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LARGE DEFLECTION OF A CANTILEVER BEAM UNDER POINT LOAD USING THE HOMOTOPY PERTURBATION METHOD

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Abstract. In this study, the homotopy-perturbation method (HPM) is used to investigate the large deflection of a cantilever beam under point load at the free end. The vertical and horizontal displacements of the cantilever beam are conveniently obtained in explicit analytical forms The main objective of this study is to propose an alternative method of solution, which does not require small parameters and avoid linearization and physically unrealistic assumptions. The results show that this method is very efficient and convenient and can be applied to a large class of practical problems.

1. Introduction

In recent years, it has turned out that many phenomena in engineering, physics, chemistry and other branches of science can be described very successfully by nonlinear models using mathematical tools. In most cases, it is difficult to solve these nonlinear problems, especially analytically. Perturbation method [1] is one of the well-known methods to solve nonlinear equations. On the other hand, since, using the common perturbation method is based upon the existence of a small parameter, developing the method for different applications is difficult. Very recently, some promising approximate analytical solutions are proposed,

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such as Exp-function method [15], [16], Adomian decomposition method [18], [24], [25], variational iteration method [4], [8], [9] and homotopy-perturbation method [5], [10]–[12].

Other methods are reviewed in [13] and [14].

HPM is the most effective and convenient method for both linear and nonlinear equations. This method does not depend on a small parameter. Using homotopy technique in topology, a homotopy is constructed with an embedding parameter $p \in [0,1]$, which is considered as a "small parameter". HPM has been used to solve effectively, easily and accurately a large class of linear and nonlinear problems with components converging rapidly to accurate solutions. HPM was first proposed by He [10] and was successfully applied to various engineering problems [1], [3], [17], [19], [20].

The aim of this work is to employ HPM to obtain the analytic expressions for the rotation angle of a cantilever beam at the free end, which provide a straightforward approach to calculate the vertical and horizontal displacements of the beam. Different from ADM, where specific algorithms are usually used to determine the Adomian polynomials, HPM handles linear and nonlinear problems in a simple manner by deforming a difficult problem into a simple one.

The bending equation of a uniform cross-section beam with large deflection is [23]:

$$\frac{d\theta}{ds} = \frac{P}{EI}(l_1 - x), \quad \theta(0) = 0, \quad \theta'(l) = 0, \tag{1.1}$$

where s is the arc-coordinate of the neutral axis of the beam, x is the horizontal coordinate from the fixed end, l is the length of the beam, P is the point load at the free end, EI is the bending stiffness of the beam, θ is the rotation of cross-section of the beam and l_1 is the horizontal distance of two ends. The axial elongation of the beam is neglected, because it is much smaller than the deflection at the free tip.

After differentiating (1.1) with respect to s and then using the dimensionless variable $\gamma = s/l$, the original equation becomes

$$\frac{d^2\theta}{d\gamma^2} + \alpha\cos\theta = 0, \quad \theta(0) = 0, \quad \theta'(l) = 0, \tag{1.2}$$

where $\alpha = Pl^2/(EI)$. The rotation angle of cross-section of the beam at free tip is denoted by $\theta_B = \theta(1)$. The dimensionless exact vertical and horizontal displacements of the free tip are given by [23]:

$$\frac{f_B}{l} = 1 - \frac{2}{\sqrt{\alpha}}[E(\mu) - E(\phi,\mu)], \tag{1.3}$$

$$\frac{\delta_B}{l} = \frac{l - l_1}{l} = 1 - \sqrt{\frac{2\sin\theta_B}{\alpha}}, \tag{1.4}$$

where $E(\mu)$ is the complete elliptic integral of the second kind, $E(\phi, \mu)$ is the elliptic integral of the second kind and

$$\mu = \sqrt{\frac{1 + \sin\theta_B}{2}}, \qquad \phi = \arcsin\left(\frac{1}{\sqrt{2}\mu}\right),$$

For infinitesimal deflection, we can assume that the linear equation in the form of

$$\frac{d^2}{d\gamma^2} + \alpha = 0, \quad \theta(0) = 0, \quad \acute{\theta}(1) = 0, \tag{1.5}$$

is enough to model the problem. The solution of the equation (1.5) is

$$\theta(\gamma) = \frac{\alpha}{2}(2 - \gamma)\gamma. \tag{1.6}$$

2. Basic idea of the homotopy-perturbation method

To illustrate the basic ideas of this method, we consider the following equation [10]:

$$A(u) - f(r) = 0, \quad r \in \Omega,$$

with the boundary condition:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma,$$

where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and G is the boundary of the domain Ω.

A can be divided into two parts which are L and N, where L is linear and N is nonlinear. Equation (1.3) can therefore be rewritten as follows:

$$L(u) + N(u) - f(r) = 0, \quad r \in \Omega, \tag{2.1}$$

Homotopy perturbation structure is shown as follows:

$$H(v,p) = (1 - p) * [L(v) - L(u_0)] + p\ [A(v) - \ f(r)] = 0,$$

where $v(r,p)\colon \Omega \times [0,1] \to \mathbb{R}$.

In equation (2.1) $p \in [0, 1]$ is an embedding parameter and is the first approximation that satisfies the boundary conditions. We can assume that the solution of (1.3) can be written as a power series in p, as following:

$$v = v_0 + p\ v_1 + p^2 v_2 + \dots\,, \tag{2.2}$$

and the best approximation is: $u = \lim_{p\to 1}\ v = v_0 +\ v_1 +\ v_2 + \dots$ The above convergence is discussed in [10].

3. Implementation of HPM

In order to solve nonlinear boundary value problem (1.2), using HPM, one may construct the following homotopy:

$$(1-p)\bigg(\frac{d^2\theta(\gamma)}{d\gamma^2}-\frac{d^2\theta_0(x)}{dx^2}\bigg)+p\bigg(\frac{d^2\theta(\gamma)}{d\gamma^2}+\alpha\cos\theta(\gamma)\bigg)=0,$$

Computing the Taylor series expansion of $\cos(\theta(\gamma))$, about the point zero and substituting into it (2.3) transforms (2.3) to:

$$(3.1)\quad (1-p)\bigg(\frac{d^2\theta(\gamma)}{d\gamma^2}-\frac{d^2\theta_0(x)}{dx^2}\bigg) + p\bigg(\frac{d^2\theta(\gamma)}{d\gamma^2}+\alpha-\frac{1}{2!}\alpha\theta^2(\gamma)+\frac{1}{4!}\alpha\theta^4(\gamma)-\ldots\bigg)=0,$$

Substituting $\theta(\gamma)$ from (2.2) into (3.1) and rearranging powers of p-terms, we have:

$$(3.2)\quad \begin{aligned}
p^0 &: \quad \frac{d^2\theta_0(\gamma)}{d\gamma^2}-\frac{d^2\theta_0(x)}{dx^2}=0,\\
p^1 &: \quad \frac{d^2\theta_1(\gamma)}{d\gamma^2}+\frac{d^2\theta_0(x)}{dx^2}-\frac{1}{2!}\alpha\theta_0^2(\gamma)+\frac{1}{4!}\alpha\theta_0^4(\gamma)+\alpha=0,\\
p^2 &: \quad \frac{d^2\theta_2(\gamma)}{d\gamma^2}-\alpha\theta_0(\gamma)\theta_1(\gamma)+\frac{1}{3!}\alpha\theta_0^3(\gamma)\theta_1(\gamma)=0,\\
p^3 &: \quad \frac{d^2\theta_3(\gamma)}{d\gamma^2}-\alpha\theta_0(\gamma)\theta_2(\gamma)-\frac{1}{2!}\alpha\theta_1^2(\gamma)\\
&\qquad +\frac{1}{4!}\alpha\theta_0^2(\gamma)\theta_1^2(\gamma)+\frac{1}{3!}\alpha\theta_0^3(\gamma)\theta_2(\gamma)=0.
\end{aligned}$$

First, we start with an initial approximation. It is obvious that equation (1.6) obtained by the linear equation for infinitesimal deflection is a good initial guess. So, we choose the initial approximation in the following form:

$$\theta_0(x)=\frac{\alpha}{2}(2-x)x,$$

After solving equations (3.2), the solution of the nonlinear boundary value equation, when $p\to 1$, will be as follows:

$$\theta(\gamma)=\theta_0(\gamma)+\theta_1(\gamma)+\theta_2(\gamma)+\theta_3(\gamma).$$

For example, the 4th-order approximation of θ_B is:

$$\theta_B=0.5\alpha-0.04583333\alpha^3+0.008993105\alpha^5+0.00222473\alpha^7 + 0.00018046\alpha^9-0.00000489\alpha^11+4.138754\times 10^-8\alpha^13,$$

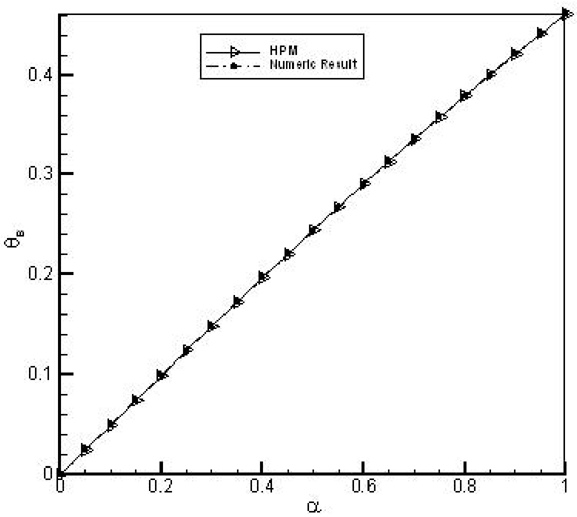

Figure 1. The rotation angle of the beam at the free tip

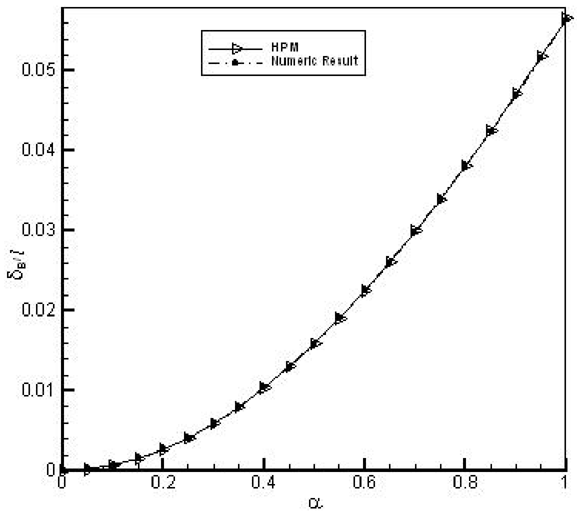

Figure 2. The dimensionless horizontal displacement of the beam at the free tip

The result obtained by HPM is shown in Figure 1 and compared with the numerical result obtained by finite difference technique using Richardson extrapolation. As shown in this figure, the expression for rotation angle obtained by HPM is very accurate and convenient.

Now, as explicit analytical expression for rotation angle of cross-section of the beam is obtained, the vertical and horizontal displacements can readily be calculated form equations (1.3) and (1.4). Figure 2 shows the dimensionless horizontal displacement at the free end attained by HPM and finite difference technique using Richardson extrapolation, which admits excellent agreement.

4. Conclusion

In this paper, the homotopy perturbation method (HPM) was successfully applied to study time-fractional equations. The solution obtained by means of homotopy perturbation method is an infinite power series with respect to appropriate initial condition, which can be, in turn, expressed in a closed form. The obtained results reinforce the conclusions made by many researchers about the efficiency of HPM. The results show that homotopy perturbation method is a powerful and efficient technique in finding exact and approximate solutions for nonlinear partial differential equations of fractional order.

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