Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 31, 2008, 287–293

ANALYTICAL APPROACH TO KAWAHARA EQUATION USING VARIATIONAL ITERATION METHOD AND HOMOTOPY PERTURBATION METHOD

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Abstract. Variational iteration method and homotopy perturbation method are introduced to solve the Kawahara equation. Comparison of the obtained results with the numerical solution shows that both methods lead to remarkably accurate solutions. The main property of the both methods is its flexibility and ability to solve nonlinear equations accurately and conveniently.

1. Introduction

Consider the Kawahara equation given by [21]

$$u_t + uu_x + u_{xxx} - u_{xxxxx} = 0, \tag{1.1}$$

subject to the initial condition

$$u(x,0) = \frac{105}{169}\operatorname{sech}^4\left(\frac{x}{2\sqrt{13}}\right).$$

The Kawahara equation plays an important role in describing motions of plasma waves, capillary-gravity water waves, water waves with surface tension, shallow

2000 *Mathematics Subject Classification.* 35Q35; 65D99.

Key words and phrases. Variational iteration method, homotopy perturbation method, Kawahara equation, approximat solution, exact solution.

The author would like to thank the referees for their comments and Dr. Ji-Huan He for his kind help during preparing this paper.

The work was supported by the Youth Talent Fund of Zhejiang Gongshang University.

water waves and so forth [1], [5], [13], [14]. This equation has been the subject of extensive research work in past decades using various methods, such as tanh-function method, extended tanh-function method, sine-cosine method, direct algebraic method and Adomian decomposition method.

The variational iteration method was first proposed by J. H. He [8], [9] and systematically illustrated in 1999 [10]. It was successfully applied to Burger's equation and coupled Burger's equation, to generalized KdV and coupled Schrödinger- KdV, to delay differential equations, to Duffing equation and mathematical pendulum, to autonomous ordinary differential systems, to construction of solitary solution and compacton-like solution, and to various other problems [3], [4], [8], [10], [12], [15]–[17], [20], [23], while the homotopy perturbation method proposed by J. H. He is constantly being developed and applied to solve various nonlinear problems [2], [6], [7], [11], [18], [19], [22], [24].

In this paper, we apply both methods to solve the Kawahara equation. The main advantage of the methods is that they can give the approximate solutions without unrealistic nonlinear assumptions, linearization, discretization or calculation of the complicated Adomian polynomials, therefore, the methods provide efficient approaches to solve the Kawahara equation. Numerical results are presented to show the efficiencies of the methods.

The rest of this paper is organized as follows. In Section 2, the variational iteration method and homotopy perturbation method are applied to solve the Kawahara equation. The numerical results for the Kawahara equation are presented in Section 3. Finally, we give the conclusion in Section 4.

2. Numerical examples

In this section, we apply the variational iteration method [8]–[10] and homotopy perturbation method [11], [12] to solve the Kawahara equation (1.1). Note that the exact solution of (1.1) is given by [21]

$$u(x,t) = \frac{105}{169}\operatorname{sech}^4\left(\frac{1}{2\sqrt{13}}\left(x - \frac{36}{169}t\right)\right).$$

2.1. Variational iteration method for the Kawahara equation. According to the variational iteration method [8]–[10], we can construct the following correct functional:

$$(2.1)\quad \begin{aligned} u_{n+1}(x,t) &= u_n(x,t) \\ &+ \int_0^t \lambda\{u_{n\xi}(x,\xi) + \widetilde{u}_n(x,\xi)\widetilde{u}_{nx}(x,\xi) + \widetilde{u}_{nxxx}(x,\xi) - \widetilde{u}_{nxxxxx}(x,\xi)\}d\xi, \end{aligned}$$

where $\widetilde{u}_n$ is considered as a restricted variation, i.e. $\delta\widetilde{u}_n = 0$, and λ is the general Lagrange multiplier.

Making the above correct functional stationary, and noticing that $\delta\widetilde{u}_n = 0$, we have

$$\begin{aligned}\delta u_{n+1}(x,t) &= \delta u_n(x,t) \\ &+ \delta \int_0^t \lambda\{u_{n\xi}(x,\xi) + \widetilde{u}_n(x,\xi)\widetilde{u}_{nx}(x,\xi) + \widetilde{u}_{nxxx}(x,\xi) - \widetilde{u}_{nxxxxx}(x,\xi)\}d\xi \\ &= \delta u_n(x,t) + \lambda\delta u_n(x,\xi)|_{\xi=t} - \int_0^t \lambda^{'}\delta u_n(x,\xi)d\xi,\end{aligned}$$

which yields the following stationary conditions

$$1+\lambda = 0, \quad \lambda^{'} = 0.$$

Therefore, the general Lagrange multiplier can be readily identified as $\lambda = -1$. Then substituting this value of the Lagrange multiplier into functional (2.1)

$$\begin{aligned}u_{n+1}(x,t) &= u_n(x,t) \\ &- \int_0^t \{u_{n\xi}(x,\xi) + u_n(x,\xi)u_{nx}(x,\xi) + u_{nxxx}(x,\xi) - u_{nxxxxx}(x,\xi)\}\, d\xi.\end{aligned} \tag{2.2}$$

Beginning with $u_0 = (105/169)\mathrm{sech}^4(x/(2\sqrt{13}))$, by the above iteration formula, we obtain

$$\begin{aligned}u_1 &= \frac{105}{169}\mathrm{sech}^4\left(\frac{x}{2\sqrt{13}}\right) + \frac{7560}{28561\sqrt{13}}t\mathrm{sech}^4\left(\frac{x}{2\sqrt{13}}\right)\tanh\left(\frac{x}{2\sqrt{13}}\right), \\ u_2 &= \frac{105}{169}\mathrm{sech}^4\left(\frac{x}{2\sqrt{13}}\right) + \frac{7560}{28561\sqrt{13}}t\mathrm{sech}^4\left(\frac{x}{2\sqrt{13}}\right)\tanh\left(\frac{x}{2\sqrt{13}}\right) \\ &+ \frac{68040}{62748517}t^2\mathrm{sech}^4\left(\frac{x}{2\sqrt{13}}\right)\left(4-5\mathrm{sech}^2\left(\frac{x}{2\sqrt{13}}\right)\right) \\ &+ \frac{9525600}{10604499373}t^3\mathrm{sech}^8\left(\frac{x}{2\sqrt{13}}\right)\tanh\left(\frac{x}{2\sqrt{13}}\right)\left(4-5\mathrm{sech}^2\left(\frac{x}{2\sqrt{13}}\right)\right).\end{aligned}$$

In the same manner, the rest of components of the iteration formula (2.2) can be obtained by using Maple or Mathematica package.

2.2. Homotopy perturbation method for the Kawahara equation. By the homotopy perturbation method [11], [12], we can construct the homotopy which satisfies

$$u_t - y_{0t} + py_{0t} + p(uu_x + u_{xxx} - u_{xxxxx}) = 0, \tag{2.3}$$

with the initial approximation $y_0 = u(x,0) = (105/169)\mathrm{sech}^4(x/(2\sqrt{13}))$.

Suppose that the solution of (2.3) has the form:

$$u = u_0 + pu_1 + p^2u_2 + \dots . \tag{2.4}$$

Substituting (2.4) into (2.3), and equating the terms of the same power of p, it follows that

$$p^0: \ u_{0t} - y_{0t} = 0, \quad y_0 = \frac{105}{169}\mathrm{sech}^4\left(\frac{x}{2\sqrt{13}}\right), \tag{2.5}$$

$$p^1: \ u_{1t} + u_0u_{0x} + u_{0xxx} - u_{0xxxxx} + y_{0t} = 0, \qquad u_1(x,0) = 0,$$
$$p^2: \ u_{2t} + u_0u_{1x} + u_1u_{0x} + u_{1xxx} - u_{1xxxxx} = 0, \quad u_2(x,0) = 0.$$

Setting $u_0 = y_0$ and solving the above equations results in $u(x,t)$. According to (2.4) and the assumption $p = 1$, we have

$$u(x,t) = \frac{105}{169}\mathrm{sech}^4\left(\frac{x}{2\sqrt{13}}\right) + \frac{7560}{28561\sqrt{13}}t\,\mathrm{sech}^4\left(\frac{x}{2\sqrt{13}}\right)\tanh\left(\frac{x}{2\sqrt{13}}\right) + \frac{68040}{62748517}t^2\mathrm{sech}^4\left(\frac{x}{2\sqrt{13}}\right)\left(4 - 5\mathrm{sech}^2\left(\frac{x}{2\sqrt{13}}\right)\right).$$

3. Numerical results for the Kawahara equation

For the purpose of comparison, we give the absolute errors of the approximate solutions obtained by variational iteration method and homotopy perturbation method in Table 1. The evolution results for the approximate solutions and the exact solutions of the Kawahara equation with $-20 \le x \le 20$ and $0 \le t \le 2$ are given in Figure 1. From the numerical results, we can easily conclude that both methods yield high accuracy of the approximate solutions for the Kawahara equation. It's important to note that both methods don't involve the unrealistic assumptions. In addition, we obtain the high accuracy of the approximate solutions by two iterations only, see (2.2) and (2.5), respectively. The accuracy can be further improved if the solution procedure continues to a higher order.

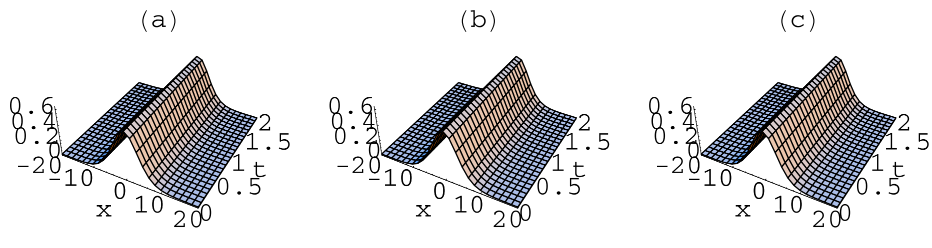

Figure 1. The evolution results for the approximate solutions obtained by VIM and HPM, and the exact solutions of the Kawahara equation

	$x = -5$		$x = -2.5$	
t	VIM	HPM	VIM	HPM
0.01	1.13523E−11	3.17016E−10	4.43000E−11	4.61000E−10
0.02	3.12801E−10	3.80149E−10	8.84000E−11	1.52800E−10
0.03	7.31032E−10	3.79879E−10	4.18000E−11	7.97000E−11
0.04	1.74229E−9	8.22488E−10	4.66700E−10	1.41200E−9
0.05	3.52950E−9	1.26447E−9	7.95400E−10	2.98700E−9

	$x = 0$		$x = 2.5$	
t	VIM	HPM	VIM	HPM
0.01	1.00000E−10	1.00000E−10	1.17500E−10	1.04900E−10
0.02	1.00000E−10	1.00000E−10	1.80000E−11	3.60700E−10
0.03	0.00000E+00	0.00000E+00	1.58200E−10	6.95900E−10
0.04	0.00000E+00	0.00000E+00	4.10100E−10	1.64030E−9
0.05	0.00000E+00	0.00000E+00	9.21400E−10	3.77840E−9

	$x = 5$		$x = 7.5$	
t	VIM	HPM	VIM	HPM
0.01	3.05612E−11	3.17466E−10	1.96880E−10	1.14560E−10
0.02	3.24375E−10	1.27076E−10	1.19000E−11	1.82200E−10
0.03	7.84474E−10	5.08666E−10	7.72000E−11	3.87600E−10
0.04	1.72904E−9	2.54513E−10	2.21800E−10	3.09100E−10
0.05	3.53590E−9	1.14612E−9	3.25800E−10	8.64000E−10

Table 1. The numerical results for the approximate solutions obtained by VIM and HPM in comparison with the exact solutions of (1.1)

4. Conclusions

In this paper, the variational iteration method and homotopy perturbation method have been applied to solve the Kawahara equation. Numerical results have been presented to show the efficiencies of both methods. A clear conclusion can be drawn from the numerical results that both methods lead to high accuracy of the obtained analytical solutions for the Kawahara equation. Therefore, both methods can be seen as promising and powerful methods for solving various nonlinear equations.

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Manuscript received September 18, 2007

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