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HOMOTOPY PERTURBATION METHOD FOR THE NONLINEAR RELATIVISTIC TODA LATTICE EQUATIONS

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Abstract. The work presents a derivation of solitary wave solutions of the nonlinear relativistic Toda lattice equations using the homotopy perturbation method.

1. Introduction

Since the work of Fermi, Pasta and Ulam in the 1950s [7], the investigation of exact solutions of the nonlinear differential-difference equations (DDEs) has played a crucial role in the modeling of many phenomena in different fields. It is known to all that discrete solitons exist in atomic chains [26] (discrete lattices) with on-site cubic nonlinearities, molecular crystals [5], biophysical systems [23], electrical lattices [21] and in arrays of coupled nonlinear optical wave guides [6], [24]. Recent study also reveals that discrete solitons appear in photorefractive optically induced photonic lattices [8], observation of lattice solitons in two-dimensional systems was reported by Fleischer et al. [4] and Chen et al. [28]. Therefore, the properties of the solitons in nonlinear lattices have been the focus of considerable studies in various fields of natural science [31] and [32]. Unlike difference equations which are fully discretized, differential-difference equations

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are semi-discretized, with some (or all) of their spatial variables discretized, while time variable is usually kept continuous. A wealth of information about integrable nonlinear differential-difference equations (NDDEs) can be found in papers by Suris [12], [28], [30].

The homotopy perturbation method (HPM) [13]–[15] is powerful in investigating the approximate or analytical solutions of the nonlinear differential equations. The method does not depend on a small parameter in the equation. Using homotopy technique in topology, a homotopy is constructed with an embedding parameter $p \in [0,1]$ which is considered as an expanding parameter. The method was successfully applied to nonlinear oscillators with discontinuities [16] and bifurcation of nonlinear problems [17]. In [18], a comparison of HPM and homotopy analysis method was made, revealing that the former is more powerful than the latter. The HPM was proposed to search for limit cycles or bifurcation curves of nonlinear equations [19]. In [20], a heuristic example was given to illustrate the basic idea of the homotopy perturbation method and its advantages over the Adomian-method, and also this method was applied to solve boundary value problems [9] and heat radiation equations [11]. Recently, many researchers did a lot of significant work about the homotopy perturbation method [1], [3], [10], [25].

In this paper, we extend HPM to solve the Relativistic Toda Lattice equations, and the accuracy of the extended method is investigated as well. The first Relativistic Toda Lattice equation is as follows [27]

$$
\begin{aligned}
\frac{du_n}{dt} &= (u_{n+1} - v_n)v_n - (u_{n-1} - v_{n-1})v_{n-1}, \\
\frac{dv_n}{dt} &= v_n(u_{n-1} - u_n),
\end{aligned}
\tag{1.1}
$$

where the subscript n represents the nth lattice. Its integrability was studied in [27]. The polynomial traveling wave solution in tanh can be found in [2] and [22]. Zhu [31], [32] applied the Exp-function method to the nonlinear differential-difference equations (DDEs), and found many new solitary solutions.

2. Basic idea of He's homotopy perturbation method

We consider the following nonlinear differential equation:

$$
A(u) - f(r) = 0, \quad r \in \Omega, \tag{2.1}
$$

with the boundary conditions

$$
B(u, \partial u/\partial n) = 0, \quad r \in \Gamma,
$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytical function and Γ is the boundary of the domain Ω.

Generally speaking, the operator A can be decomposed into two operators, L and N, where L is linear, and N is a nonlinear operator. Equation (2.1) can therefore be rewritten as follows:

$$L(u) + N(u) - f(r) = 0.$$

By the homotopy technique, we construct a homotopy $V\colon \Omega \times [0,1] \to \mathbb{R}$ and we let:

$$H(V,p) = (1-p)[L(u) - l(u_0)] + p[A(V) - f(r)] = 0, \quad p \in [0,1],\ r \in \Omega \tag{2.2}$$

or

$$H(V,p) = L(u) - l(u_0) + pl(u_0) + p[N(V) - f(r)] = 0, \tag{2.3}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation of (2.1), which satisfies the boundary conditions. Obviously, from (2.2) and (2.3), we will have:

$$H(V,0) = L(u) - l(u_0) = 0, \qquad H(V,1) = A(V) - f(r) = 0.$$

The changing process of p from zero to unity is just that of $V(r,p)$ from $u_0(r)$ to $u(r)$.

According to the HPM, we can first use the embedding parameter p as a "small parameter", and assume that the solution of (2.2)–(2.3) can be written as a power series in p:

$$V = V_0 + pV_1 + p^2V_2 + \dots$$

Setting $p = 1$ results in the approximate solution of (2.1):

$$V = \lim_{p=1} V = V_0 + V_1 + V_2 + \dots$$

The combination of the perturbation method and the homotopy method is called HPM, which has eliminated the limitations of the traditional perturbation methods. On the other hand, this technique can have full advantage of the traditional perturbation techniques.

3. Analysis of the HPM

To investigate the traveling wave solution of (1.1), we first construct a homotopy as follows:

$$\begin{aligned} (1-p)&\left(\frac{du_n}{dt} - \frac{du_{n0}}{dt}\right) \\ &+ p\left[\frac{du_n}{dt} - (u_{n+1} - v_n)v_n + (u_{n-1} - v_{n-1})v_{n-1}\right] = 0, \end{aligned} \tag{3.1}$$

$$(1-p)\left(\frac{dv_n}{dt}-\frac{dv_{n0}}{dt}\right)+p\left[\frac{dv_n}{dt}-v_n(u_{n+1}-u_n)\right]=0. \tag{3.2}$$

Suppose the solution of (3.1)–(3.2) and the initial approximations are as follows:

$$u_{n0}(n,t)=u_n(n,0),\quad v_{n0}(n,t)=v_n(n,0), \tag{3.3}$$

$$\begin{aligned} u_n(n,t) &= U_n(n,t)=u_{n0}+pu_{n1}+p^2u_{n2}+p^3u_{n3}+\dots, \\ v_n(n,t) &= V_n(x,t)=v_{n0}+pv_{n1}+p^2v_{n2}+p^3v_{n3}+\dots \end{aligned} \tag{3.4}$$

where u_{ni}, v_{ni} $(i=1,2,\dots)$ are functions of (n,t) yet to be determined. Substituting (3.4) into (3.1)–(3.2), and equating the coefficients of the terms with the identical powers of p, we have

$$\begin{aligned} &\left(\frac{du_{n1}}{dt}+\frac{du_{n0}}{dt}+u_{(n-1)0}v_{(n-1)0}-u_{(n-1)0}v_{n0}-v_{(n-1)0}^2+v_{n0}^2\right)p \\ &\quad+\left(\frac{du_{n2}}{dt}-u_{(n+1)1}v_{n0}+2v_{n0}v_{n1}+v_{(n-1)1}v_{(n-1)0}\right. \\ &\quad\left.-u_{(n+1)0}v_{n1}+u_{(n-1)1}v_{(n-1)0}-2v_{(n-1)0}v_{(n-1)1}\right)p^2+\dots=0, \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\left[\frac{dv_{n1}}{dt}+\frac{dv_{n0}}{dt}-v_{n0}(u_{(n+1)0}-u_{n0})\right]p \\ &\quad+\left[\frac{dv_{n2}}{dt}-v_{n0}(u_{(n+1)1}-u_{n1})-v_{n1}(u_{(n+1)0}-u_{n0})\right]p^2+\dots=0. \end{aligned} \tag{3.6}$$

In order to obtain the unknowns of u_{ni}, v_{ni}, $(i=1,2,\dots)$, we have to construct and solve the following system, considering the initial approximations of equations (3.3)

$$\frac{du_{n1}}{dt}+\frac{du_{n0}}{dt}+u_{(n-1)0}v_{(n-1)0}-u_{(n-1)0}v_{n0}-v_{(n-1)0}^2+v_{n0}^2=0, \tag{3.7}$$

$$\begin{aligned} &\frac{du_{n2}}{dt}-u_{(n+1)1}v_{n0}+2v_{n0}v_{n1}+v_{(n-1)1}v_{(n-1)0} \\ &\qquad-u_{(n+1)0}v_{n1}+u_{(n-1)1}v_{(n-1)0}-2v_{(n-1)0}v_{(n-1)1}=0, \end{aligned} \tag{3.8}$$

$$\frac{dv_{n1}}{dt}+\frac{dv_{n0}}{dt}-v_{n0}(u_{(n+1)0}-u_{n0})=0, \tag{3.9}$$

$$\frac{dv_{n2}}{dt}-v_{n0}(u_{(n+1)1}-u_{n1})-v_{n1}(u_{(n+1)0}-u_{n0})=0. \tag{3.10}$$

If the first three approximations are sufficient, we will obtain:

$$u_n(n,t)=\lim_{p\to1}U_n(n,t)=\sum_{k=0}^{k=2}u_{nk}(n,t), \tag{3.11}$$

$$v_n(n,t)=\lim_{p\to1}V_n(n,t)=\sum_{k=0}^{k=2}v_{nk}(n,t). \tag{3.12}$$

4. Application

Firstly, we consider the solution of (1.1) with the initial conditions:

$$u_{n0}(n,t) = c[\coth(k) + \tanh(kn)], \quad v_{n0}(n,t) = c[\coth(k) + \tanh(kn)], \tag{4.1}$$

where k and c are arbitrary constants. Equation (1.1) has a kink-type soliton solution [22], which reads

$$\begin{aligned} u_n(n,t) &= c[\coth(k) + \tanh(kn+ct)], \\ v_n(n,t) &= c[\coth(k) + \tanh(kn+ct)]. \end{aligned} \tag{4.2}$$

To calculate the terms of the homotopy series (3.11)–(3.12) for $u_n(n,t)$ and $v_n(n,t)$, we substitute the initial conditions (4.1) and equations (3.11)–(3.12) into the (3.7)–(3.10), the solutions of the equation can be obtained as follows:

$$\begin{aligned}
u_{n0}(n,t) =& c[\coth(k) + \tanh(kn)], \\
u_{n1}(n,t) =& \frac{2c^2 t}{1+\cosh(2nk)}, \\
u_{n2}(n,t) =& [\cosh(4nk+4k) - 3\cosh(2k) - 3\cosh(4nk+2k) + 4\cosh(2nk+2k) \\
& - \cosh(4nk+2k) - \cosh(2nk+2k) + 3]4c^3t^2/[\sinh(6nk+2k) \\
& - \sinh(6nk+4k) + 3\sinh(2nk-2k) - 3\sinh(2nk+4k) \\
& + 3\sinh(4nk) - \sinh(4k) - 3\sinh(4nk+4k) \\
& - 8\sinh(2k) - 6\sinh(2nk+2k) + 6\sinh(2nk)], \\
v_{n0}(n,t) =& c[\coth(k) + \tanh(kn)], \\
v_{n1}(n,t) =& \frac{2c^2 t}{1+\cosh(2nk)}, \\
v_{n2}(n,t) =& [\cosh(2nk+4k) - \cosh(4nk+4k) + 3\cosh(2k) + 3\cosh(2nk) \\
& - 4\cosh(2nk+2k) + \cosh(4nk+4k) + 3]4c^3t^3/[\sinh(6nk+4k) \\
& - \sinh(6nk+2k) + 3\sinh(2nk+4k) - 3\sinh(2nk-2k) \\
& - 3\sinh(4nk) + \sinh(4k) + 3\sinh(4nk+4k) \\
& + 8\sinh(2k) + 6\sinh(2nk+2k) - 6\sinh(2nk)].
\end{aligned}$$

Similarly we can easily obtain $u_{n3}, v_{n3}, u_{n4}, v_{n4}, \ldots$ Substituting u_{n0}, u_{n1}, u_{n2}, $v_{n0}, v_{n1}, v_{n2}, \ldots$ into equations (3.11)–(3.12) yields

$$u_n(n,t) = u_{n0} + u_{n1} + u_{n2} + \ldots, \quad v_n(n,t) = v_{n0} + v_{n1} + v_{n2} + \ldots$$

In order to verify numerically whether the proposed methodology leads to high accuracy, we evaluate the numerical solution using the 3-term approximation and compare it with the exact analytical solution (4.2). The accuracy of the HPM for the first relativistic Toda lattice equation is controllable, and absolute errors are very small. These results are listed in Tables 1–3, it is seen that the implemented method achieves a minimum accuracy for the first three approximations for the initial condition (3.11)–(3.12). It is also evident that when more terms are computed the numerical results get much closer to the corresponding exact solutions with the initial conditions (3.11)–(3.12) of (1.1).

n	$\lvert u_{ne} - u_{nh}\rvert$	$\lvert(u_{ne} - u_{nh})/u_{ne}\rvert$	$\lvert v_{ne} - v_{nh}\rvert$	$\lvert(v_{ne} - v_{nh})/v_{ne}\rvert$
-15	0	0	0	0
-5	2.0E−12	2.012144094E−10	2.0E−12	2.012144094E−10
-4	3.96E−10	3.984044906E−08	3.96E−10	3.984044906E−08
-3	1.4532E−07	1.461957711E−05	1.4532E−07	1.461957711E−05
3	5.3E−08	1.321715656E−08	5.3E−08	1.321715656E−08
4	0	0	0	0
5	0	0	0	0
15	0	0	0	0

TABLE 1. The HPM results for u_n for the first three approximation in comparison with the analytical solutions when $c = 2$, $k = 3$, $t = 0.5$, for the solitary wave solutions with the initial conditions (4.1) of equations (1.1), respectively. (u_{ne} equal to $u_{n\text{exact}}$, u_{nh} is $u_{n\text{homotopy}}$, v_{ne} equal to $v_{n\text{exact}}$, v_{nh} is $v_{n\text{homotopy}}$)

n	$\lvert u_{ne} - u_{nh}\rvert$	$\lvert(u_{ne} - u_{nh})/u_{ne}\rvert$	$\lvert v_{ne} - v_{nh}\rvert$	$\lvert(v_{ne} - v_{nh})/v_{ne}\rvert$
-15	0	0	0	0
-5	4.0E−12	4.024288189E−10	4.0E−12	4.024288189E−10
-4	6.188E−09	6.225568818E−07	6.188E−09	6.225568818E−07
-3	2.533961E−06	2.548494555E−04	2.533961E−06	2.548494555E−04
3	3.03E−07	7.556223455E−08	3.03E−07	7.556223455E−08
4	0	0	0	0
5	0	0	0	0
15	0	0	0	0

TABLE 2. The HPM results for u_n for the first three approximation in comparison with the analytical solutions when $c = 2$, $k = 3$, $t = 1$, for the solitary wave solutions with the initial conditions (4.1) of equations (1.1), respectively. (u_{ne} equal to $u_{n\text{exact}}$, u_{nh} is $u_{n\text{homotopy}}$, v_{ne} equal to $v_{n\text{exact}}$, v_{nh} is $v_{n\text{homotopy}}$)

n	$\lvert u_{ne} - u_{nh}\rvert$	$\lvert (u_{ne} - u_{nh})/u_{ne}\rvert$	$\lvert v_{ne} - v_{nh}\rvert$	$\lvert (v_{ne} - v_{nh})/v_{ne}\rvert$
-15	0	0	0	0
-5	9.0E−12	9.054648425E−10	9.0E−12	9.054648425E−10
-4	5.7376E−08	5.772403553E−06	5.7376E−08	5.772403553E−06
-3	2.3053922E−05	2.313669816E−03	2.3053922E−05	2.313669816E−03
3	7.92E−07	1.975092071E−07	7.92E−07	1.975092071E−07
4	2.0E−09	4.98760624E−10	2.0E−09	4.98760624E−10
5	0	0	0	0
15	0	0	0	0

Table 3. The HPM results for u_n for the first three approximation in comparison with the analytical solutions when $c = 2$, $k = 3$, $t = 1.5$, for the solitary wave solutions with the initial conditions (4.1) of equations (1.1), respectively (u_{ne} equal to $u_{n\text{exact}}$, u_{nh} is $u_{n\text{homotopy}}$, v_{ne} equal to $v_{n\text{exact}}$, v_{nh} is $v_{n\text{homotopy}}$)

5. Conclusions

In this paper, the HPM was used for finding soliton solutions of the nonlinear Relativistic Toda Lattice equations with initial conditions. It can be concluded that the HPM is a very powerful and efficient technique in finding exact solutions for wide classes of problems. It is worth pointing out that the HPM presents a rapid convergence for the solutions. The HPM does not require small parameters in the equation, so that the limitations of the traditional perturbation methods can be eliminated, and also the calculations in the HPM are simple and straightforward. The reliability of the method and the reduction in the size of computational domain give to this method a wider applicability. The results show that the HPM is a powerful mathematical tool for solving systems of nonlinear partial differential equations.

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