary 3.7 the functions $\lambda_k(\,\cdot\,)$ are continuous on $(1,\infty)$.

When $p = 2$, the operator $K_{\lambda,2}$ is in fact the linear operator $\lambda\Delta_2^{-1} \colon Y \to Y$. By Lemma 3.8 in [16], all the characteristic values of $-\Delta_2^{-1}$ have algebraic multiplicity 1, so in this case the result follows from the Leray–Schauder index theorem (see, for example, Proposition 14.5 in [Zeidler]).

Now fix $k \geq 1$, $p > 2$ and $\lambda \in (\lambda_k(p), \lambda_{k+1}(p))$ (the cases $1 < p < 2$ and $\lambda < \lambda_1(p)$ are similar). By Theorem 3.5 and Corollary 3.7, we can choose a continuous function $\rho: [2,p] \to \mathbb{R}$ such that $\rho(p) = \lambda$ and

$$\lambda_k(q) < \rho(q) < \lambda_{k+1}(q), \quad q \in [2,p].$$

Now, by Lemma 3.3, the homotopy

$$H(q,u) := K_{\rho(q),q}(u): [2,p] \times Y \to Y$$

is completely continuous and, by construction, for each $q \in [2,p]$ the equation $u + H(q,u) = 0$ has no non-trivial solution u, since $\rho(q)$ is not an eigenvalue of $-\Delta_p$. Hence the result follows from the homotopy invariance of the Leray–Schauder degree. $\square$

4. Global bifurcation theory

In this section we consider the bifurcation problem,

$$-\Delta_p(u) = \lambda f(u), \quad (\lambda, u) \in \mathbb{R} \times X. \tag{4.1}$$

Clearly, $u \equiv 0$ is a solution of (4.1) for any $\lambda \in \mathbb{R}$; such solutions will be called *trivial*. We will prove a Rabinowitz-type global-bifurcation result for the solution set of (4.1).

We first observe that, by Theorem 3.1, equation (4.1) is equivalent to the equation

$$u + \Delta_p^{-1} \circ (\lambda f)(u) = 0, \quad (\lambda, u) \in \mathbb{R} \times Y. \tag{4.2}$$

The following result gives the degree of the operator on the left of this equation.

Lemma 4.1. *If λf_0 is not an eigenvalue of* (3.3)*, then for sufficiently small $r > 0$.*

$$\deg(I + \Delta_p^{-1} \circ (\lambda f), B_r, 0) = \deg(I + K_{\lambda f_0}, B_r, 0).$$

Proof. The operator $\Delta_p^{-1}: Y \to Y$ is completely continuous, by Theorem 3.1, and $f: Y \to Y$ is bounded, so the operator $\Delta_p^{-1} \circ (\lambda f): Y \to Y$ is completely continuous. Since $|f(u)|_0 = f_0 \phi_p(|u|_0) + \mathrm{o}(|u|_0^{p-1})$, as $|u|_0 \to 0$, for $u \in Y$, the result follows from standard continuity properties of the degree. $\square$

We now prove various results on the set of non-trivial solutions of (4.1).

Definition 4.2. Let $\mathcal{S} \subset \mathbb{R} \times X$ denote the the set of non-trivial solutions (λ, u) of (4.1), and let $\overline{\mathcal{S}}$ denote the closure of $\mathcal{S}$ in $\mathbb{R} \times X$.

Lemma 4.3. *If* $(\lambda, u) \in \mathcal{S}$ *then* $u \in T_k$ *for some* $k \geq 1$.

Proof. The properties of solutions of (4.1) proved in Theorem 2.3 are, essentially, the properties used to prove the corresponding result for the case $p = 2$ in Proposition 4.1 in [16]. Thus, to prove the result in the current $p \neq 2$ setting we simply follow the proof in [16]. The only modification required here is in the verification that property (ii) in the definition of the sets T_k holds. In view of this, we simply describe this verification.

Suppose there exists $x_0 \in (0,1)$ such that $\phi_p(u'(x_0)) = 0$, $\phi_p(u'(x_0))' = 0$. Then, by equation (4.1) and the assumptions on f, $u(x_0) = u'(x_0) = 0$, and so by Theorem 2.3, $u \equiv 0$ on $[0,1]$. However, this contradicts the non-triviality of u, so we conclude that u satisfies condition (ii) in the definition of T_k (for some k). □

Lemma 4.4.

(a) *For each* $k \geq 1$ *there is a neighbourhood* O_k *of* $(\lambda_k/f_0, 0)$ *in* $\mathbb{R} \times X$ *such that* $\mathcal{S} \cap O_k \subset \mathbb{R} \times T_k$.

(b) $\overline{\mathcal{S}} \cap (\mathbb{R} \times \{0\}) \subset \bigcup_{k=1}^{\infty} \{(\lambda_k/f_0, 0)\}$.

Proof. Let $(\mu_i, u_i) \in \mathcal{S}$, $i = 1, 2, \ldots$, be a sequence of solutions such that $u_i \notin T_k$, and $\lim_{i\to\infty}(\mu_i, u_i) = (\lambda_k/f_0, 0)$. Then, for each $i \geq 1$, the function $v_i := u_i/|u_i|_0$ satisfies

$$v_i = -\Delta_p^{-1}(\mu_i(f(u_i)/|u_i|_0^{p-1})), \tag{4.3}$$

and $f(u_i)/|u_i|_0^{p-1}$ is bounded in Y, by assumption. Hence, by Theorem 3.1, after taking a subsequence if necessary, there exists $v_\infty \neq 0$ such that $v_i \to v_\infty$ in Y. Therefore, by (1.4), $f(u_i)/|u_i|_0^{p-1} \to f_0 v_\infty$ in Y, and by (4.3) and Theorem 3.1 again, $v_i \to v_\infty$ in X, and in fact $-\Delta_p v_\infty = \lambda_k v_\infty$. Thus, v_∞ is an eigenfunction of $-\Delta_p$ corresponding to the eigenvalue λ_k, and hence $v_\infty \in T_k$, by Theorem 3.5. However, each $v_i \notin T_k$ so, since T_k is open in X, this is a contradiction. This proves part (a) of the lemma; similar arguments prove part (b). □

We now prove the following global bifurcation result. Here, a *continuum* is a closed, connected set.

Theorem 4.5. *For each* $k \geq 1$ *there exists a continuum* $\mathcal{C}_k \subset \mathbb{R}^+ \times X$ *of solutions of* (4.1) *with the properties:*

(a) $(\lambda_k/f_0, 0) \in \mathcal{C}_k$;

(b) $\mathcal{C}_k \setminus \{(\lambda_k/f_0, 0)\} \subset \mathbb{R}^+ \times T_k$;

(c) $\mathcal{C}_k$ *is unbounded in* $\mathbb{R}^+ \times Y$.

Proof. For fixed $k \geq 1$, it follows from Lemmas 3.8 and 4.1 and Theorem 2.9 in [9] that there exists a continuum $\mathcal{C}_k \subset \overline{\mathcal{S}}$ of solutions of (4.2) containing $(\lambda_k/f_0, 0)$, such that one of the following alternatives holds:

(i) $\mathcal{C}_k$ is unbounded in $\mathbb{R} \times Y$;
(ii) $(\lambda_j/f_0, 0) \in \mathcal{C}_k$ for some $j \neq k$.

Remarks 4.6. (a) When $p = 2$, that is, when Δ_p is linear, this result is analogous to Theorem 2.3 in [12] (which deals with separated boundary conditions); when $p \neq 2$, that is, when Δ_p is non-linear but positively homogeneous, the result can be proved in a similar manner to the proof in [12], see for example Theorem 2.9 in [9].

(b) The cited global bifurcation results actually show that $\mathcal{C}_k$ is a continuum in the space $\mathbb{R} \times Y$. However, by (4.2) and the continuity of the operator $\Delta_p^{-1} \circ f : Y \to X$, the continuum $\mathcal{C}_k$ can also be regarded as a continuum in $\mathbb{R} \times X$.

Now, by Lemmas 4.3, 4.4, and the fact that the sets T_i, $i \geq 1$, are open and disjoint in X, we see that property (b) in the theorem holds. It follows from this that alternative (ii) above cannot hold. Thus, $\mathcal{C}_k$ is unbounded in $\mathbb{R} \times Y$.

Finally, it follows from Theorem 3.1 that (4.2) cannot have a non-trivial solution with $\lambda = 0$. Thus, since $\mathcal{C}_k$ is connected and $\lambda_k > 0$, it follows that $\mathcal{C}_k \subset \mathbb{R}^+ \times X$. □

Definition 4.7. For $k \geq 1$, let $\mathcal{C}_k^\pm := (\mathcal{C}_k \cap (\mathbb{R} \times T_k^\pm)) \cup \{(\lambda_k/f_0, 0)\}$.

Theorem 4.8. *For each $k \geq 1$, each set $\mathcal{C}_k^\pm$ is closed, connected and unbounded in $\mathbb{R}^+ \times Y$.*

Proof. Since the sets $T_k^\pm$ are open, disjoint subsets of X, it follows from the proof of Lemma 28.6 in [WIL] that the sets $\mathcal{C}_k^\pm$ are connected, and it is clear that they are also closed. The argument in the proof of [15, Theorem 6.1] shows that each of the sets $\mathcal{C}_k^\pm$ are unbounded. □

5. A non-resonance condition and nodal solutions

In this section we obtain solutions of the problem

$$-\Delta_p(u) = f(u) + h, \quad u \in X, \tag{5.1}$$

for arbitrary $h \in Y$, and also nodal solutions of the problem (1.1), (1.2), which is a special case of (5.1), and can be rewritten as

$$-\Delta_p(u) = f(u), \quad u \in X. \tag{5.2}$$

We first suppose that the following limit exists,

$$f_\infty := \lim_{|\xi| \to \infty} \frac{f(\xi)}{\phi_p(\xi)},$$

where we allow $f_\infty = \infty$ (our basic hypotheses on f imply that $f_\infty \geq 0$).

Theorem 5.1. *Suppose that $f_\infty < \infty$. If f_∞ is not an eigenvalue of* (3.3) *then equation* (5.1) *has a solution $u \in X$ for any $h \in Y$.*

Proof. The proof is similar to the proof of Theorem 4.1 in [15], using the above properties of the operator Δ_p^{-1}. $\square$

Remark 5.2. Theorem 5.1 can be extended to yield a solution $u \in \widetilde{X}$ of (5.1), for any $h \in \widetilde{Y} = L^1(0,1)$, by using the operator $\widetilde{\Delta}_p^{-1}$ described in Remark 3.2.

The hypothesis in Theorem 5.1 that f_∞ is not an eigenvalue is a 'non-resonance' condition. Non-resonance conditions have been extensively investigated for general, separated boundary condition problems, both for the semilinear problem $p = 2$ and the general p-Laplacian problem with $1 < p \neq 2$ (see, for example, [14], [15] and the references therein). However, non-resonance conditions for the m-point problem (5.1) have received less attention, since the spectral theory of the operator $-\Delta_p$ has not previously been derived. The paper [7], and the references therein, consider 'resonant' problems.

Next, we consider nodal solutions of (5.2). We note that (5.2) has the trivial solution $u = 0$, so Theorem 5.1 tells us nothing about this problem.

Theorem 5.3. *Suppose that $f_\infty < \infty$. If, for some $k \geq 1$,*

$$(\lambda_k - f_0)(\lambda_k - f_\infty) < 0, \tag{5.3}$$

then (5.2) *has solutions $u_k^\pm \in T_k^\pm$.*

Proof. We shall show the existence of a solution $u_k^+ \in T_k^+$, and note that the existence of u_k^- follows by similar arguments. Let $\mathcal{C}_k^+ \subset \mathbb{R}^+ \times X$ denote the subcontinuum of solutions of (4.1) given by Theorem 4.8 Clearly, any non-trivial solution $(1, u) \in \mathcal{C}_k^+$ yields a non-trivial solution of (5.2), so we will show that $\mathcal{C}_k^+$ intersects the set $\{1\} \times X$.

Let $(\mu_i, u_i) \in \mathcal{C}_k^+$, $i = 1, 2, \dots$, be a sequence of non-trivial solutions which is unbounded in $\mathbb{R}^+ \times Y$. Suppose, firstly, that $f_\infty > 0$ and $\mu_i \to \infty$ as $i \to \infty$. Then, for each $i \geq 1$, (μ_i, u_i) satisfies the differential equation

$$-\phi_p(u_i'(x))' = \mu_i G_i(x) \phi_p(u_i(x)), \quad x \in [0,1],$$

where, for $x \in [0,1]$,

$$G_i(x) := \begin{cases} f(u_i(x))/\phi_p(u_i(x)) & \text{if } u_i(x) \neq 0, \\ f_0 & \text{if } u_i(x) = 0. \end{cases}$$

It follows from our assumptions on f and $f_0 > 0$, $f_\infty > 0$, that $G_i \in C^0[0,1]$, and that there exists constants C, c such that $C \geq G_i \geq c > 0$, for all i. Hence, by the Sturm comparison theorem (Theorem 6 in [8]), for sufficiently large i the solution u_i has more than $k+2$ interior zeros, which contradicts $u_i \in T_k$. Thus, if $f_\infty > 0$ then the sequence (μ_i) must be bounded.

Now suppose that $\mu_i \to \mu_\infty < \infty$ and $|u_i|_0 \to \infty$ (after taking a subsequence if necessary). For each $i \geq 1$, letting $v_i := u_i/|u_i|_0$, $h_i := f(u_i)/|u_i|_0^{p-1}$, we see that

$$v_i = \Delta_p^{-1}(\mu_i h_i). \tag{5.4}$$

By our assumptions on f, the sequence (h_i) is bounded in Y, so by (5.4) and Theorem 3.1 we may suppose that $v_i \to v_\infty \neq 0$ in Y. We now have the following result.

Lemma 5.4. *$h_i \to f_\infty \phi_p(v_\infty)$ in Y.*

Proof. Let $\varepsilon > 0$ and define the set $S(\varepsilon) := \{x \in [0,1] : |v_\infty(x)| < \varepsilon\}$. Since $v_i \to v_\infty$ in Y, there exists N_1 such that, for $i > N_1$,

$$|v_i - v_\infty|_0 + |\phi_p(v_i) - \phi_p(v_\infty)|_0 < \frac{1}{2}\varepsilon,$$

and hence, since $|h_i(x)| = |G_i(x)||\phi_p(v_i(x))|$, $x \in [0,1]$,

$$|h_i(x) - f_\infty\phi_p(v_\infty(x))| < (C + |f_\infty|)\phi_p(2\varepsilon), \quad x \in S(\varepsilon). \tag{5.5}$$

Also, since $|u_i|_0 \to \infty$, there exists $N_2 \geq N_1$ such that, for $i > N_2$,

$$|G_i(x) - f_\infty| < \varepsilon, \quad x \in [0,1] \setminus S(\varepsilon),$$

it follows that

$$\begin{aligned}|h_i(x) - f_\infty\phi_p(v_\infty(x))| &\leq |G_i(x) - f_\infty||\phi_p(v_i(x))| + |f_\infty||\phi_p(v_i(x)) - \phi_p(v_\infty(x))| \\ &\leq (\sup_{i\in\mathbb{N}} \phi_p(|v_i|_0) + |f_\infty|)\varepsilon, \quad x \in [0,1] \setminus S(\varepsilon).\end{aligned}$$

Combining this with (5.5) proves Lemma 5.4.

Now, by Lemma 5.4, letting $i \to \infty$ in (5.4) shows that $v_i \to v_\infty$ in X, and

$$-\Delta_p(v_\infty) = \mu_\infty f_\infty \phi_p(v_\infty).$$

Therefore, since $v_i \in T_k^+$, $i \geq 1$, it follows from Lemma 4.4 that $v_\infty \in T_k^+$, and hence $\mu_\infty = \lambda_k/f_\infty$. Now, since $\mathcal{C}_k^+$ is connected and bifurcates from $(\lambda_k/f_0, 0)$, condition (5.3) implies that $\mathcal{C}_k^+$ intersects the hyperplane $\{1\} \times X$ at a non-trivial solution. This completes the proof of Theorem 5.3 when $f_\infty > 0$.

On the other hand, if $f_\infty = 0$ then the preceding argument in fact shows that the case $\mu_i \to \mu_\infty < \infty$ cannot hold. Hence, in this case $\mu_i \to \infty$, and so the final part of the above argument again shows that $\mathcal{C}_k^+$ intersects the hyperplane $\{1\} \times X$ at a non-trivial solution. This completes the proof. $\square$

Next, we consider the case where $f_\infty = \infty$.

THEOREM 5.5. *Suppose that $f_\infty = \infty$. If $\lambda_{k_0}/f_0 > 1$, for some $k_0 \geq 1$, then* (5.2) *has solutions $u_k^\pm \in T_k^\pm$, for all $k \geq k_0$.*

PROOF. We follow the proof of Theorem 5.3. The initial argument there shows that the sequence (μ_i) must be bounded in this case, and so we have $|u_i|_0 \to \infty$. Next, the following result can be proved as in Lemmas 3.12 and 4.5 of [9].

LEMMA 5.6. *For any compact interval $J \subset (0,\infty)$, there exists $M_J > 0$ such that if $\lambda \in J$ and u is a solution of* (4.2)*, then $|u|_0 \leq M_J$.*

Combining these results shows that we must have $\mu_i \to 0$, and the final argument in the proof of Theorem 5.3 now shows that $\mathcal{C}_k^+$ intersects the hyperplane $\{1\} \times X$ at a non-trivial solution of (4.2). This completes the proof of Theorem 5.5. □

Finally, we no longer suppose that the limit f_∞ exists, and we allow different asymptotic behaviour of $f(\xi)$ as $\xi \to \infty$ and $\xi \to -\infty$. To make this precise, we define

$$\overline{\gamma}^\pm := \limsup_{\xi \to \pm\infty} \frac{f(\xi)}{\phi_p(\xi)}, \quad \underline{\gamma}^\pm := \liminf_{\xi \to \pm\infty} \frac{f(\xi)}{\phi_p(\xi)},$$

and we assume that $0 < \underline{\gamma}^\pm \leq \overline{\gamma}^\pm < \infty$ (the numbers $\underline{\gamma}^\pm$, $\overline{\gamma}^\pm$ may all be different).

For any $k \geq 1$, let

$$\begin{aligned}
(\underline{\lambda}_k^+)^{1/p} &:= (p-1)^{1/p}\pi_p\left[\frac{(k+1)/2}{(\underline{\gamma}^+)^{1/p}} + \frac{k/2}{(\underline{\gamma}^-)^{1/p}}\right],\\
(\overline{\lambda}_k^+)^{1/p} &:= (p-1)^{1/p}\pi_p\left[\frac{k/2}{(\overline{\gamma}^+)^{1/p}} + \frac{(k-1)/2}{(\overline{\gamma}^-)^{1/p}}\right],\\
(\underline{\lambda}_k^-)^{1/p} &:= (p-1)^{1/p}\pi_p\left[\frac{(k+1)/2}{(\underline{\gamma}^-)^{1/p}} + \frac{k/2}{(\underline{\gamma}^+)^{1/p}}\right],\\
(\overline{\lambda}_k^-)^{1/p} &:= (p-1)^{1/p}\pi_p\left[\frac{k/2}{(\overline{\gamma}^-)^{1/p}} + \frac{(k-1)/2}{(\overline{\gamma}^+)^{1/p}}\right].
\end{aligned}$$

Clearly, $\overline{\lambda}_k^\pm \leq \underline{\lambda}_k^\pm$.

THEOREM 5.7. *If, for some $k \geq 1$ and $\nu \in \{\pm\}$,*

$$\lambda_k/f_0 < 1 \quad \text{and} \quad \overline{\lambda}_k^\nu > 1 \qquad \text{or} \qquad \lambda_k/f_0 > 1 \quad \text{and} \quad \underline{\lambda}_k^\nu < 1, \tag{5.6}$$

then (5.2) *has a solution $u_k^\nu \in T_k^\nu$.*

PROOF. We again follow the proof of Theorem 5.3. In this case $\mathcal{C}_k^\nu$ contains a sequence (μ_i, u_i), $i = 1, 2, \ldots$, such that $\mu_i \to \mu_\infty$, $|u_i|_0 \to \infty$ and $v_i = u_i/|u_i|_0 \to v_\infty \neq 0$ in Y. In addition, the sequence $h_i = f(u_i)/\phi_p(|u_i|_0)$ is bounded in $L^2(0,\pi)$, so an extension of the argument on p. 648 of [5] proves that, after choosing a subsequence if necessary,

$$h_i \rightharpoonup q^+\phi_p(v_\infty^+) - q^-\phi_p(v_\infty^-),$$

in $L^2(0,1)$, where $v_\infty^\pm(x) = \max\{\pm v_\infty(x), 0\}$, $x \in [0,1]$, and

$$\underline{\gamma}^\pm \le q^\pm(x) \le \overline{\gamma}^\pm, \quad x \in [0,1]. \tag{5.7}$$

Now, by Lemma 3.3, $v_\infty \in \widetilde{X}$ (recall Remark 3.2) and satisfies

$$-\phi_p(v_\infty')' = \mu_\infty q^+\phi_p(v_\infty^+) - \mu_\infty q^-\phi_p(v_\infty^-). \tag{5.8}$$

We can now estimate μ_∞.

Lemma 5.8. $\overline{\lambda}_k^\nu \le \mu_\infty \le \underline{\lambda}_k^\nu$.

Proof. It follows from uniqueness of the solutions of the initial value problem for (5.8) (see Theorem 1 in [8]) that v_∞ has only simple zeros in $[0,1]$. In addition, by its construction as the limit of the sequence $u_i/|u_i|_2 \in T_k^\nu$, $i = 1,2,\ldots$, the function v_∞ has the general shape described in Theorem 2.3. Specifically, v_∞ consists of a sequence of positive and negative bumps (together with a truncated bump at the right end of the interval $[0,1]$), such that all the positive (respectively, negative) bumps have the same shape (the shapes of the positive and negative bumps may be different), and all these bumps are symmetric about their mid-points (so the idea of a 'half bump' makes sense). Now, recalling the definition of the set T_k^ν, and 'counting' the truncated bump, we can say, heuristically, that v_∞ has between $k-1/2$ and $k+1/2$ bumps. Also, letting d^+ (respectively, d^-) denote the width of a complete (untruncated) positive (respectively, negative) bump of v_∞, it follows from the estimate (5.7) that

$$\frac{(p-1)^{1/p}\pi_p}{(\mu_\infty\overline{\gamma}^\pm)^{1/p}} \le d^\pm \le \frac{(p-1)^{1/p}\pi_p}{(\mu_\infty\underline{\gamma}^\pm)^{1/p}},$$

by (5.8), the properties of π_p and $\sin_p$, and the Sturm comparison theorem (Theorem 6 in [8]).

Now suppose that $k = 2l$, for some integer l, and $\nu = +$. Then v_∞ has between l and $l+1/2$ positive bumps, and between $l-1/2$ and l negative bumps, and hence

$$\begin{aligned}(p-1)^{1/p}\pi_p\biggl[\frac{l}{(\mu_\infty\overline{\gamma}^+)^{1/p}} + \frac{(l-1/2)}{(\mu_\infty\overline{\gamma}^-)^{1/p}}\biggr]& \\ \le 1 \le (p-1)^{1/p}\pi_p\biggl[&\frac{(l+1/2)}{(\mu_\infty\underline{\gamma}^+)^{1/p}} + \frac{l}{(\mu_\infty\underline{\gamma}^-)^{1/p}}\biggr],\end{aligned}$$

from which we deduce that $\overline{\lambda}_{2l}^+ \le \mu_\infty \le \underline{\lambda}_{2l}^+$. The other cases can be dealt with similarly. $\square$

The proof of Theorem 5.7 can now completed by noting that the condition (5.6), together with the estimates in Lemma 5.8, ensures that the bifurcation point λ_k/f_∞ and the asymptote μ_∞ of $\mathcal{C}_k^\nu$ are on opposite sides of 1, and hence $\mathcal{C}_k^\nu$ must cross the hyperplane $\{1\} \times X$. $\square$

Remark 5.9. Most of the results in this paper can be extended to the case where the boundary condition at $x = 0$ is replaced by $u'(0) = 0$. The only changes required to deal with this case are to redefine X to incorporate this alternative boundary condition, to modify conditions (i) and (ii) in the definition of the sets T_k^ν, to:

(i) $u'(0) = 0$, $\nu u(0) > 0$ and $u'(1) \neq 0$,

(ii) $\phi_p(u')$ has only simple zeros in $[0,1)$, and has exactly k such zeros,

and to redefine the numbers $\underline{\lambda}^\nu$, $\overline{\lambda}^\nu$ appropriately. All results then follow by similar arguments, with the exception of part (ii) in Theorem 3.5, where ϕ_1 is no longer strictly positive, but ϕ_1' is now strictly negative on $(0,1)$.

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