Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 31, 2008, 263–270

SIMULATION OF THE PREDATOR-PREY PROBLEM BY THE HOMOTOPY-PERTURBATION METHOD REVISED

M. S. H. Chowdhury — I. Hashim — R. Roslan

Abstract. In this paper, the predator-prey problem is revisited. Previous solution by homotopy-perturbation method (HPM) is improved by treating the homotopy-perturbation method as an algorithm in a sequence of intervals (i.e. time steps) called the multistage homotopy-perturbation method (shortly MHPM). Numerical results show that the multistage homotopy-perturbation method and the classical fourth-order Rungge–Kutta (RK4) methods are in complete agreement.

1. Introduction

All models of biological systems are essentially based on systems of nonlinear ordinary differential equations (ODEs). Both mathematical modelling and simulation are very important in recent studies of biological mathematics. In this analysis, we study the mathematical model of the prey and predator problem in which some rabbits and foxes are considered living together. Foxes eat the rabbits and rabbits eat clover, and when the number of foxes increases, the number of rabbits decreases and the number of foxes decreases, the rabbits will be safe. The relationship of increasing and decreasing in the population of these

2000 *Mathematics Subject Classification.* Primary 34E10, 65C20; Secondary 34E05.
Key words and phrases. Homotopy-perturbation method, non-linear ODEs, prey and predator problem.
The authors would like to acknowledge the financial supports received from the UKM Computational Biology Group, Universiti Malaysia Sabah, the Malaysian Technical Cooperation Program and the International Islamic University Chittagong, Bangladesh.

two kinds of animals can be described by the so called mathematical model of the problem of prey and predator i.e. the following system of nonlinear equations

$$x' = x(a - by), \quad y' = -y(c - dx), \tag{1.1}$$

where $x(t)$ and $y(t)$ represent the populations of rabbits and the foxes at the time t, respectively, and a, b, c, d are known coefficients. For more details on the mathematical modelling to the above systems of nonlinear equations, we refer the readers to [32]. The problem was solved by Biazar et al. using the power series method (PSM) [3] and the classical Adomian decomposition method [4], and by Rafei et al. using the homotopy perturbation method [30]. Like ADM and based on our observations, it is hopeless to find the homotopy-perturbation method solutions of system (1.1) valid globally in time.

The time evolution of the reaction can be determined from the traditional purely numerical methods like the classical fourth-order Runge–Kutta method (RK4), but we are interested in this work to find numeric-analytic solution of the prey and predator problems (1.1) by the homotopy-perturbation method. To do so, we shall adopt the idea in the hybrid numeric-analytic procedure of the Adomian decomposition method (ADM) [1], successfully applied in [18], [27], to develop a new hybrid numeric-analytic technique based on the standard homotopy-perturbation method [19]. We shall call this technique as the multistage homotopy-perturbation method. Some applications of the standard homotopy-perturbation method on nonlinear problems can be found in [2], [5], [6], [13]–[15], [20]–[26], [31], [33]–[35]. We shall demonstrate that the lack of global convergence of the standard homotopy-perturbation method can be overcome by the multistage homotopy-perturbation method. Recently, the applicability of the homotopy-perturbation method was extended to singular second-order differential equations [7], time-dependent Emden–Fowler type equations [8], general time-independent Emden–Fowler equations [9], Klein–Gordon and Sine–Gordon equations [10], nonlinear population dynamics models [11], traveling wave solution of the Korteweg–de Vries equation [28], determining frequency-amplitude relation of a nonlinear oscillator with discontinuities [29], K(2,2), KdV and modified KdV equations [36] and nonlinear system of second order boundary value problems [37]. Very recently, Chowdhury et al. [12], Hashim and Chowdhury [16] and Hashim et al. [17] were the first to apply successfully the multistage homotopy-perturbation method to the chaotic Lorenz system and a class of systems of ODEs.

In this paper, the total time evolution in the prey and predator problem is simulated by the numeric-analytic multistage homotopy-perturbation method. Numerical comparisons with the classical homotopy-perturbation method [30] and the classical fourth-order Rungge–Kutta (RK4) methods are presented.

2. Method of solution

Since the homotopy-perturbation method is now standard and for brevity, the reader is referred to [19]–[26] for basic ideas of the homotopy-perturbation method. In this section, we shall demonstrate the applicability and accuracy of the multistage homotopy-perturbation method to the systems (1.1).

According to the homotopy-perturbation method, we can construct a homotopy of system (1.1) as follows:

$$\begin{aligned} v_1' - x_0' + p(x_0' - av_1 + bv_1v_2) &= 0, \\ v_2' - y_0' + p(y_0' + cv_2 - dv_1v_2) &= 0. \end{aligned} \tag{2.1}$$

Let us choose the initial approximations as

$$\begin{aligned} v_{1,0}(t) &= x_{t^*}(t) = x(t^*) = k_1, \\ v_{2,0}(t) &= y_{t^*}(t) = y(t^*) = k_2, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} v_1(t) &= v_{1,0}(t) + pv_{1,1}(t) + p^2v_{1,2}(t) + p^3v_{1,3}(t) + \dots, \\ v_2(t) &= v_{2,0}(t) + pv_{2,1}(t) + p^2v_{2,2}(t) + p^3v_{2,3}(t) + \dots, \end{aligned} \tag{2.3}$$

where $v_{i,j}$ $(i = 1, 2;\ j = 1, 2, \dots)$ are functions yet to be determined.

Substituting (2.2)–(2.3) into (2.1) and collecting terms of the same powers of p, we have

$$\begin{aligned} v_{1,1}' + x_0' - av_{1,0} + bv_{1,0}v_{2,0} &= 0, & v_{1,1}(0) &= 0, \\ v_{2,1}' + y_0' + cv_{2,0} - dv_{1,0}v_{2,0} &= 0, & v_{2,1}(0) &= 0, \\ v_{1,2}' - av_{1,1} + bv_{1,1}v_{2,0} + bv_{1,0}v_{2,1} &= 0, & v_{1,2}(0) &= 0, \\ v_{2,2}' + cv_{2,1} - dv_{1,1}v_{2,1} - dv_{1,0}v_{2,1} &= 0, & v_{2,2}(0) &= 0, \\ v_{1,3}' - av_{1,2} + bv_{2,0}v_{1,2} + bv_{2,1}v_{1,1} + bv_{2,2}v_{1,0} &= 0, & v_{1,3}(0) &= 0, \\ v_{2,3}' + cv_{2,2} - dv_{1,0}v_{2,3} - dv_{1,1}v_{2,2} - dv_{1,2}v_{2,1} - dv_{1,3}v_{2,0} &= 0, & v_{2,3}(0) &= 0. \end{aligned} \tag{2.4}$$

Solving the differential equations (2.4) by applying the inverse operator $L^{-1}(\,\cdot\,) = \int_{t^*}^{t} (\,\cdot\,)\,dt$, we obtain

$$\begin{aligned} v_{1,1}(t) &= k_1(a - k_2b)(t - t^*), \\ v_{2,1}(t) &= k_2(dk_1 - c)(t - t^*), \\ v_{1,2}(t) &= -\frac{1}{2}k_1(-a^2 + 2abk_2 - {k_2}^2b^2 - bck_2 + bdk_1k_2)(t - t^*)^2, \\ v_{2,2}(t) &= \frac{1}{2}k_2(c^2 - 2cdk_1 + adk_1 - bdk_1k_2 + d^2{k_1}^2)(t - t^*)^2, \end{aligned}$$

$$\begin{aligned}v_{1,3}(t) = {}& -\frac{1}{6}k_1(-3abck_2 + 4abdk_1k_2 + 3b^2ck_2{}^2 - 4b^2dk_1k_2{}^2 - 3ab^2k_2{}^2\\ &+ 3a^2bk_2 + bc^2k_2 - 2bcdk_1k_2 + bd^2k_1{}^2k_2 + k_2{}^3b^3 - a^3)(t-t^*)^3,\\ v_{2,3}(t) = {}& -\frac{1}{6}k_2(3dc^2k_1 - 3cd^2k_1{}^2 - 3acdk_1 + 3ad^2k_1{}^2 - c^3 + 4bcdk_1k_2\\ &- 4bd^2k_1{}^2k_2 + a^2dk_1 - 2abdk_1k_2 + b^2dk_1k_2{}^2 + d^3k_1{}^3)(t-t^*)^3.\end{aligned}$$

The 5-term approximations are considered:

$$x(t) = \lim_{p\to 1} v_1(t) = \sum_{k=0}^{4} v_{1,k}(t), \quad y(t) = \lim_{p\to 1} v_2(t) = \sum_{k=0}^{4} v_{2,k}(t).$$

To find out the iterations on every subinterval of equal length $\triangle t$, $[0,t_1)$, $[t_1,t_2)$, $\ldots$ $[t_{i-1},t)$, we need to know the values of the following

$$k_1 = x(t^*), \quad k_2 = y(t^*).$$

But, in general, we do not have these except for the initial point $t^* = t_0 = 0$. A simple way of obtaining the necessary values could be by means of the previous i-term approximations ϕ_i and ψ_i of the preceding subinterval, i.e.

$$k_1 \backsimeq \phi_i(t^*), \quad k_2 \backsimeq \psi_i(t^*).$$

We note that in this analysis we use the 5-term multistage homotopy-perturbation method solutions for x and y denoted as

$$x(t) \simeq \phi_5(t) = \sum_{i=0}^{4} v_{1,i} \quad \text{and} \quad y(t) \simeq \psi_5(t) = \sum_{i=0}^{4} v_{2,i}.$$

3. Results and discussion

The multistage homotopy-perturbation method algorithm is coded in the computer algebra package Maple. We select a fixed step sizes of $\triangle t = 0.001$ for the problems. The Maple environment variable digits controlling the number of significant digits is set to 16 in all the calculations done in this paper. The simulation done in this paper is for the time span $t \in [0,1]$.

To compare with [4], [30], we set the parameters in four cases:

Case 1. $a = 1$, $b = 1$, $c = 0.1$, $d = 1$, and the initial conditions $x(0) = 14$ and $y(0) = 18$.

Case 2. $a = 0.1$, $b = 1$, $c = 1$, $d = 1$, and the initial conditions $x(0) = 14$ and $y(0) = 18$.

Case 3. $a = 0.1$, $b = 1$, $c = 1$, $d = 1$, and the initial conditions $x(0) = 16$ and $y(0) = 10$.

Case 4. $a = 1$, $b = 1$, $c = 0.1$, $d = 1$, and the initial conditions $x(0) = 16$ and $y(0) = 10$.

From Figures 1–4, it is observed that the 5-term multistage homotopy-perturbation method solutions agree very well with the RK4 solutions for t upto $t = 1$, while the 5-term standard homotopy-perturbation method [30] solutions are only valid for $t \ll 1$.

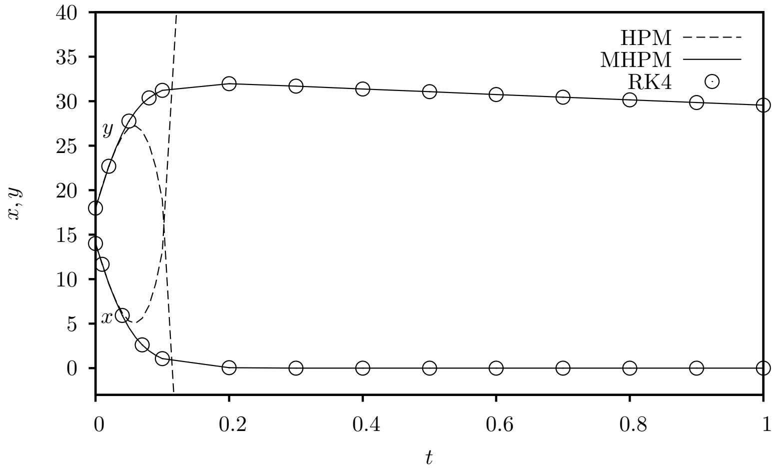

FIGURE 1. Comparison of the populations of rabbits x and foxes y at the time t for Case 1 using 5-term standard homotopy-perturbation method [30], 5-term multistage homotopy-perturbation method ($\triangle t = 0.001$) and RK4 ($\triangle t = 0.001$).

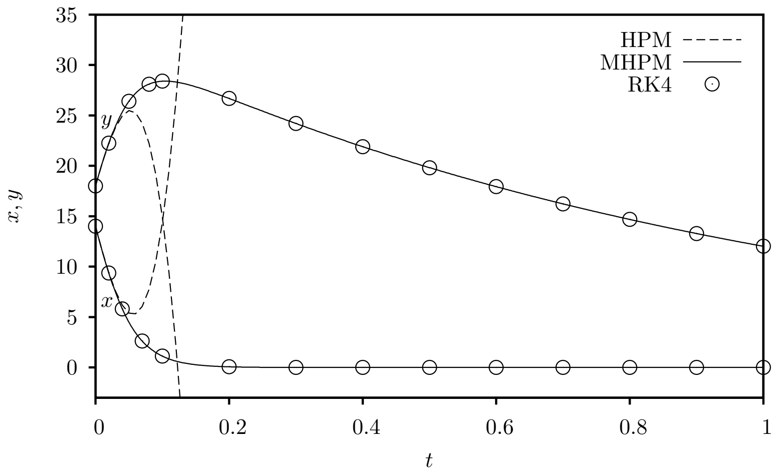

FIGURE 2. Comparison of the populations of rabbits x and foxes y at the time t for Case 2 using 5-term standard homotopy-perturbation method [30], 5-term multistage homotopy-perturbation method ($\triangle t = 0.001$) and RK4 ($\triangle t = 0.001$).

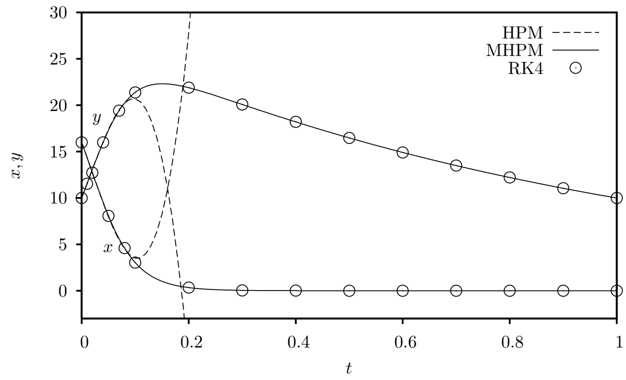

Figure 3. Comparison of the populations of rabbits x and foxes y at the time t for Case 3 using 5-term standard homotopy-perturbation method [30], 5-term multistage homotopy-perturbation method ($\triangle t = 0.001$) and RK4 ($\triangle t = 0.001$).

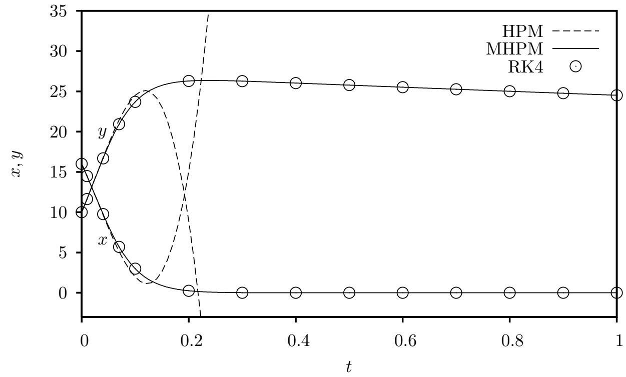

Figure 4. Comparison of the populations of rabbits x and foxes y at the time t for Case 4 using 5-term standard homotopy-perturbation method [30], 5-term multistage homotopy-perturbation method ($\triangle t = 0.001$) and RK4 ($\triangle t = 0.001$).

References

[1] G. Adomian, *Nonlinear Stochastics Systems Theory and Application to Physics*, Reidel, Dordrecht, 1988.

[2] P. D. Ariel, T. Hayat and S. Asghar, *Homotopy perturbation method and axisymmetric flow over a stretching sheet*, Internat. J. Nonlinear Sci. Numer. Simul. **7** (2006), 399–406.

[3] J. Biazar, M. Ilie and A. Khoshkenar, *A new approach to the solution of the prey and predator problem and comparison of the results with Adomian decomposition method*, Appl. Math. Comput. **173** (2005), 486–491.

[4] J. Biazar and R. Montazeri, *A computational method for solution of the prey and predator problem*, Appl. Math. Comput. **163** (2005), 841–847.

[5] J. BIAZAR, M. ESLAMI AND H. GHAZVINI, *Homotopy perturbation method for systems of partial differential equations*, Internat. J. Nonlinear Sci. Numer. Simul. **8** (2007), no. 3, 413–418.

[6] A. BELÉNDEZ, HERNÁNDEZ, T. BELÉNDEZ, ET AL., *Application of He's homotopy perturbation method to the duffing-harmonic oscillator*, Internat. J. Nonlinear Sci. Numer. Simul. **8** (2007), no. 1, 79–88.

[7] M. S. H. CHOWDHURY AND I. HASHIM, *Solutions of a class of singular second-order IVPs by homotopy-perturbation method*, Phys. Lett. A **365** (2007), 439–447.

[8] ______, *Solutions of time-dependent Emden–Fowler type equations by homotopy-perturbation method*, Phys. Lett. A **368** (2007), 305–313.

[9] ______, *Solutions of Emden–Fowler equations by homotopy-perturbation method*, Nonlinear. Anal. Ser. B: Real World Appl..

[10] ______, *Application of homotopy-perturbation method to Klein–Gordon and Sine–Gordon equations*, Chaos Solitons Fractals.

[11] M. S. H. CHOWDHURY, I. HASHIM AND O. ABDULAZIZ, *Application of homotopy-perturbation method to nonlinear population dynamics models*, Phys. Lett. A **368** (2007), 251–258.

[12] M. S. H. CHOWDHURY, I. HASHIM AND S. MOMANI, *The multistage homotopy-perturbation method: A powerful scheme for handling the Lorenz system*, Chaos Solitons Fractals.

[13] M. GORJI, D. D. GANJI AND S. SOLEIMANI, *New application of He's homotopy perturbation method*, Internat. J. Nonlinear Sci. Numer. Simul. **8** (2007), no. 3, 319–328.

[14] A. GHORBANI AND J. SABERI-NADJAFI, *He's homotopy perturbation method for calculating Adomian polynomials*, Internat. J. Nonlinear Sci. Numer. Simul. **2** (2007), 229–232.

[15] D. D. GANJI AND A. SADIGHI, *Application of He's homotopy-perturbation method to nonlinear coupled systems of reaction-diffusion equations*, Internat. J. Nonlinear Sci. Numer. Simul. **7** (2006), no. 4, 411–418.

[16] I. HASHIM AND M. S. H. CHOWDHURY, *Adaptation of homotopy-perturbation method for numeric-analytic solution of system of ODEs*, Phys. Lett. A.

[17] I. HASHIM, M. S. H. CHOWDHURY AND S. MAWA, *On multistage homotopy-perturbation method applied to nonlinear biochemical reaction model*, Chaos Solitons Fractals.

[18] I. HASHIM, M. S. M. NOORANI, R. AHMAD, S. A. BAKAR, E. S. ISMAIL, AND A. M. ZAKARIA, *Accuracy of the Adomian decomposition method applied to the Lorenz system*, Chaos Solitons Fractals **28** (2006), 1149–1158.

[19] J. H. HE, *A coupling method of homotopy technique and perturbation technique for nonlinear problems*, Internat. J. Non-Linear Mech. **35** (2000), no. 1, 37–43.

[20] ______, *Homotopy perturbation method for solving boundary value problems*, Phys. Lett. A **350** (2006), 87–88.

[21] ______, *A simple perturbation approach to Blasius equation*, Appl. Math. Comput. **140** (2003), 217–222.

[22] ______, *Application of homotopy perturbation method to nonlinear wave equations*, Chaos Solitons Fractals **26** (2005), 695–700.

[23] ______, *Homotopy perturbation method for bifurcation of nonlinear problems*, Internat. J. Nonlinear Sci. Numer. Simul. **6** (2005), no. 2, 207–208.

[24] ______, *Homotopy perturbation method for solving boundary value problems*, Phys. Lett. A **350** (2006), 87–88.

[25] ______, *Non-Perturbative Methods for Strongly Nonlinear Problems*, Die Deutsche Bibliothek, Germany, 2006.

[26] ______, *Some asymptotic methods for strongly nonlinear equations*, Internat. J. Modern Phys. B **20** (2006), 1141–1199.

[27] M. S. M. Noorani, I. Hashim, R. Ahmad, S. A. Bakar, E. S. Ismail and A. M. Zakaria, *Comparing numerical methods for the solutions of the Chen system*, Chaos Solitons Fractals **32** (2007), 1296–1304.

[28] T. Ozis and A. Yildirim, *Traveling wave solution of Korteweg–de Vries equation using He's homotopy perturbation method*, Internat. J. Nonlinear Sci. Numer. Simul. **8** (2007), no. 2, 239–242.

[29] ______, *A comparative study of He's homotopy perturbation method for determining frequency-amplitude relation of a nonlinear oscillator with discontinuities*, Internat. J. Nonlinear Sci. Numer. Simul. **8** (2007), no. 2, 2343–248.

[30] M. Rafei, H. Daniali, D. D. Ganji and H. Pashaei, *Solution of the prey and predator problem by homotopy perturbation method*, Appl. Math. Comput..

[31] M. Rafei, and D. D. Ganji, *Explicit solutions of Helmholtz equation and fifth-order KdV equation using homotopy perturbation method*, Internat. J. Nonlinear Sci. Numer. Simul. **7** (2006), no. 3, 321–328.

[32] G. FC. Simmons, *Differential equations with applications and historical notes*, McGraw–Hill, New York, 1972.

[33] A. Sadighi and D. D. Ganji, *Solution of the generalized nonlinear Boussinesq equation using homotopy perturbation and variational iteration methods*, Internat. J. Nonlinear Sci. Numer. Simul. **8** (2007), no. 3, 435–443.

[34] A. M. Siddiqui, R. Mahmood and Q. K. Ghori., *Thin film flow of a third grade fluid on a moving belt by He's homotopy perturbation method*, Internat. J. Nonlinear Sci. Numer. Simul. **7** (2007), no. 1, 7–14.

[35] A. M. Siddiqui, M. Ahmed and Q. K. Ghori, *Couette and Poiseuille flows for non-newtonian fluids*, Internat. J. Nonlinear Sci. Numer. Simul. **7** (2006), no. 1, 15–26.

[36] H. Tari, D.D. Ganji and M. Rostamian, *Approximate solutions of* $K(2,2)$*, KdV and modified KdV equations by variational iteration method, homotopy perturbation method and homotopy analysis method*, Internat. J. Nonlinear Sci. Numer. Simul. **8** (2007), no. 2, 203–210.

[37] E. Yusufoglu, *Homotopy perturbation method for solving a nonlinear system of second order boundary value problems*, Internat. J. Nonlinear Sci. Numer. Simul. **8** (2007), no. 3, 353–358.

Manuscript received September 14, 2007

M. S. H. Chowdhury
School of Mathematical Sciences
University Kebangsaan Malaysia
43600 Bangi Selangor, MALAYSIA

E-mail address: sazzadhc@yahoo.com

I. Hashim
Institute of Systems Biology
University Kebangsaan Malaysia
43600 Bangi Selangor, MALAYSIA

E-mail address: ishak_h@ukm.my

R. Roslan
Universiti Malaysia Sabah
Kota Kinabalu Sabah, MALAYSIA