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HOMOTOPY PERTURBATION SOLUTION FOR PERISTALTIC FLOW OF A THIRD ORDER FLUID

A. M. Siddiqui — Q. A. Azim — A. Ashraf — Q. K. Ghori

Abstract. The peristaltic transport of a third order fluid in a planar channel as well as in an axisymmetric tube having walls that are transversely displaced by an infinite, harmonic travelling wave of large wavelength and negligibly small Reynolds number, has been analyzed using homotopy perturbation technique. Unlike perturbation method, this method does not restrict the Deborah number Γ to be very large or small and works fairly well for any choice of Γ. The expressions for stream function and pressure rise per wavelength have been obtained up to second order of approximation.

1. Introduction

Peristalsis is an important mechanism for mixing and transporting fluids generated by a progressive wave of contraction or expansion moving on the wall of the tube. Common physiological examples are the oesophagus, gastrointestinal tract, small blood vessels. Roller and finger pumps also operate on this principle.

A number of analytical, numerical and experimental studies of peristaltic flow of Newtonian fluids have been reported [10] but only limited information on the transport of non-Newtonian fluids is available. The peristaltic transport of some non-Newtonian fluids is studied in [4], [16]–[18].

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To avoid complexity of nonlinear equations, we consider the planar and pipe flows of a third order fluid for the case of a symmetric, harmonic, infinite wave train with long wavelength approximation and transverse displacements only. We then employ Homotopy perturbation method due to He [5]–[9] for the solution of the problem. This method has earlier been employed by Siddiqui et al. in [3], [11]–[15] and others in [1] for the study of some non-Newtonian fluid flow problems. However, homotopy perturbation technique has not yet been used for the solution of problems regarding peristaltic flows. Expressions for the pressure gradient per unit wavelength and the stream function up to the second order of approximation are obtained in terms of flow rate, occlusion and the Deborah number.

2. Basic equations

Basic equations governing a flow of an incompressible fluid are given by the laws of conservation of mass and conservation of momentum

$$\nabla \cdot \mathbf{V} = 0, \tag{2.1}$$

$$\rho \frac{D\mathbf{V}}{Dt} = \nabla \cdot \mathbf{S} \tag{2.2}$$

where $\mathbf{V}$ is the velocity field, $\mathbf{S}$ the stress tensor, ρ the constant fluid density and D/Dt denotes the material derivative. The constitutive equation for a thermodynamically compatible third order fluid as proposed by Rajagopal [2] is

$$\mathbf{S} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta(\mathrm{tr}\,\mathbf{A}_1^2)\mathbf{A}_1 \tag{2.3}$$

where p is the fluid pressure, μ the coefficient of viscosity, α_1, α_2, β the material constants and A_1, A_2 are the Rivlin Ericksen tensors defined by

$$\mathbf{A}_1 = (\nabla\mathbf{V}) + (\nabla\mathbf{V})^T \tag{2.4}$$

$$\mathbf{A}_n = \frac{D\mathbf{A}_{n-1}}{Dt} + \mathbf{A}_{n-1}(\nabla\mathbf{V}) + (\nabla\mathbf{V})^T\mathbf{A}_{n-1}, \quad n = 2, 3, \ldots \tag{2.5}$$

3. Problem statement

3.1. Peristaltic flow in a planar channel. As discussed in [18] on the peristaltic flow of a third order fluid in a planar channel, equations (2.1)–(2.5) reduce to a single dimensionless equation in a wave frame

$$\frac{\partial^4\Psi}{\partial y^4} + 2\Gamma\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2\Psi}{\partial y^2}\right) = 0 \tag{3.1}$$

where Ψ is the stream function and Γ is the Deborah number. For pressure gradient, which is independent of y, we have

$$\frac{\partial^3\Psi}{\partial y^3} + 2\Gamma\frac{\partial}{\partial y}\left(\frac{\partial^2\Psi}{\partial y^2}\right) = \frac{dp}{dx} \tag{3.2}$$

and the pressure rise per wavelength is given by

$$\Delta P_\lambda = \int_0^{2\pi} \frac{dp}{dx}\, dx$$

where λ is the wavelength. The boundary conditions associated with this problem are

$$\Psi = 0, \quad \frac{\partial^2 \Psi}{\partial y^2} = 0 \quad \text{at } y = 0, \tag{3.3}$$

$$\Psi = F, \quad \frac{\partial \Psi}{\partial y} = -1 \quad \text{at } y = h \tag{3.4}$$

where h is the dimensionless wall of the tube given by

$$h = 1 + \Phi \sin x$$

with amplitude ratio or occlusion Φ, and $2\pi F$ indicates the dimensionless volume flux.

3.2. Peristaltic flow in an axisymmetric tube. For the peristaltic flow of a third order fluid in an axisymmetric tube, we follow paper [4] to obtain in a dimensionless wave frame a single dimensionless equation

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left\{r\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Psi}{\partial r}\right) + 2\Gamma\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Psi}{\partial r}\right)\right)^3\right)\right\}\right] = 0 \tag{3.5}$$

where Ψ is the stream function and Γ is the Deborah number. For pressure gradient, which is independent of r, we have

$$\frac{1}{r}\frac{\partial}{\partial r}\left\{r\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Psi}{\partial r}\right) + 2\Gamma\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Psi}{\partial r}\right)\right)^3\right)\right\} = \frac{dp}{dz} \tag{3.6}$$

and the pressure rise per wavelength is given by

$$\Delta P_\lambda = \int_0^{2\pi} \frac{dp}{dz}\, dz$$

where λ is the wavelength. The boundary conditions associated with this problem are

$$\Psi = 0, \quad \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Psi}{\partial r}\right) = 0 \quad \text{at } r = 0, \tag{3.7}$$

$$\Psi = F, \quad \frac{1}{r}\frac{\partial \Psi}{\partial r} = -1 \quad \text{at } r = h \tag{3.8}$$

where h is the dimensionless wall of the tube given by $h = 1 + \Phi \sin z$ with amplitude ratio or occlusion Φ, and $2\pi F$ indicates the dimensionless volume flux.

4. Solution of the problem

4.1. Peristaltic flow in a planar channel. We apply the homotopy perturbation method to the non-linear differential equation (3.1) subject to boundary conditions (3.3) and (3.4). First we define a homotopy $\omega(y,q)\colon \Omega\times[0,1]\to\mathbb{R}$ for (3.1) which satisfies

$$(1-q)[L(\Psi)-L(\psi_0)]+q\left[\frac{\partial^4\Psi}{\partial y^4}+2\Gamma\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2\Psi}{\partial y^2}\right)^3\right]=0 \tag{4.1}$$

where $L(\,\cdot\,)=\partial^4(\,\cdot\,)/\partial y^4$ is a linear operator, q is an embedding parameter and ψ_0 is the initial guess. Equivalently, (4.1) may be written as

$$L(\Psi)-L(\psi_0)+qL(\psi_0)+q\left[2\Gamma\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2\Psi}{\partial y^2}\right)^3\right]=0 \tag{4.2}$$

We assume that (4.2) along with (3.3) and (3.4) has a solution of the form

$$\Psi=\Psi_0+q\Psi_1+q^2\Psi_2+\dots \tag{4.3}$$

We also assume that

$$F=F_0+qF_1+q^2F_2+\dots,\qquad p=p_0+qp_1+q^2p_2+\dots \tag{4.4}$$

By substituting (4.3) and (4.4) into (4.2), (3.2), (3.3) and (3.4) and collecting the coefficients of various powers of q, we obtain

4.1.1. Zeroth order problem and solution. The zeroth order system is

$$\begin{gathered}L(\Psi_0)=L(\psi_0),\\ \frac{dp_0}{dx}=\left[\frac{\partial^3\Psi_0}{\partial y^3}+2\Gamma\frac{\partial}{\partial y}\left(\frac{\partial^2\Psi_0}{\partial y^2}\right)^3\right]_{\text{at }y=0},\end{gathered}$$

$$\begin{aligned}\Psi_0&=0, & \frac{\partial^2\Psi_0}{\partial y^2}&=0 &&\text{at } y=0,\\ \Psi_0&=F_0, & \frac{\partial\Psi_0}{\partial y}&=-1 &&\text{at } y=h.\end{aligned}$$

and its solution is given by

$$\begin{aligned}\Psi_0&=\psi_0=\frac{A_1}{2}\left(\frac{y^3}{3}-h^2y\right)-y,\\ \frac{dp_0}{dx}&=A_1,\\ \Delta P_{\lambda0}&=-3[F_0I_3+I_2],\end{aligned} \tag{4.5}$$

where

$$A_1 = -\frac{3}{h^3}(F_0 + h),$$

$$I_n = \int_0^{2\pi} \frac{dx}{(1+\Phi\sin x)^n}, \quad n = 1, 2, \ldots$$

$$I_1 = \frac{2\pi}{(1-\Phi^2)^{1/2}}, \quad I_2 = \frac{2\pi}{(1-\Phi^2)^{3/2}}, \quad I_3 = \frac{\pi(\Phi^2+2)}{(1-\Phi^2)^{5/2}},$$

$$I_n = \frac{1}{1-\Phi^2}\left[\left(\frac{2n-3}{n-1}\right)I_{n-1} - \left(\frac{n-2}{n-1}\right)I_{n-2}\right], \quad n > 4.$$

4.1.2. First order problem and solution. The system of the first order is

$$\frac{\partial^4 \Psi_1}{\partial y^4} + 2\Gamma \frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 \Psi_0}{\partial y^2}\right)^3 = 0,$$

$$\frac{dp_1}{dx} = \left[\frac{\partial^3 \Psi_1}{\partial y^3} + 6\Gamma \frac{\partial}{\partial y}\left\{\left(\frac{\partial^2 \Psi_0}{\partial y^2}\right)^2 \left(\frac{\partial^2 \Psi_1}{\partial y^2}\right)\right\}\right]_{\text{at } y=0},$$

$$\Psi_1 = 0, \quad \frac{\partial^2 \Psi_1}{\partial y^2} = 0 \quad \text{at } y = 0,$$

$$\Psi_1 = F_1, \quad \frac{\partial \Psi_1}{\partial y} = 0 \quad \text{at } y = h$$

and its solution is

$$\Psi_1 = -\frac{\Gamma A_1^3}{2}\left(\frac{y^5}{5} - h^4 y\right) + \frac{B_1}{2}\left(\frac{y^3}{3} - h^2 y\right),$$

(4.6)
$$\frac{dp_1}{dx} = B_1,$$

$$\Delta P_{\lambda 1} = -3F_1 I_3 - \frac{162\Gamma}{5}[F_0^3 I_7 + 3F_0^2 I_6 + 3F_0 I_5 + I_4],$$

where

$$B_1 = -\frac{3}{h^3}\left(F_1 - \frac{2\Gamma}{5}A_1^3 h^5\right).$$

4.1.3. Second order problem and solution. The second order system is

$$\frac{\partial^4 \Psi_2}{\partial y^4} + 6\Gamma \frac{\partial^2}{\partial y^2}\left\{\left(\frac{\partial^2 \Psi_0}{\partial y^2}\right)^2 \left(\frac{\partial^2 \Psi_1}{\partial y^2}\right)\right\} = 0,$$

$$\frac{dp_2}{dx} = \left[\frac{\partial^3 \Psi_2}{\partial y^3} + 6\Gamma \frac{\partial}{\partial y}\left\{\left(\frac{\partial^2 \Psi_0}{\partial y^2}\right)^2 \left(\frac{\partial^2 \Psi_2}{\partial y^2}\right) + \left(\frac{\partial^2 \Psi_0}{\partial y^2}\right)\left(\frac{\partial^2 \Psi_1}{\partial y^2}\right)^2\right\}\right]_{at\ y=0},$$

$$\Psi_2 = 0, \quad \frac{\partial^2 \Psi_2}{\partial y^2} = 0 \quad \text{at } y = 0,$$

$$\Psi_2 = F_2, \quad \frac{\partial \Psi_2}{\partial y} = 0 \quad \text{at } y = h.$$

Its solution is given by

$$(4.7)\quad \begin{aligned} \Psi_2 &= 2\Gamma^2 A_1^5\left(\frac{y^7}{7} - h^6 y\right) - \frac{3}{2}\Gamma A_1^2 B_1\left(\frac{y^5}{5} - h^4 y\right) + \frac{C_1}{2}\left(\frac{y^3}{3} - h^2 y\right), \\ \frac{dp_2}{dx} &= C_1, \\ \Delta P_{\lambda 2} &= -3F_2 I_3 - \frac{486\Gamma}{5} F_1[F_0^2 I_7 + 2F_0 I_6 + I_5] \\ &\quad - \frac{34992\Gamma^2}{175}[F_0^5 I_{11} + 5F_0^4 I_{10} + 10F_0^3 I_9 + 10F_0^2 I_8 + 5F_0 I_7 + I_6], \end{aligned}$$

where

$$C_1 = -\frac{3}{h^3}\left(F_2 + \frac{12}{7}\Gamma^2 A_1^5 h^7 - \frac{6}{5}\Gamma A_1^2 B_1 h^5\right).$$

4.1.4. Solution up to second order of approximation. If Ψ is the solution of (4.2), then according to the construction of a homotopy in (4.1), we have

$$\Psi = \lim_{q\to 1}(\Psi_0 + q\Psi_1 + q^2\Psi_2 + \dots).$$

Using the results obtained in (4.5)–(4.7) we find that

$$\begin{aligned} \Psi &= 2\Gamma^2 A_1^5\left(\frac{y^7}{7} - h^6 y\right) - \frac{\Gamma A_1^2(A_1 + 3B_1)}{2}\left(\frac{y^5}{5} - h^4 y\right) \\ &\quad + \frac{A_1 + B_1 + C_1}{2}\left(\frac{y^3}{3} - h^2 y\right) - y, \\ \frac{dp}{dx} &= A_1 + B_1 + C_1, \\ \Delta P_\lambda &= -3[F_2 I_3 + F_1 I_3 + F_0 I_3 + I_2] - \frac{162\Gamma}{5}[F_0^3 I_7 + 3F_0^2 I_6 + 3F_0 I_5 + I_4] \\ &\quad - \frac{486\Gamma}{5}F_1[F_0^2 I_7 + 2F_0 I_6 + I_5] \\ &\quad - \frac{34992\Gamma^2}{175}[F_0^5 I_{11} + 5F_0^4 I_{10} + 10F_0^3 I_9 + 10F_0^2 I_8 + 5F_0 I_7 + I_6]. \end{aligned}$$

4.2. Peristaltic flow in an axisymmetric tube. We apply the homotopy perturbation method to the non-linear differential equation (3.5) subject to boundary conditions (3.7) and (3.8). First we define a homotopy $\omega(r, q)\colon \Omega \times [0,1] \to \mathbb{R}$ for (3.5) which satisfies

$$(4.8)\quad (1-q)[L(\Psi) - L(\psi_0)] + q\left[\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left\{r\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Psi}{\partial r}\right) + 2\Gamma\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Psi}{\partial r}\right)\right)^3\right)\right\}\right]\right] = 0$$

where

$$L(\,\cdot\,) = \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left\{r\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(\,\cdot\,)}{\partial r}\right)\right)\right\}\right]$$

is a linear operator, q is an embedding parameter and ψ_0 is the initial guess. Equivalently, (4.8) may be written as

$$L(\Psi) - L(\psi_0) + qL(\psi_0) + q\left[\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left\{r2\Gamma\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Psi}{\partial r}\right)\right)^3\right\}\right]\right] = 0. \tag{4.9}$$

We assume that (4.9) along with (3.7) and (3.8) has a solution of the form

$$\Psi = \Psi_0 + q\Psi_1 + q^2\Psi_2 + \dots \tag{4.10}$$

We also assume that

$$F = F_0 + qF_1 + q^2F_2 + \dots, \qquad p = p_0 + qp_1 + q^2p_2 + \dots \tag{4.11}$$

Substituting (4.10) and (4.11) into (4.9), (3.6)–(3.8) and collecting the coefficients of various powers of q, then solving the zeroth, first and second order problems subject to their respective boundary conditions we obtain the following stream function, pressure gradient and pressure rise per wavelength. Substituting and collecting the coefficients of various powers of q, we obtain

$$\begin{aligned}
\Psi =& \Gamma^2 A_2^5\left(\frac{r^8}{4} - h^6r^2\right) - \frac{\Gamma A_2^2}{4}(A_2 + 3B_2)\left(\frac{r^6}{3} - h^4r^2\right) \\
&+ \frac{A_2 + B_2 + C_2}{4}\left(\frac{r^4}{2} - h^2r^2\right) - \frac{r^2}{2}, \\
\frac{dp}{dz} =& 2A_2 + 2B_2 + 2C_2, \\
\Delta P_\lambda =& -8I_2 - 16[I_4F_0 + F_1I_4 + F_2I_4] \\
&- \frac{512}{3}\Gamma[8F_0^3I_{10} + 12F_0^2I_8 + 6F_0I_6 + I_4] \\
&- 1024\Gamma F_1[4F_0^2I_{10} + 4F_0I_8 + I_6] \\
&+ \frac{4096}{3}\Gamma^2[32F_0^5I_{16} + 80F_0^4I_{14} + 80F_0^3I_{12} + 40F_0^2I_{10} + 10F_0I_8 + I_6]
\end{aligned}$$

where

$$\begin{aligned}
A_2 &= -\frac{8}{h^4}\left(F_0 + \frac{h^2}{2}\right), \\
B_2 &= -\frac{8}{h^4}\left(F_1 - \frac{\Gamma}{6}A_2^3h^6\right), \\
C_2 &= -\frac{8}{h^4}\left(F_2 + \frac{3}{4}\Gamma^2A_2^5h^8 - \frac{\Gamma}{2}A_2^2B_2h^6\right)
\end{aligned}$$

and

$$I_n = \int_0^{2\pi}\frac{dz}{(1 + \Phi\sin z)^n}, \quad n = 1, 2, \dots$$

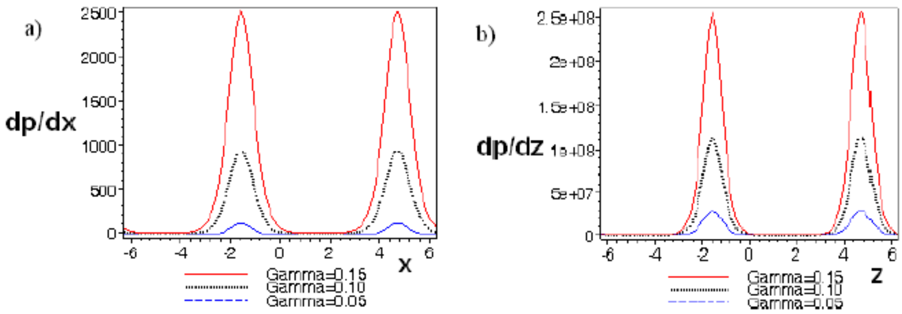

Figure 1. Pressure gradient against axis of symmetry in case of (a) planar channel and (b) axisymmetric tube for fixed values of dimensionless flow rate and amplitude ratio

5. Conclusions

The solutions for the peristaltic transport of a third order fluid are obtained by employing the homotopy perturbation method. Similar results have earlier been solved by the perturbation method in [18] and [4] for small values of the Deborah number Γ. On the other hand, the Adomian decomposition method applied to these problems yields similar results for the planar channel case but requires tedious calculations in the case of axisymmetric tube. Since the homotopy perturbation method does not demand any parameter to be small or large, the solutions obtained in this paper are independent of choice of Γ. This shows that the homotopy perturbation method has a wider area of application.

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A. M. Siddiqui
Pennsylvania State University
York Campus
York, PA 17403, USA

E-mail address: ams5@psu.edu

Q. A. Azim, A. Ashraf and Q. K. Ghori
COMSATS Institute of Information Technology
30 H-8/1, Islamabad, PAKISTAN

E-mail address: quratulain_azim@yahoo.com
afeera_ashraf@comsats.edu.pk
ghori@comsats.edu.pk