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FREQUENCY-AMPLITUDE RELATIONSHIP OF THE DUFFING-HARMONIC OSCILLATOR

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Abstract. The variational iteration method, the variational method and the parameter-expanding method are applied to obtain the frequency-amplitude relationship of the Duffing-harmonic oscillator. The obtained results reveal that all the three methods are very effective and convenient.

1. Introduction

In [10], Ji-Huan He gave a very lucid as well as elementary discussion of the variational iteration method and the parameter-expansion method for various nonlinear equations. In particular, He used unheard-of simple numerical procedures to arrive at surprisingly accurate predictions of frequency-amplitude relationships for nonlinear oscillators [10]. In addition, He gave a great effort to give sophisticated interpretation of the numerical results.

In the present work, we will follow He's spirit of simplicity, while aiming at more accurate determination of the frequency-amplitude relationship of the Duffing-harmonic oscillator [2], [13]–[16], which reads

$$\frac{d^2x}{dt^2} + \frac{x^3}{1+x^2} = 0 \tag{1.1}$$

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with initial conditions

$$x(0) = A \quad \text{and} \quad \frac{dx}{dt}(0) = 0.$$

Equation (1.1) is an example of a conservative nonlinear oscillatory system having a rational form for the non-dimensional restoring force. All the motions corresponding to (1.1) are periodic. And the angular frequency ω increases from 0 to 1 as the initial value of $x(0) = A$ increases [2]. Equation (1.1) is not amenable to exact treatment and, therefore, approximate techniques must be resorted to. In this paper we will apply the variational iteration method [7], [8], [12], the variational method [11] and the parameter-expanding method [9], [10], to the discussed problem.

2. Variational iteration method

Equation (1.1) can be re-written in the form

$$\frac{d^2x}{dt^2} + \omega^2 x = g(x),$$

where $g(x) = \omega^2 x - x^3 - x^2(d^2x/dt^2)$, and ω is the unknown angular frequency of the nonlinear oscillator. Applying the variational iteration method [7], [8], [12], we have the following functional

$$x_{n+1}(t) = x_n(t) + \int_0^t \lambda(x_n''(\tau) + \omega^2 x_n(\tau) - \widetilde{g}(x_n))\, d\tau, \tag{2.1}$$

where $\widetilde{g}$ is considered as a restricted variation, i.e. $\delta\widetilde{g}(x_n) = 0$. The method can find wide applications [3], [4], [18], [23], [26].

Calculating variation with respect to x_n, and noting that $\delta\widetilde{g} = 0$, we have the following stationary conditions:

$$\begin{cases} \lambda''(\tau) + \omega^2\lambda(\tau) = 0, \\ \lambda(\tau)|_{\tau=t} = 0, \\ 1 - \lambda'(\tau)|_{\tau=t} = 0. \end{cases}$$

The multiplier, therefore, can be identified as

$$\lambda = \frac{1}{\omega}\sin\omega(\tau - t).$$

Substituting the identified multiplier into (2.1) results in the following iteration formula:

$$x_{n+1}(t) = x_n(t) + \frac{1}{\omega}\int_0^t \sin\omega(\tau - t)\left(x_n''(\tau) + x_n^3 + x_n^2\frac{d^2x_n}{dt^2}\right) d\tau. \tag{2.2}$$

Assuming that its initial approximate solution has the form:

$$x_0(t) = A\cos\omega t, \tag{2.3}$$

and substituting (2.3) into (1.1) leads to the following residual

$$R_0(t) = \left(\frac{3}{4}A^2 - \omega^2 - \frac{3}{4}\omega^2 A^2\right) A\cos\omega t + (1-\omega^2)\frac{A^3}{4}\cos 3\omega t.$$

By formula (2.2), we have

$$x_1(t) = A\cos\omega t + \frac{1}{\omega}\int_0^t R_0(t)\sin\omega(\tau - t)\,d\tau.$$

In the case of no secular, we find the relation between frequency and amplitude of the Duffing-harmonic oscillator

$$\omega^2 = \frac{3A^2/4}{1+3A^2/4} = 1 - \frac{1}{1+3A^2/4}. \tag{2.4}$$

This is valid for the whole range of values of A. Equation (2.4) is the same as that obtained by homotopy perturbation method in [2, equations (16), (17), (70) and (71)].

Equation (1.1) can also be written in the form

$$\frac{d^2x}{dt^2} + \omega^2 x + g(x) = 0,$$

where $g(x) = x^3/(1+x^2) - \omega^2 x$, and ω is the unknown angular frequency of the nonlinear oscillator.

Assuming $x_0(t) = A\cos\omega t$, by the same manipulation as illustrated in the above section, we obtain the following formula

$$x_{n+1}(t) = x_n(t) + \frac{1}{\omega}\int_0^t \sin\omega(\tau - t)\left(x_n''(\tau) + \frac{x_n^3}{1+x_n^2}\right)d\tau.$$

$$R_0(t) = -\omega^2 A\cos\omega t + \frac{A^3\cos^3\omega t}{1+A^2\cos^2\omega t}.$$

Use Fourier series expansion

$$\frac{A^3\cos^3\omega t}{1+A^2\cos^2\omega t} = \sum_{n=0}^{\infty} a_{2n+1}\cos((2n+1)\omega t) = a_1\cos\omega t + a_3\cos\omega t + \dots \tag{2.5}$$

Here, the coefficient a_1 can be obtained by means of the following equation

$$\begin{aligned} a_1 &= \frac{2}{\pi}\int_0^\pi \frac{A^3\cos^3\tau}{1+A^2\cos^2\tau}\cos\tau\,d\tau \\ &= \frac{2A}{\pi}\int_0^\pi \frac{A^2\cos^4\tau}{1+A^2\cos^2\tau}\,d\tau = A + \frac{2}{A}\left(\frac{1}{\sqrt{1+A^2}} - 1\right), \end{aligned} \tag{2.6}$$

where $\tau = \omega t$. Therefore,

$$R_0(t) = -\omega^2 A\cos\omega t + \left\{A + \frac{2}{A}\left(\frac{1}{\sqrt{1+A^2}} - 1\right)\right\}\cos\omega t + \sum_{n=1}^{\infty} a_{2n+1}\cos((2n+1)\omega t).$$

No secular requires

$$-\omega^2 A + A + \frac{2}{A}\left(\frac{1}{\sqrt{1+A^2}} - 1\right) = 0.$$

So the relation between frequency and amplitude of the Duffing-harmonic oscillator is

$$\omega^2 = 1 + \frac{2}{A^2}\left(\frac{1}{\sqrt{1+A^2}} - 1\right). \tag{2.7}$$

This result coincides with that obtained in [2, equations (33), (74)–(76)].

3. Variational method

Assume the solution of equation (1.1) can be expressed as

$$x(t) = A\cos\omega t,$$

where A and ω are the amplitude and frequency of the oscillator, respectively.

Using the novel variational method [11], we obtain the following

$$J(x) = \frac{1}{2}\int_0^{T/4}\left\{\left(\frac{dx}{dt}\right)^2 + x^2 - \ln(1+x^2)\right\}dt,$$

where T is the period of the nonlinear oscillator.

$$J(A) = \int_0^{\pi/2}\left\{-\frac{1}{2}A^2\omega\sin^2 t + \frac{1}{2\omega}A^2\cos^2 t - \frac{1}{2\omega}\ln(1+A^2\cos^2 t)\right\}dt,$$

The stationary condition with respect to A reads

$$\frac{dJ}{dA} = \int_0^{\pi/2}\left\{-A\omega\sin^2 t + \frac{1}{\omega}\left(A\cos^2 t - \frac{A\cos^2 t}{1+A^2\cos^2 t}\right)\right\}dt = 0,$$

and leads to the result

$$\omega^2 = 1 + \frac{2}{A^2}\left(\frac{1}{\sqrt{1+A^2}} - 1\right).$$

It is equal with (2.7).

4. Parameter expanding method

Now rewrite equation (1.1) in the form

$$\frac{d^2x}{dt^2} + 0 \cdot x + 1 \cdot \frac{x^3}{1+x^2} = 0. \tag{4.1}$$

According to the parameter-expanding method [5], [9], [10], the solution can be expressed as a power series in a bookkeeping parameter p:

$$x = x_0 + px_1 + p^2x_2 + \dots\,, \tag{4.2}$$

where p is a bookkeeping parameter $p = 1$.

According to He's parameter-expanding method, the coefficients 0 and 1 in the left hand side of (4.1) should be respectively expanded to series in p:

$$0 = \omega^2 + p\omega_1 + p^2\omega_2 + \dots\,, \tag{4.3}$$

$$1 = pc_1 + p^2c_2 + \cdots\,. \tag{4.4}$$

Substituting (4.2)–(4.4) into (4.1) and equating the terms with the identical powers of p, we have

$$x_0'' + \omega^2 x_0 = 0, \qquad x_0(0) = A, \quad x_0'(0) = 0, \tag{4.5}$$

$$x_1'' + \omega^2 x_1 + \omega_1 x_0 + c_1 \frac{x_0^3}{1+x_0^2} = 0, \quad x_1(0) = 0, \quad x_1'(0) = 0. \tag{4.6}$$

Solving equation (4.5), we can easily obtain the result:

$$x_0 = A\cos\omega t.$$

Substituting x_0 into (4.6) yields

$$x_1'' + \omega^2 x_1 + \omega_1 A\cos\omega t + c_1 \frac{A^3\cos^3\omega t}{1+(A\cos\omega t)^2} = 0.$$

Combine equations (2.5) and (2.6) with the no secular requirement, we have

$$\omega_1 A + c_1 a_1 = 0.$$

If the first-order approximation is enough, then setting $p = 1$, from (4.3) and (4.4), we have

$$0 = \omega^2 + \omega_1, \quad 1 = c_1.$$

Therefore, we obtain the relation between frequency and amplitude of the Duffing-harmonic oscillator, which reads

$$\omega^2 = 1 + \frac{2}{A^2}\left(\frac{1}{\sqrt{1+A^2}} - 1\right).$$

This is the same as equation (2.7).

The method is very effective [24], [25], and can lead to the same iteration scheme as that obtained by the homotopy perturbation method [1], [6], [17], [19]–[22], [27].

5. Conclusion

He's variational iteration method, variational method and parameter-expanding method are all proved to be powerful, convenient and efficient mathematical tools for searching for frequency-amplitude relationship of nonlinear oscillators.

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