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APPLICATIONS OF VARIATIONAL ITERATION AND HOMOTOPY PERTURBATION METHODS TO FRACTIONAL EVOLUTION EQUATIONS

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Abstract. In this paper, variational iteration and homotopy perturbation methods that developed for integer-order differential equations are directly extended to derive explicit and numerical solutions of various evolution equations with time-fractional derivatives. The results reveal that the two methods are very effective and convenient for solving nonlinear partial differential equations of fractional order.

1. Introduction

In recent years, considerable interest in fractional differential equation has been stimulated due to their numerous applications in the areas of physics and engineering [21]. Many important phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry and material science are well described by fractional differential equation [10], [11], [21], [23]. The solution of fractional differential equation is of significant interest. In general, there exists no method that yields an exact solution for fractional differential equation. Only approximate solutions can be derived using linearization or series solutions methods.

The aim of this paper is to extend directly the variational iteration and homotopy perturbation methods to consider the numerical solution of the fractional

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evolution equation with time-fractional derivatives of the form

$$D_t^{\alpha} u(x,t) = L(u) + N(u) + g(x,t), \tag{1.1}$$

where L is a linear operator, N represents a nonlinear operator, $g(x,t)$ is the source term, and D_t^{α} is the Caputo fractional derivative of order α with $m-1 < \alpha \leq m$. The function $u(x,t)$ is assumed to be a causal function of time, i.e. vanishing for $t < 0$.

The variational iteration method [6], [7] and the homotopy perturbation method [8], [9] are relatively new approaches to provide an analytical approximation to linear and nonlinear problems, and they are particulary valuable as tools for scientists and applied mathematicians, because they provide immediate and visible symbolic terms of analytic solutions, as well as numerical approximate solutions to both linear and nonlinear differential equations without linearization or discretization. In this paper, the two methods are implemented to derive explicit and numerical solutions to nonlinear fractional evolution equations of the form given in equation (1.1).

2. Variational iteration method

The principles of the variational iteration method (VIM) and its applicability for various kinds of differential equations are given in [1], [3], [6], [7], [22], [24], [25]. The variational iteration method for solving the time-fractional partial differential equation (1.1) has been considered in the work of Momani and Odibat [12], [13], [16], [17]. For the sake of clarity the present section gives a brief account of their approach. For $m = 1$, the variational iteration formula for equation (1.1) can be expressed in the form:

$$u_{k+1}(x,t) = u_k(x,t) - \int_0^t \left(\frac{\partial^{\alpha}}{\partial \xi^{\alpha}} u_k(x,\xi) - L(u_k) - N(u_k) - g(x,\xi) \right) d\xi, \tag{2.1}$$

and for $m = 2$, we obtain the following iteration formula:

$$u_{k+1}(x,t) = u_k(x,t) + \int_0^t (\xi - t) \left(\frac{\partial^{\alpha}}{\partial \xi^{\alpha}} u_k(x,\xi) - L(u_k) - N(u_k) - g(x,\xi) \right) d\xi.$$

If we begin with the initial approximation $u_0(x,t) = u(x,0)$, in the case of $m = 1$, and the initial approximation $u_0(x,t) = u(x,0) + tu_t(x,t)$, in the case of $m = 2$, then the approximations $u_n(x,t)$, for $n \geq 1$, can be completely determined. Finally, we approximate the solution of the time-fractional equation (1.1) $u(x,t) = \lim_{n \to \infty} u_n(x,t)$ by the Nth term $u_N(x,t)$.

3. Homotopy perturbation method

The homotopy perturbation method (HPM) which provides an analytical approximate solution is applied to various nonlinear problems [2], [4], [5], [8], [9],

[20], [22], [24], [25]. In the work of Odibat and Momani [14], [15], [18], a new modification has been presented for solving differential equations of fractional order. For the sake of clarity the present section gives a brief account of their modification. For $m-1 < \alpha \le m$, the new modification constructs the following homotopy

$$\frac{\partial u^m}{\partial t^m} - L(u) - g(x,t) = p\left[\frac{\partial u^m}{\partial t^m} + N(u) - D_t^\alpha u\right], \tag{3.1}$$

or

$$\frac{\partial u^m}{\partial t^m} - g(x,t) = p\left[\frac{\partial u^m}{\partial t^m} + L(u) + N(u) - D_t^\alpha u\right], \tag{3.2}$$

where $p \in [0,1]$. The homotopy parameter p always changes from zero to unity. In case $p = 0$, equation (3.1) becomes the linearized equation

$$\frac{\partial u^m}{\partial t^m} - L(u) = g(x,t),$$

and equation (3.2) becomes the linearized equation

$$\frac{\partial u^m}{\partial t^m} = g(x,t),$$

and when it is one, equation (3.1) or equation (3.2) turns out to be the original fractional differential equation (1.1). The basic assumption is that the solution of equation (3.1) or equation (3.2) can be written as a power series in p:

$$u = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \dots . \tag{3.3}$$

Finally, we approximate the solution $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$ by the truncated series

$$\phi_N(x,t) = \sum_{n=0}^{N-1} u_n(x,t).$$

4. Applications

In this section we consider two examples that demonstrate the performance and efficiency of the two methods for solving evolution equations with time-fractional derivatives.

Example 4.1. Consider the nonlinear time-fractional RLW equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} - u_{xxt} + \left(\frac{u^2}{2}\right)_x = 0, \quad t > 0,\ 0 < \alpha \le 1 \tag{4.1}$$

subject to the initial condition

$$u(x,0) = x. \tag{4.2}$$

Variational iteration method. The iteration formula for equation (4.1), according to the formula (2.1), can be constructed as

$$u_{k+1}(x,t) = u_k(x,t) - \int_0^t \left(\frac{\partial^\alpha}{\partial \xi^\alpha} u_k(x,\xi) - (u_k)_{xx\xi}(x,\xi) + \left(\frac{u_k^2}{2}\right)_x (x,\xi) \right) d\xi,$$

By the above variational iteration formula, begin with the initial approximation $u_0 = x$, we can obtain the following iterations

$$\begin{aligned}
u_0 &= x,\\
u_1 &= x(1-t),\\
u_2 &= x\left(1 - 2t + t^2 - \frac{t^3}{3} + \frac{t^{2-\alpha}}{\Gamma(3-\alpha)}\right),\\
u_3 &= x\left(1 - 3t + 3t^2 - \frac{7t^3}{3} + \frac{7t^4}{6} - \frac{7t^5}{15} + \frac{t^6}{9} - \frac{t^7}{63} + \frac{3t^{2-\alpha}}{\Gamma(3-\alpha)}\right.\\
&\quad - \frac{4t^{3-\alpha}}{\Gamma(4-\alpha)} - \frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)} + \left(\frac{2}{\Gamma(5-\alpha)} + \frac{4\Gamma(4-\alpha)}{\Gamma(3-\alpha)\Gamma(5-\alpha)}\right)t^{4-\alpha}\\
&\quad \left. - \frac{2\Gamma(5-\alpha)t^{5-\alpha}}{\Gamma(3-\alpha)\Gamma(6-\alpha)} + \frac{2\Gamma(6-\alpha)t^{6-\alpha}}{3\Gamma(3-\alpha)\Gamma(7-\alpha)} - \frac{\Gamma(5-2\alpha)t^{5-2\alpha}}{\Gamma(3-\alpha)^2\Gamma(6-2\alpha)}\right),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Homotopy perturbation method. In view of the homotopy (4.1), we can construct the following homotopy

$$\frac{\partial u}{\partial t} = p\left[\frac{\partial u}{\partial t} + u_{xxt} - \left(\frac{u^2}{2}\right)_x - D_t^\alpha u\right]. \tag{4.3}$$

Substituting (3.3) and the initial condition (4.2) into the homotopy (4.3) and equating the terms with identical powers of p, we obtain a set of differential equations. Solving this set, we obtain the following approximations

$$\begin{aligned}
u_0 &= x,\\
u_1 &= -xt,\\
u_2 &= x\left(-t + t^2 + \frac{t^{2-\alpha}}{\Gamma(3-\alpha)}\right),\\
u_3 &= x\left(-t + 2t^2 - t^3 + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{4t^{3-\alpha}}{\Gamma(4-\alpha)} - \frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)}\right),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Table 1 shows the approximate solutions for equation (4.1) obtained for different values of α using VIM and HPM. The value of $\alpha = 1$ is the only case for which we know the exact solution $u(x,t) = x/(t+1)$ and our approximate solution using the VIM is more accurate than the approximate solution obtained using the HPM. It is to be noted that only the fourth-order term of the approximate solutions were used in Table 1. It is evident that the efficiency of this

		$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1.0$		
t	x	u_{VIM}	u_{HPM}	u_{VIM}	u_{HPM}	u_{VIM}	u_{HPM}	u_{Exact}
0.2	0.25	0.166654	0.168080	0.188458	0.189222	0.208242	0.208000	0.208333
	0.50	0.333308	0.336160	0.376915	0.378443	0.416483	0.416000	0.416667
	0.75	0.499962	0.504240	0.565373	0.567665	0.624725	0.624000	0.625000
	1.0	0.666616	0.672320	0.753830	0.756886	0.832967	0.832000	0.833333
0.4	0.25	0.139655	0.146281	0.164885	0.167084	0.177521	0.174000	0.178571
	0.50	0.279311	0.292562	0.329771	0.334167	0.355041	0.348000	0.357143
	0.75	0.418966	0.438843	0.494656	0.501251	0.532562	0.522000	0.535714
	1.0	0.558622	0.585124	0.659542	0.668335	0.710082	0.696000	0.714286
0.6	0.25	0.138768	0.149304	0.155643	0.153867	0.152305	0.136000	0.156250
	0.50	0.277536	0.298608	0.311287	0.307734	0.304610	0.272000	0.312500
	0.75	0.416304	0.447912	0.466930	0.461600	0.456915	0.408000	0.468750
	1.0	0.555072	0.597216	0.622574	0.615467	0.609220	0.544000	0.625000

Table 1. Numerical values when $\alpha = 0.5$, 0.75 and 1.0 for equation (4.1)

approach can be dramatically enhanced by computing further terms of $u(x,t)$ when the VIM and HPM are used.

Example 4.2. Consider the linear time-fractional equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = u_{xxxx}, \quad t > 0,\ 0 < \alpha \le 1 \tag{4.4}$$

subject to the initial condition

$$u(x,0) = \sin(x).$$

Variational iteration method. The iteration formula for equation (4.4), according to the formula (2.1), can constructed as

$$u_{k+1}(x,t) = u_k(x,t) - \int_0^t \left(\frac{\partial^\alpha}{\partial \xi^\alpha} u_k(x,\xi) - (u_k)_{xxxx}(x,\xi)\right) d\xi. \tag{4.5}$$

Homotopy perturbation method. In view of the homotopy (4.1), we can construct the following homotopy

$$\frac{\partial u}{\partial t} = p\left[\frac{\partial u}{\partial t} + u_{xxxx} - D_t^\alpha u\right]. \tag{4.6}$$

Using the variational iteration formula (4.5) or the homotopy (4.6), begin with the initial approximation $u_0 = \sin(x)$, we can obtain the same approximate solution

$$u(x,t) = \left(1 - 4t + 3t^2 - \frac{2t^3}{3} + \frac{t^4}{24} + \frac{6t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{8t^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{3t^{4-\alpha}}{\Gamma(5-\alpha)} - \frac{4t^{3-2\alpha}}{\Gamma(4-2\alpha)} + \frac{3t^{4-2\alpha}}{\Gamma(5-2\alpha)} + \frac{t^{4-3\alpha}}{\Gamma(5-3\alpha)}\right) \sin(x). \tag{4.7}$$

If $\alpha = 1$, then the solution (4.7) reduces to

$$u(x,t) = \left(1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24}\right) \sin(x),$$

which converges to the exact solution $\exp(-t)\sin(x)$.

5. Remarks

In [19], the present authors have shown that the VIM and the HPM are equivalent for linear differential equations of fractional order. If we start with the initial approximation

$$u_0 = \int_0^t g(x,\xi)\,d\xi + u(x,0),$$

when $m = 1$, or the initial approximation

$$u_0 = \int_0^t \int_0^\xi g(x,\tau)\,d\tau\,d\xi + u(x,0)t + u_t(x,0),$$

when $m = 2$, for VIM, then the nth-order term approximate solutions for equation (1.1), when $N = 0$, obtained using VIM or HPM are equal.

For nonlinear differential equations of fractional order the methods are not identical, but they are equivalent as powerful and efficient techniques for obtaining analytic as well as approximate solutions for wide classes of nonlinear problems. They provide more realistic series solutions that converge very rapidly in real physical problems.

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