APPLICATION
OF HE’S FREQUENCY-AMPLITUDE FORMULATION
TO THE DUFFING-HARMONIC OSCILLATOR

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Abstract. The work presents a derivation of frequency-amplitude of the Duffing-harmonic oscillator from a formulation suggested by Ji-Huan He. The obtained result is valid for all amplitudes, and its maximal error is less than 2.2%.

1. Introduction

Consider the Duffing-harmonic oscillator [1]–[4], [12] as follows

\[ \frac{d^2 u}{dt^2} + \frac{u^3}{1 + u^2} = 0, \quad u(0) = A, \quad \frac{du}{dt}(0) = 0. \]

Usually, it is difficult to find an accurate analytical approximation for (1.1). Several new methods have been applied to dealing with (1.1), such as the variational iteration method [6], [16], the homotopy perturbation method [9]–[11], [14], [17], the parameter-expanding method [10], [15], the exp-function method [13], [18], [19] and harmonic balance based methods [1], [12].

In this work, He’s frequency-amplitude formulation [7], [8], [11] originated from ancient Chinese mathematics was employed to solve the nonlinear oscillator. It is a rather simple and relatively accurate way to get an analytical approximate solution of the Duffing-harmonic nonlinear oscillator.

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2. Solution procedure

According to He’s frequency-amplitude formulation, we choose two trial-functions (initial solutions) [11]:

\[ u_1(t) = A \cos t \quad \text{and} \quad u_2(t) = A \cos \omega t. \]

Submitting the above trial-functions to (1.1) results in the following residuals:

\[ R_1(t) = -A \cos t \quad \text{and} \quad R_2(t) = -A\omega^2 \cos \omega t + (1 - \omega^2)A^3 \cos^3 \omega t. \]

He’s frequency-amplitude formulation requires that [11]:

\[ \omega^2 = \frac{\omega^2 R_2(t_2) - \omega^2 R_1(t_1)}{R_2(t_2) - R_1(t_1)} \]

where \( \omega_1 = 1 \) and \( \omega_2 = \omega \), are respectively the frequency of \( u_1 \) and \( u_2 \), and \( \omega \) is the frequency of the Duffing-harmonic oscillator, \( t_1 \) and \( t_2 \) are location points. Generally we let

\[ t_1 = \frac{T_1}{N}, \quad t_2 = \frac{T_2}{N} \]

where \( T_1 \) and \( T_2 \) are periods of the trial solutions \( u_1(t) = A \cos t \) and \( u_2(t) = A \cos \omega t \), respectively. In [11] \( N = 0 \), and in [5] \( N = 12 \). Setting \( N = 12 \), we obtain

\[
\omega^2 = \frac{-A \omega^2 \cos \frac{T_2}{N} + (1 - \omega^2)A^3 \cos^3 \frac{T_2}{N} - \omega^2 \left( -A \cos \frac{T_1}{N} \right)}{-A \omega^2 \cos \frac{T_2}{N} + (1 - \omega^2)A^3 \cos^3 \frac{T_2}{N} - \left( -A \cos \frac{T_1}{N} \right)} = \frac{3}{4} \frac{A^2}{1 + \frac{3}{4} A^2}
\]

i.e.

\[
\omega = \sqrt{\frac{\frac{3}{4} A^2}{1 + \frac{3}{4} A^2}}.
\]

The approximate period is

\[ T = 2\pi \sqrt{\frac{4}{3A^2} + 1}. \]

The approximate analytical solution has a considerable accuracy by comparing it with the numerical solution. See comparison of approximate solution \( u = A \cos \omega t \) with the numerical solution in Figures 1–5 (exact solution — continued line, approximate solution — dashed line).

Figures 1–5 show that the accuracy increases with the increase of the amplitude \( A \). In order to illustrate the accuracy of the approximate analytical result,
we compare the approximate solution with the exact solution. The exact period is \([2], [12]\)

\[
T_e(A) = 4A \int_0^1 \frac{du}{\sqrt{A^2(1-u^2) + \log((1 + A^2u^2)/(1 + A^2))}}.
\]

When \(A \to 0\),

\[
T_e \approx \frac{2\sqrt{2}K(-1)}{A} + \ldots = \frac{7.4163}{A} + \ldots
\]

So we have

\[
\lim_{A \to 0} \frac{T}{T_e} = \frac{4\pi \sqrt{3}}{3.4/7.4163} = 0.9783.
\]
The accuracy of 2.2% is a remarkable accuracy.

In the case $A \to \infty$, the original equation (1.1) can be reduced to

$$\frac{d^2u}{dt^2} + u = 0.$$ 

Its period is $T = 2\pi$. When $A \to \infty$, the approximate period is

$$\lim_{A \to \infty} T = \lim_{A \to \infty} 2\pi \sqrt{\frac{4}{3A^2} + 1} = 2\pi.$$ 

It agrees exactly with the exact period.
3. Conclusions

The He’s frequency-amplitude formulation is of remarkable convenience and of excellent accuracy, it can be easily applied to other nonlinear oscillators without any difficulty.

References


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