Topological Methods in Nonlinear Analysis
Journal of the Juliusz Schauder Center
Volume 31, 2008, 389–394

APPLICATION OF HE'S FREQUENCY-AMPLITUDE FORMULATION TO THE DUFFING-HARMONIC OSCILLATOR

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Abstract. The work presents a derivation of frequency-amplitude of the Duffing-harmonic oscillator from a formulation suggested by Ji-Huan He. The obtained result is valid for all amplitudes, and its maximal error is less than 2.2%.

1. Introduction

Consider the Duffing-harmonic oscillator [1]–[4], [12] as follows

$$\frac{d^2u}{dt^2} + \frac{u^3}{1+u^2} = 0, \quad u(0) = A, \quad \frac{du}{dt}(0) = 0. \tag{1.1}$$

Usually, it is difficult to find an accurate analytical approximation for (1.1). Several new methods have been applied to dealing with (1.1), such as the variational iteration method [6], [16], the homotopy perturbation method [9]–[11], [14], [17], the parameter-expanding method [10], [15], the exp-function method [13], [18], [19] and harmonic balance based methods [1], [12].

In this work, He's frequency-amplitude formulation [7], [8], [11] originated from ancient Chinese mathematics was employed to solve the nonlinear oscillator. It is a rather simple and relatively accurate way to get an analytical approximate solution of the Duffing-harmonic nonlinear oscillator..

2000 *Mathematics Subject Classification.* 01A05, 01A25 , 20C20.

Key words and phrases. Ancient Chinese mathematics, nonlinear oscillator, He's frequency-amplitude formulation.

2. Solution procedure

According to He's frequency-amplitude formulation, we choose two trial-functions (initial solutions) [11]:

$$u_1(t) = A\cos t \quad \text{and} \quad u_2(t) = A\cos \omega t.$$

Submitting the above trial-functions to (1.1) results in the following residuals:

$$R_1(t) = -A\cos t \quad \text{and} \quad R_2(t) = -A\omega^2 \cos \omega t + (1-\omega^2)A^3 \cos^3 \omega t.$$

He's frequency-amplitude formulation requires that [11]:

$$\omega^2 = \frac{\omega_1^2 R_2(t_2) - \omega_2^2 R_1(t_1)}{R_2(t_2) - R_1(t_1)}$$

where $\omega_1 = 1$ and $\omega_2 = \omega$, are respectively the frequency of u_1 and u_2, and ω is the frequency of the Duffing-harmonic oscillator, t_1 and t_2 are location points. Generally we let

$$t_1 = \frac{T_1}{N}, \quad t_2 = \frac{T_2}{N}$$

where T_1 and T_2 are periods of the trial solutions $u_1(t) = A\cos t$ and $u_2(t) = A\cos \omega t$, respectively. In [11] $N = 0$, and in [5] $N = 12$. Setting $N = 12$, we obtain

$$\omega^2 = \frac{-A\omega^2 \cos\dfrac{\omega T_2}{N} + (1-\omega^2)A^3\cos^3\dfrac{\omega T_2}{N} - \omega^2\Big(-A\cos\dfrac{T_1}{N}\Big)}{-A\omega^2 \cos\dfrac{\omega T_2}{N} + (1-\omega^2)A^3\cos^3\dfrac{\omega T_2}{N} - \Big(-A\cos\dfrac{T_1}{N}\Big)} = \frac{\dfrac{3}{4}A^2}{1+\dfrac{3}{4}A^2}$$

i.e.

$$\omega = \sqrt{\frac{\dfrac{3}{4}A^2}{1+\dfrac{3}{4}A^2}}.$$

The approximate period is

$$T = 2\pi\sqrt{\frac{4}{3A^2} + 1}.$$

The approximate analytical solution has a considerable accuracy by comparing it with the numerical solution. See comparison of approximate solution $u = A\cos\omega t$ with the numerical solution in Figures 1–5 (exact solution — continued line, approximate solution — dashed line).

Figures 1–5 show that the accuracy increases with the increase of the amplitude A. In order to illustrate the accuracy of the approximate analytical result,

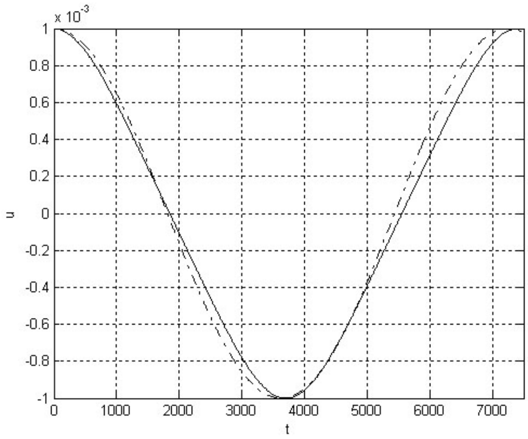

Figure 1. $A = 0.001$

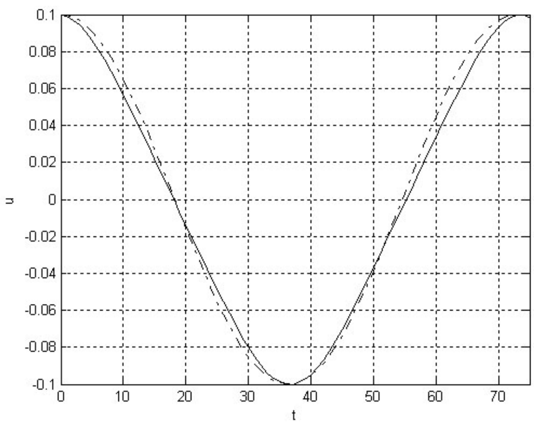

Figure 2. $A = 0.1$

we compare the approximate solution with the exact solution. The exact period is [2], [12]

$$T_e(A) = 4A \int_0^1 \frac{du}{\sqrt{A^2(1-u^2) + \log((1 + A^2u^2)/(1 + A^2))}}.$$

When $A \to 0$,

$$T_e \cong \frac{2\sqrt{2}K(-1)}{A} + \ldots = \frac{7.4163}{A} + \ldots$$

So we have

$$\lim_{A\to 0} \frac{T}{T_e} = \frac{\dfrac{4\pi\sqrt{3}}{3A}}{\dfrac{7.4163}{A}} = 0.9783.$$

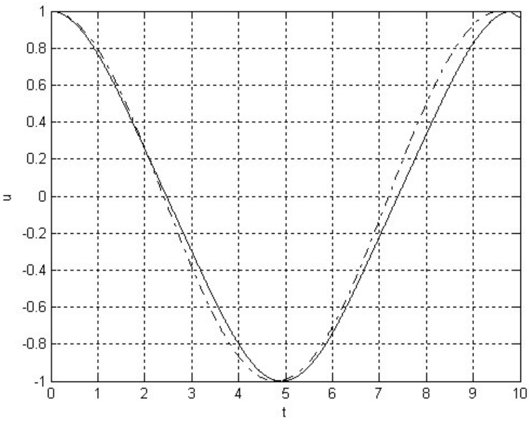

FIGURE 3. $A = 1$

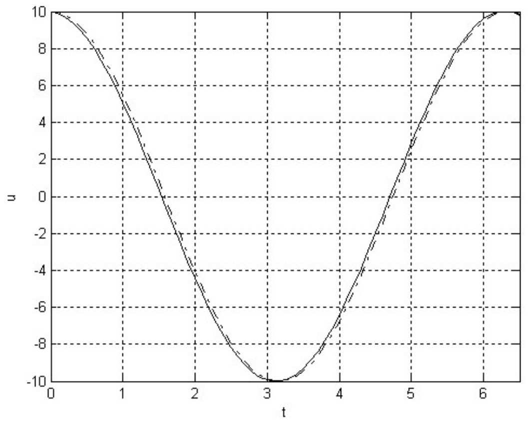

FIGURE 4. $A = 10$

The accuracy of 2.2% is a remarkable accuracy.

In the case $A \to \infty$, the original equation (1.1) can be reduced to

$$\frac{d^2u}{dt^2} + u = 0.$$

Its period is $T = 2\pi$. When $A \to \infty$, the approximate period is

$$\lim_{A\to\infty} T = \lim_{A\to\infty} 2\pi\sqrt{\frac{4}{3A^2} + 1} = 2\pi.$$

It agrees exactly with the exact period.

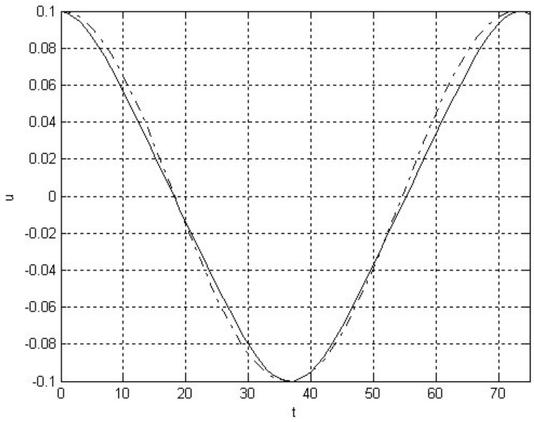

FIGURE 5. $A = 1000$

3. Conclusions

The He's frequency-amplitude formulation is of remarkable convenience and of excellent accuracy, it can be easily applied to other nonlinear oscillators without any difficulty.

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Manuscript received October 11, 2007

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