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CHINESE MATHEMATICS FOR NONLINEAR OSCILLATORS

Ling Zhao

Abstract. Ancient Chinese mathematicians made dramatic progress toward answering one of the oldest, most fundamental problem of how to solve approximately a real root of a nonlinear algebra equation in about 2nd century BC. The idea was further extended to nonlinear differential equations by J. H. He in 2002. In this paper, J. H. He's frequency-amplitude formation is used to find periodic solution of a pure nonlinear oscillator (without a linear term). The obtained result is of remarkable accuracy.

1. Introduction

In this paper, we consider the following nonlinear oscillator,

$$u'' + \varepsilon u^3, \quad u(0) = A, \quad u'(0) = 0. \tag{1.1}$$

No linear term is included in the above equation, and the classical perturbation method becomes invalid even when $\varepsilon \ll 1$.

Recently, many new approaches to nonlinear oscillators have been proposed. For example, the variational iteration method [3], [11], [15], the homotopy perturbation method [1], [5], [6], [12], [13], [16], the parameter-expanding method [7], [14], exp-function method [10], [17], [18]. A review on recently developed analytical methods is available in papers [7] and [8]. In this paper we will use an ancient Chinese method to solve the problem.

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2. An ancient Chinese method

Jiu Zhang Suan Shu, "Nine Chapters on the Art of Mathematics", comprised of nine chapters and hence its title, is the oldest and most influential work in the history of Chinese mathematics. The Chapter 7 of the "Nine Chapters" is the Ying Buzu Shu (literally "Method of Surplus and Deficiency") [4], [9], an ancient Chinese algorithm, which is the oldest method for approximating real roots of a nonlinear equation in about the 2nd century B.C. known as the rule of double false position in the West after 1202 A.D.

To illustrate the basic idea of the method, we consider an algebraic equation,

$$f(x) = 0.$$

Let x_1 and x_2 be the approximate solutions of the equation, which lead to the remainders and, respectively, the ancient Chinese algorithm leads to the result

$$x = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}.$$

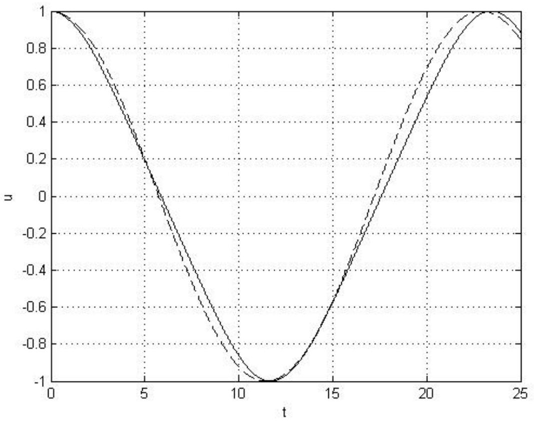

FIGURE 1. $\varepsilon A^2 = 0.1$

3. He's frequency-amplitude relationship

We consider a generalized nonlinear oscillator in the form

$$u'' + f(u) = 0, \quad u(0) = A, \quad u'(0) = 0.$$

We use two trial functions $u_1(t)A\cos t$ and $u_2 = A\cos\omega t$, which are, respectively, the solutions of the following linear oscillator equations (see [7])

$$u'' + \omega_1^2 u = 0, \quad \omega_1^2 = 1 \quad \text{and} \quad u'' + \omega_2^2 u = 0, \quad \omega_2^2 = \omega^2$$

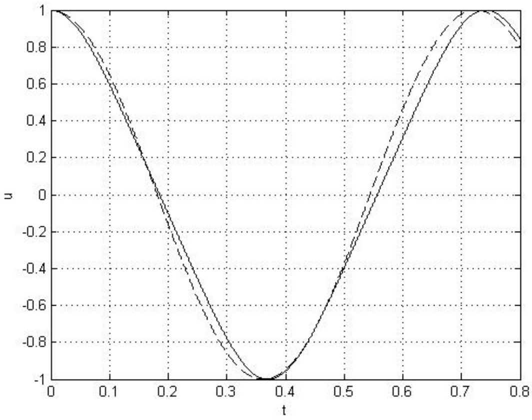

Figure 2. $\varepsilon A^2 = 100$

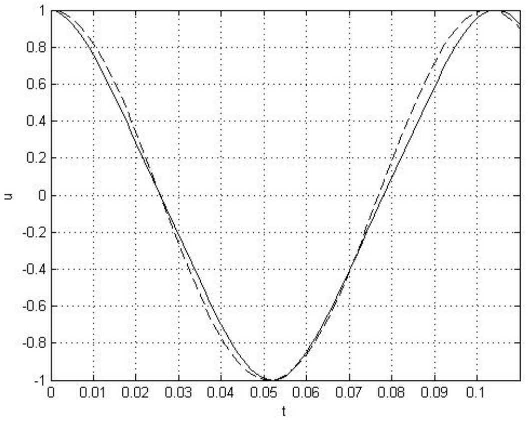

Figure 3. $\varepsilon A^2 = 5000$

where ω is assumed to be the frequency of the nonlinear oscillator. The residuals are

$$R_1(t) = -cost + f(A\cos t), \quad R_2(t) = -\omega^2 \cos t + f(A\cos\omega t).$$

According to the ancient Chinese method, we have (see [7]):

$$\omega^2 = \frac{\omega_1^2 R_2(T_2/N) - \omega_2^2 R_1(T_1/N)}{R_2(T_2/N) - R_1(T_1/n)}$$

where $T_1 = 2\pi$ and $T_2 = 2\pi/\omega$, $N \in (0, \infty)$. In practice, the value of N is chosen as $N = 12$.

For equations (1.1), we have

$$R_1(t) = \varepsilon A^3 \cos^3 t - A\cos t, \quad R_2(t) = -A\omega^2 \cos\omega t + \varepsilon A^3 \cos^3 \omega t.$$

Using frequency formulation, and setting $N = 12$ (see [2]), we have

$$\omega^2 = \frac{\omega_1^2 R_2(T_2/12) - \omega_2^2 R_1(T_1/12)}{R_2(T_2/12) - R_1(T_1/12)} = frac3\varepsilon A^2 4$$

or

$$\omega = \sqrt{\frac{3\varepsilon A^2}{4}}$$

which has high accurancy for all $A > 0$.

Figures 1–3 show comparison of the approximate solution $u = A\cos wt$ of the equation above with the exact one. Exact solution is denotet by —; approximate solution by - - -.

4. Conclusion

The present work is a short note on He's frequency-amplitude formation. It is obvious that the formulation is a unifying framework for finding approximately periodic solutions of nonlinear oscillators. We conjecture that in the future He's formulation will be playing a major role.

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LING ZHAO
Modern Textile Institute
Donghua University
1882 Yan-an Xilu Road
Shanghai 200051, P. R. CHINA

E-mail address: ijnsns@yahoo.com.cn