Topological Methods in Nonlinear Analysis
Journal of the Juliusz Schauder Center
Volume 31, 2008, 211–226

ALGORITHMS FOR NONLINEAR FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS: A SELECTION OF NUMERICAL METHODS

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This paper is dedicated to the great Egyptian scientist Mohamed El Naschie on the occasion of his 65^{th} *birthday.*

Abstract. Fractional order partial differential equations, as generalization of classical integer order partial differential equations, are increasingly used to model problems in fluid flow, finance and other areas of applications. In this paper we present a collection of numerical algorithms for the solution of nonlinear partial differential equations with space- and time-fractional derivatives. The fractional derivatives are considered in the Caputo sense. Two numerical examples are given to demonstrate the effectiveness of the present methods. Results show that the numerical schemes are very effective and convenient for solving nonlinear partial differential equations of fractional order.

1. Introduction

In recent years fractional derivatives have found numerous applications in many fields of physics, finance, and hydrology [27]. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. Fractional space derivatives are used to model anomalous diffusion or dispersion, where a particle plume spreads at

2000 *Mathematics Subject Classification.* 26A33, 35A15, 35A35.

Key words and phrases. Fractional differential equations, differential transform method, homotopy perturbation method, variational iteration method, Adomian decomposition method.

The authors would like to thank the referee for his valuable suggestions and comments.

a rate inconsistent with the classical Brownian motion model, and the plume may be asymmetric. When a fractional derivative replaces the second derivative in a diffusion or dispersion model, it leads to enhanced diffusion (also called superdiffusion) [11], [28]. A review of some applications of fractional derivatives to many problems in physics is given by Metzler and Klafter [13].

A great deal of effort has been expended over the last 10 years or so in attempting to find robust and stable numerical and analytical methods for solving fractional partial differential equations of physical interest. Numerical and analytical methods have included finite difference method [10]–[12], [29], Adomian decomposition method [1], [14]–[18], [23], variational iteration method [5], [19], [21], [24], homotopy perturbation method [20], [21], and generalized differential transform method [22], [25]. Among these solution techniques, the variational iteration method (VIM) and the Adomian decomposition method (ADM) are the most transparent methods of solution of fractional partial differential equations, because they provide immediate and visible symbolic terms of analytic solutions, as well as numerical approximate solutions to both linear and nonlinear differential equations without linearization or discretization. The homotopy perturbation method (HPM), which is a coupling of the traditional perturbation method and homotopy in topology, deforms continuously to a simple problem which is easily solved. This method, which doesn't require a small parameter in an equation, has a significant advantage in that it provides an analytical approximate solution to a wide range of nonlinear problems in applied sciences. The finite difference scheme for fractional derivatives is based on the definition in the Grünwald–Letnikov form. This from can make the scheme more flexible and straightforward. Recently the authors developed a semi-numerical method for solving linear partial differential equations of fractional order. This method is named as generalized differential transform method (GDTM) [22], [25] and is based on the two-dimensional differential transform method [2], [4], [30] and generalized Taylor's formula [26].

In this paper, we present a numerical comparison between ADM, VIM, and HPM and the generalized differential transform method (GDTM) [22], [25] for solving nonlinear partial differential equations with space- and time-fractional derivatives of the form

$$\frac{\partial^{\mu} u}{\partial t^{\mu}} = \frac{\partial^{\nu} u}{\partial x^{\nu}} + N_f(u(x,t)), \quad m-1 < \mu \le m,\ n-1 < \nu \le n,\ n,m \in N, \tag{1.1}$$

where μ and ν are parameters describing the order of the fractional time- and space-derivatives in the Caputo sense, respectively, and N_f is a nonlinear operator which might include other fractional derivatives with respect to the variables x and t. The function $u(x,t)$ is assumed to be a causal function of time and space, i.e. vanishing for $t < 0$ and $x < 0$. The general response expression contains parameters describing the order of the fractional derivatives that can be varied to obtain various responses.

There are several definitions of a fractional derivative of order $\nu > 0$ [3], [27]. The two most commonly used definitions are the Riemann–Liouville and Caputo. The Riemann–Liouville fractional integration of order ν is defined as

$$J_a^\nu f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} f(t)\, dt, \quad \nu > 0,\ x > 0.$$

The next two equations define the Riemann–Liouville and Caputo fractional derivatives of order ν, respectively,

$$\begin{aligned} D_a^\nu f(x) &= \frac{d^m}{dx^m}(J_a^{m-\nu} f(x)), \\ D_{*a}^\nu f(x) &= J_a^{m-\nu}\left(\frac{d^m}{dx^m} f(x)\right), \end{aligned}$$

where $m-1 < \nu \leq m$ and $m \in N$. For now, the Caputo fractional derivative will be denoted by D_*^ν to maintain a clear distinction from the Riemann–Liouville fractional derivative.

The Caputo fractional derivative is considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem. In this paper, we consider the one-dimensional space- and time-fractional nonlinear partial differential equation (1.1), where the unknown function $u = u(x,t)$ is a assumed to be a causal function of space and time, respectively, and the fractional derivatives are taken in the Caputo sense as follows:

Definition 1.1. For m to be the smallest integer that exceeds μ, the Caputo time-fractional derivative operator of order $\mu > 0$ is defined as

$$\begin{aligned} D_{*t}^\mu u(x,t) &= \frac{\partial^\mu u(x,t)}{\partial t^\mu} \\ &= \begin{cases} \dfrac{1}{\Gamma(m-\mu)} \displaystyle\int_0^t (t-\tau)^{m-\mu-1} \frac{\partial^m u(x,\tau)}{\partial \tau^m}\, d\tau & \text{for } m-1 < \mu < m, \\ \dfrac{\partial^m u(x,t)}{\partial t^m} & \text{for } \mu = m \in \mathcal{N}, \end{cases} \end{aligned}$$

and the space-fractional derivative operator of order $\nu > 0$ is defined as

$$\begin{aligned} D_{*x}^\nu u(x,t) &= \frac{\partial^\nu u(x,t)}{\partial x^\nu} \\ &= \begin{cases} \dfrac{1}{\Gamma(n-\nu)} \displaystyle\int_0^x (x-\theta)^{n-\nu-1} \frac{\partial^n u(\theta,t)}{\partial \theta^n}\, d\theta & \text{for } n-1 < \nu < n, \\ \dfrac{\partial^n u(x,t)}{\partial x^n} & \text{for } \nu = n \in \mathcal{N}. \end{cases} \end{aligned}$$

For mathematical properties of fractional derivatives and integrals one can consult the mentioned references.

2. Generalized two-dimensional differential transform method

Recently the authors developed the generalized two-dimensional differential transform method (GDTM) for solving linear partial differential equations of fractional order [22], [25]. This method is based on the two-dimensional differential transform method [2], [4], [30] and generalized Taylor's formula [26]. In this section we shall use the analysis presented in [25] to construct our numerical method for solving the following nonlinear partial differential equation with space- and time-fractional derivatives

$$\frac{\partial^\mu u}{\partial t^\mu} = \frac{\partial^\nu u}{\partial x^\nu} + N_f(u(x,t)), \quad m-1 < \mu \le m,\ n-1 < \nu \le n,\ n,m \in N, \tag{2.1}$$

where N_f is a nonlinear operator which might include other fractional derivatives with respect to the variables x and t.

Firstly, if $0 < \mu \le 1$ and $0 < \nu \le 1$, we suppose that the solution of the nonlinear equation (2.1) can be written as a product of single-valued functions. In this case, selecting $\alpha = \mu$, $\beta = \nu$ and applying Theorem 2.1 in [25] to both sides of (2.1), equation (2.1) transforms to

$$\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)} U_{\alpha,\beta}(k,h+1) = \frac{\Gamma(\beta(k+1)+1)}{\Gamma(\beta k+1)} U_{\alpha,\beta}(k+1,h) + F_{\alpha,\beta}(k,h),$$

where $F_{\alpha,\beta}(k,h)$ is the generalized differential transformation of $N_f(u(x,t))$.

Secondly, if $m-1 < \mu = m_1/m_2 \le m$ and $0 < \nu \le 1$, we suppose that the solution of the nonlinear equation (2.1) can be written as a product of single-valued functions $u(x,t) = v(x)w(t)$, where the function $w(t)$ satisfies the conditions given in Theorem 2.2 in [25]. In this case, selecting $\alpha = 1/m_2$, $\beta = \nu$, and applying Theorem 2.3 in [21] to both sides of (2.1), equation (2.1) transforms to

$$\frac{\Gamma(\alpha(h+1)+m_1)}{\Gamma(\alpha h+1)} U_{\alpha,\beta}(k,h+m_1) = \frac{\Gamma(\beta(k+1)+1)}{\Gamma(\beta k+1)} U_{\alpha,\beta}(k+1,h) + F_{\alpha,\beta}(k,h).$$

Finally, if $m-1 < \mu = m_1/m_2 \le m$ and $n-1 < \nu = n_1/n_2 \le n$, we suppose that the solution of the nonlinear equation (2.1) can be written as a product of single-valued functions $u(x,t) = v(x)w(t)$, where the functions $v(x)$ and $w(t)$ satisfy the conditions given in Theorem 2.2. In this case, selecting $\alpha = 1/m_2$, $\beta = 1/n_2$, and applying Theorem 2.3 in [25] to both sides of (2.1), equation (2.1) transforms to

$$\frac{\Gamma(\alpha(h+1)+m_1)}{\Gamma(\alpha h+1)}U_{\alpha,\beta}(k,h+m_1) \\ = \frac{\Gamma(\beta(k+1)+n_1)}{\Gamma(\beta k+1)}U_{\alpha,\beta}(k+n_1,h)+F_{\alpha,\beta}(k,h).$$

In all the above cases, the solution of the nonlinear space-time fractional equation (2.1), using definition (2.1), can be written as:

$$u(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U_{\alpha,\beta}(k,h)x^{k\alpha}y^{h\beta}.$$

3. Adomian decomposition method

The decomposition method of Adomian is another technique for obtaining the solution of equations, especially nonlinear equations. This method proposed by the American mathematician, G. Adomian (1923–1996) has been applied for solving various problems [1], [14]–[18], [23]. The advantage of this method is that it provides a direct scheme for solving the problem, i.e. without the need for linearization, perturbation, massive computation and any transformation.

The decomposition method requires that the nonlinear fractional differential equation (1.1) be expressed in terms of operator from as

$$\frac{\partial^\mu u}{\partial t^\mu}=\frac{\partial^\nu u}{\partial x^\nu}+N_f(u(x,t)),\quad m-1<\mu\le m,\ n-1<\nu\le n,\ n,m\in N, \tag{3.1}$$

where N_f is a nonlinear operator which might include other fractional derivatives with respect to the variables x and t.

Applying the operator J^μ, the inverse of the operator D^μ_{*t}, to both sides of equation (3.1) yields

$$u(x,t)=\sum_{k=0}^{m-1}\frac{\partial^k u}{\partial t^k}(x,0^+)\frac{t^k}{k!}+J^\mu[D^\nu_{*x}u(x,t)+N_f(u(x,t))]. \tag{3.2}$$

The Adomian decomposition method suggests the solution $u(x,t)$ be decomposed into the infinite series of components

$$u(x,t)=\sum_{i=0}^{\infty}u_i(x,t), \tag{3.3}$$

and the nonlinear function in equation (3.2) is decomposed as follows:

$$N_f(u(x,t))=\sum_{i=0}^{\infty}A_i, \tag{3.4}$$

where A_i are so-called the Adomian polynomials.

Substituting the decomposition series (3.3) and (3.4) into both sides of (3.2) gives

$$\sum_{i=0}^{\infty} u_i(x,t) = \sum_{k=0}^{m-1} \frac{\partial^k u}{\partial t^k}(x,0^+)\frac{t^k}{k!} + J^\mu\bigg[D^\nu_{*x}\bigg(\sum_{i=0}^{\infty} u_i(x,t)\bigg) + \sum_{i=0}^{\infty} A_i\bigg].$$

From this equation, the iterates are determined by the following recursive way

$$\begin{aligned} u_0(x,t) &= \sum_{k=0}^{m-1} \frac{\partial^k u}{\partial t^k}(x,0^+)\frac{t^k}{k!}, \\ u_1(x,t) &= J^\mu(D^\nu_{*x}u_0 + A_0), \\ u_2(x,t) &= J^\mu(D^\nu_{*x}u_1 + A_1), \\ &\cdots\cdots\cdots\cdots\cdots \\ u_{i+1}(x,t) &= J^\mu(D^\nu_{*x}u_i + A_i). \end{aligned} \tag{3.5}$$

The Adomian polynomial A_i can be calculated for all forms of nonlinearity according to specific algorithms constructed by Adomian [1], [14]–[18], [23]. The general form of formula for A_i Adomian polynomials is

$$A_i = \frac{1}{i!}\bigg[\frac{d^i}{d\lambda^i} N_f\bigg(\sum_{k=0}^{\infty}\lambda^k u_k\bigg)\bigg]_{\lambda=0}.$$

This formula is easy to be computed by using Mathematica software or by writing a computer code to get as many polynomials as we need in the calculation of the numerical as well as explicit solutions.

Finally, we approximate the solution $u(x,t)$ by the truncated series

$$\phi_I(x,t) = \sum_{i=0}^{I-1} u_i(x,t) \quad \text{and} \quad \lim_{I\to\infty}\phi_I(x,t) = u(x,t). \tag{3.6}$$

4. Variational iteration method

Variational iteration method was first proposed by the Chinese mathematician He [6], [7]. This method has been employed to solve a large variety of linear and nonlinear problems with approximations converging rapidly to accurate solutions [5], [19], [21], [24]. Some advantages of this technique are: (i) the initial condition can be chosen freely with some unknown parameters, (ii) the unknown parameters in the initial condition can be easily identified, (iii) the calculation is simple and straightforward.

To apply the variational iteration method to nonlinear partial differential equations of fractional order, we consider the following equation

$$D^\mu_{*t}u(x,t) = D^\nu_{*x}u(x,t) + N_f(u(x,t)), \quad x>0,\ t>0, \tag{4.1}$$

The initial and boundary conditions associated with (4.1) are of the from

$$u(x,0)=f(x), \quad 0<\mu\le 1,$$

and

$$u(x,0)=f(x), \quad \frac{\partial u(x,0)}{\partial t}=g(x), \quad 1<\mu\le 2.$$

The correction functional for equation (4.1) can be approximately expressed as follows:

$$u_{k+1}(x,t)=u_k(x,t) + \int_0^t \lambda(\xi)\Big(\frac{\partial^m}{\partial \xi^m}u_k(x,\xi)-N_f(\widetilde{u}_k(x,\xi))-D^{\nu}_{*x}\widetilde{u}_k(x,\xi)\Big)\,d\xi,$$

where λ is a general Lagrange multiplier, which can be identified optimally via variational theory, here $\widetilde{u}_k$ is considered as a restricted variation. Making the above functional stationary,

$$\delta u_{k+1}(x,t)=\delta u_k(x,t)+\delta\int_0^t \lambda(\xi)\Big(\frac{\partial^m}{\partial \xi^m}u_k(x,\xi)\Big)\,d\xi,$$

yields the following Lagrange multipliers

$$\lambda=-1, \quad \text{for } m=1, \qquad \lambda=\xi-t, \quad \text{for } m=2.$$

Therefore, for $m=1$, we obtain the following iteration formula:

$$u_{k+1}(x,t)=u_k(x,t)-\int_0^t \Big(\frac{\partial^\mu}{\partial \xi^\mu}u_k(x,\xi)-N_f(u_k(x,\xi))-D^{\nu}_{*x}u_k(x,\xi)\Big)\,d\xi. \tag{4.2}$$

In this case, we begin with the initial approximation $u_0(x,t)=f(x)$.

For $m=2$, we obtain the following iteration formula:

$$u_{k+1}(x,t)=u_k(x,t) + \int_0^t (\xi-t)\Big(\frac{\partial^\mu}{\partial \xi^\mu}u_k(x,\xi)-N_f(u_k(x,\xi))-D^{\nu}_{*x}u_k(x,\xi)\Big)\,d\xi, \tag{4.3}$$

and in this case, we begin with the initial approximation $u_0(x,t)=f(x)+tg(x)$.

5. Homotopy perturbation method

He's homotopy perturbation technique [20], [21] has recently been used to solve linear and nonlinear fractional partial differential equations, and it has been claimed that this techniques is valid for nonlinear problems regardless of the presence or absence of a small parameter in the differential equations.

For convenience of the reader, we will present a review of the HPM [8], [9], then we will apply the method to solve the nonlinear problem (1.1). To achieve our goal, we consider the nonlinear differential equation

$$L(u)+N(u)=f(r), \quad r\in\Omega, \tag{5.1}$$

with boundary conditions

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma,$$

where L is a linear operator, while N is nonlinear operator, B is a boundary operator, Γ is the boundary of the domain Ω and $f(r)$ is a known analytic function.

He's homotopy perturbation technique (see [8] and [9]) defines the homotopy $v(r,p)\colon \Omega \times [0,1] \to \mathcal{R}$ which satisfies

$$\mathcal{H}(v,p) = (1-p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(r)] = 0, \tag{5.2}$$

or

$$\mathcal{H}(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{5.3}$$

where $r \in \Omega$ and $p \in [0,1]$ is an impeding parameter, u_0 is an initial approximation which satisfies the boundary conditions. Obviously, from equations (5.2) and (5.3), we have

$$\mathcal{H}(v,0) = L(v) - L(u_0) = 0, \qquad \mathcal{H}(v,1) = L(v) + N(v) - f(r) = 0.$$

The changing process of p from zero to unity is just that of $v(r,p)$ from u_0 to $u(r)$. In topology, this is called a deformation, $L(v) - L(u_0)$ and $L(v) + N(v) - f(r)$ are homotopic. The basic assumption is that the solution of equations (5.2) and (5.3) can be expressed as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + \ldots .$$

The approximate solution of equation (5.1), therefore, can be readily obtained:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \ldots . \tag{5.4}$$

The convergence of the series (5.4) has been proved in [8], [9].

To use the homotopy perturbation method to solve equation (1.1), we apply a modified form of this method suggested by Momani and Odibat in [20] for solving linear and nonlinear fractional partial differential equations as follows:

At first, rewrite (1.1) in the form

$$D^{\mu}_{*t} u(x,t) = D^{\nu}_{*x} u(x,t) + N_f(u(x,t)), \quad x > 0,\ dt > 0, \tag{5.5}$$

subject to the initial conditions

$$u^k(x,0) = g_k(x), \quad k = 0, \ldots, m-1.$$

In view of the homotopy technique, we can construct the following homotopy:

$$\frac{\partial u^m}{\partial t^m} - D^{\nu}_{*x} u(x,t) = p\left[\frac{\partial u^m}{\partial t^m} + N(u, u_x, u_{xx}) - D^{\mu}_{*t} u\right], \tag{5.6}$$

or

$$\frac{\partial u^m}{\partial t^m} = p\left[\frac{\partial u^m}{\partial t^m} + D^{\nu}_{*x}u(x,t) + N(u,u_x,u_{xx}) - D^{\mu}_{*t}u\right], \tag{5.7}$$

where $p \in [0,1]$. The homotopy parameter p always changes from zero to unity. In case $p = 0$, equation (5.6) becomes the linearized equation

$$\frac{\partial u^m}{\partial t^m} = D^{\nu}_{*x}u(x,t),$$

and equation (5.7) becomes the linearized equation

$$\frac{\partial u^m}{\partial t^m} = 0, \tag{5.8}$$

and when it is one, equation (5.6) or (5.7) turns out to be the original fractional differential equation (5.5). The basic assumption is that the solution of equation (5.6) or (5.7) can be written as a power series in p:

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \dots \tag{5.9}$$

Finally, we approximate the solution $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ by the truncated series

$$\phi_I(x,t) = \sum_{i=0}^{I-1} u_i(x,t).$$

6. Numerical experiments

In this section we consider two examples that demonstrate the performance and efficiency of the four algorithms for solving nonlinear partial differential equations with time- or space-fractional derivatives.

EXAMPLE 6.1. Consider the following nonlinear time-fractional KdV equation

$$\frac{\partial^{\mu} u}{\partial t^{\mu}} - (u^2)_x + [u(u)_{xx}]_x = 0, \quad t > 0,\ x > 0,\ 0 < \mu \leq 1, \tag{6.1}$$

subject to the initial condition

$$u(x,0) = \sinh^2(x/2). \tag{6.2}$$

6.1.1. Adomian decomposition method. To solve the problem using the decomposition method, we simply substitute (6.1) and the initial condition (6.2) into (3.5), to obtain the following recurrence relation

$$\begin{aligned} u_0(x,t) &= u(x,0) = \sinh^2(x/2), \\ u_{i+1}(x,t) &= J^{\mu}((A_i)_x - (B_i)_x), \quad i \geq 0, \end{aligned} \tag{6.3}$$

where A_i and B_i are the Adomian polynomials for the nonlinear functions $N = u^2$ and uu_{xx}, respectively.

In view of (6.3), the first three terms of the decomposition series (3.6) are given by

$$u_{\mathrm{ADM}}(x,t) = \sinh\left(\frac{x}{2}\right)^2 - \frac{t^\mu}{4\Gamma(\mu+1)}\sinh(x) + \frac{t^{2\mu}}{8\Gamma(2\mu+1)}\cosh(x).$$

6.1.2. Variational iteration method. According to the formula (4.2), the iteration formula for equation (6.1) is given by

$$u_{k+1}(x,t) = u_k(x,t) - \int_0^t \left(\frac{\partial^\mu}{\partial\xi^\mu}u_k(x,\xi) - (u_k^2)_x + [u_k(u_k)_{xx}]_x\right)d\xi.$$

By the above variational iteration formula, begin with $u_0 = \sinh^2(x/2)$, we can obtain the following third-term approximate solution

$$\begin{aligned}u_{\mathrm{VIM}}(x,t) = \frac{1}{2}(\cosh(x)-1) - \frac{1}{2}\sinh(x)t \\ + \frac{1}{4}\sinh(x)\frac{t^{2-\mu}}{\Gamma(3-\mu)} + \frac{1}{8}\cosh(x)\frac{t^{\mu+1}}{\Gamma(\mu+2)}.\end{aligned}$$

6.1.3. Homotopy perturbation method. In view of equation (5.7), the homotopy for equation (6.1) can be constructed as

$$\frac{\partial u}{\partial t} = p\left[\frac{\partial u}{\partial t} + (u^2)_x - [u(u)_{xx}]_x - D^\mu_{*t}u\right], \tag{6.4}$$

Substituting (5.9) and the initial condition (6.2) into (6.4) and equating the terms with identical powers of p, we obtain the following set of linear partial differential equations

$$\begin{aligned}
\frac{\partial u_0}{\partial t} &= 0, & u_0(x,0) &= \sinh^2\left(\frac{x}{2}\right),\\
\frac{\partial u_1}{\partial t} &= \frac{\partial u_0}{\partial t} + (u_0^2)_x - [u_0(u_0)_{xx}]_x - D^\mu_{*t}u_0, & u_1(x,0) &= 0,\\
\frac{\partial u_2}{\partial t} &= \frac{\partial u_1}{\partial t} + (2u_0u_1)_x & &\\
&\quad - [u_0(u_1)_{xx} + u_1(u_0)_{xx}]_x - D^\mu_{*t}u_1, & u_2(x,0) &= 0.\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & &
\end{aligned}$$

Consequently, the third-term of the homotopy perturbation solution for equation (6.1) is given by

$$u_{\mathrm{HPM}}(x,t) = \sinh\left(\frac{x}{2}\right)^2 - \frac{t^\mu}{4\Gamma(\mu+1)}\sinh(x) + \frac{t^{2\mu}}{8\Gamma(2\mu+1)}\cosh(x).$$

So, the homotopy solution for equation (6.1) is the same solution obtained using the decomposition method.

6.1.4. Generalized two-dimensional differential transform method. Suppose that the solution $u(x,t)$ of equation (6.1) can be represented as a product of two single-variable functions. Applying the generalized two-dimensional differential to both sides of (6.1), we get the following recurrence relation

$$\tag{6.5} \begin{aligned} \frac{\Gamma(\mu(h+1)+1)}{\Gamma(\mu h+1)}U(k,h+1) &- (k+1)\sum_{r=0}^{k+1}\sum_{s=0}^{h}U(r,h-s)U(k-r+1,s) \\ &+ (k+1)\sum_{r=0}^{k+1}\sum_{s=0}^{h}U(r,h-s)(k-r+2)(k-r+3)U(k-r+3,s). \end{aligned}$$

The generalized two-dimensional differential transform of the initial condition (6.2) is given by

$$\tag{6.6} U(k,h) = \begin{cases} \dfrac{1}{(2k)!} - \delta(k) & \text{for } k \text{ even,} \\ 0 & \text{otherwise.} \end{cases}$$

Utilizing the recurrence relation (6.5) and the transformed initial condition (6.6), we can obtain the following approximation, for $k,h<4$,

$$\begin{aligned} u_{\mathrm{GDTM}}(x,t) = &-\frac{1}{2}+\frac{x^2}{4}-\frac{1}{4}\left(x+\frac{x^3}{3!}\right)\frac{t^{\mu}}{\Gamma(\mu+1)} \\ &+\frac{1}{8}\left(1+\frac{x^2}{2}\right)\frac{t^{2\mu}}{\Gamma(2\mu+1)}-\frac{1}{16}\left(x+\frac{x^3}{3!}\right)\frac{t^{3\mu}}{\Gamma(3\mu+1)}, \end{aligned}$$

which converges to the closed form solution

$$\tag{6.7} \begin{aligned} u_{\mathrm{GDTM}}(x,t) = \frac{1}{2}\Bigg[&\cosh(x)\sum_{n=0}^{\infty}\frac{(t^{\mu}/2)^{2n}}{\Gamma(2n\mu+1)} \\ &- \sinh(x)\sum_{n=0}^{\infty}\frac{(t^{\mu}/2)^{2n+1}}{\Gamma((2n+1)\mu+1)}-1\Bigg]. \end{aligned}$$

Table 1 shows the approximate solutions for equation (6.1), in case of $\mu = 0.5$ and $\mu = 1.0$, using the four methods. It is to be noted that only the first twenty terms of the series in (6.7) were used in evaluating the approximate solutions using the generalized differential transform method. The results obtained using the GDTM compare well with those obtained by the ADM, VIM, and HPM.

EXAMPLE 6.2. Consider the following nonlinear space-fractional hyperbolic equation

$$\tag{6.8} \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(u(x,t)\frac{\partial^{\beta}u}{\partial x^{\beta}}\right), \quad t>0,\ x>0,$$

where $0<\beta\le 1$, subject to the initial conditions

$$\tag{6.9} u(x,0)=x^{2\beta}, \quad u_t(x,0) = -2x^{2\beta}.$$

		$\mu = 0.5$				$\mu = 1.0$			
t	x	u_{GDTM}	u_{ADM}	u_{VIM}	u_{HPM}	u_{GDTM}	u_{ADM}	u_{VIM}	u_{HPM}
0.2	0.25	0.00919	0.00962	0.00336	0.00962	0.00563	0.00565	0.00565	0.00565
	0.50	0.02474	0.02626	0.02995	0.02626	0.04053	0.04057	0.04057	0.04057
	0.75	0.07326	0.07596	0.08983	0.07596	0.10939	0.10946	0.10946	0.10946
	1.00	0.15779	0.16185	0.18676	0.16185	0.21654	0.21663	0.21663	0.21663
0.4	0.25	0.02174	0.02221	0.00173	0.02221	0.00062	0.00075	0.00075	0.00075
	0.50	0.02368	0.02722	0.01121	0.02722	0.02266	0.022979	0.022979	0.022979
	0.75	0.05852	0.06536	0.05279	0.06336	0.07755	0.07805	0.07805	0.07805
	1.00	0.12846	0.13902	0.12912	0.13902	0.16871	0.16945	0.16945	0.16945
0.6	0.25	0.03810	0.03786	0.00707	0.03786	0.00062	0.00102	0.00102	0.00102
	0.50	0.02909	0.03452	0.00231	0.03452	0.01003	0.01102	0.01102	0.01102
	0.75	0.05332	0.06475	0.02910	0.06475	0.05148	0.05312	0.05312	0.05312
	1.00	0.11232	0.13047	0.08913	0.13047	0.12758	0.12997	0.12997	0.12997

TABLE 1. Numerical values when $\mu = 0.5$ and 1.0 for equation (6.1)

6.2.1. Adomian decomposition method. To solve the problem using the decomposition method, we simply substitute (6.8) and the initial conditions (6.9) into (3.5), to obtain the following recurrence relation

$$\begin{aligned} u_0(x,t) &= u(x,0) + tu_t(x,0) = x^{2\beta} - 2x^{2\beta}t, \\ u_{i+1}(x,t) &= J^2(A_j), \quad j \geq 0. \end{aligned} \tag{6.10}$$

where A_j are the Adomian polynomials for the nonlinear function $N = u(\partial^\beta u/\partial x^\beta)$.

In view of (6.10), the first three terms of the decomposition series (3.6) are given by

$$\begin{aligned} u_{\mathrm{ADM}}(x,t) &= (1-2t)x^{2\beta} + \frac{3\beta\Gamma(2\beta+1)}{\Gamma(\beta+1)}\left[\frac{t^2}{2} - \frac{2t^3}{3} + \frac{t^4}{3}\right]x^{3\beta-1} \\ &+ \frac{3\beta(4\beta-1)\Gamma(2\beta+1)}{\Gamma(\beta+1)}\left(\frac{\Gamma(3\beta)}{\Gamma(2\beta)} + \frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)}\right)\left[\frac{t^4}{24} - \frac{t^5}{12} + \frac{t^6}{18} - \frac{t^7}{63}\right]x^{4\beta-2}. \end{aligned}$$

6.2.2. Variational iteration method. According to the formula (4.3), the iteration formula for equation (6.8) is given by

$$u_{k+1}(x,t) = u_k(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_k}{\partial t^2} - \frac{\partial}{\partial x}\left(u_k(x,\xi)\frac{\partial^\beta u_k}{\partial x^\beta}\right)\right) d\xi.$$

By the above variational iteration formula, begin with $u_0 = x^{2\beta} - 2x^{2\beta}t$, we can obtain the following third-term approximate solution

$$u_{\mathrm{VIM}}(x,t) = u_{\mathrm{ADM}}(x,t).$$

6.2.3. Homotopy perturbation method. In view of equation (5.7), the homotopy for equation (6.8) can be constructed as

$$\frac{\partial^2 u}{\partial t^2} = p\left[\frac{\partial}{\partial x}\left(u(x,t)\frac{\partial^\beta u}{\partial x^\beta}\right)\right], \tag{6.11}$$

Substituting (5.9) and the initial conditions (6.9) into (6.11) and equating the terms with identical powers of p, we obtain the following set of linear partial differential equations

$$\begin{aligned}
\frac{\partial^2 u_0}{\partial t^2} &= 0, & u_0(x,0) &= x^{2\beta} & (u_0)_t(x,0) &= -2x^{2\beta},\\
\frac{\partial^2 u_1}{\partial t^2} &= \frac{\partial}{\partial x}\left(u_0\frac{\partial^\beta u_0}{\partial x^\beta}\right), & u_1(x,0) &= 0, & (u_1)_t(x,0) &= 0,\\
\frac{\partial^2 u_2}{\partial t^2} &= \frac{\partial}{\partial x}\left(u_0\frac{\partial^\beta u_1}{\partial x^\beta} + u_1\frac{\partial^\beta u_0}{\partial x^\beta}\right), & u_2(x,0) &= 0 & (u_2)_t(x,0) &= 0.\\
&\cdots\cdots\cdots\cdots
\end{aligned}$$

Consequently, the third-term of the homotopy perturbation solution for equation (6.8) is given by

$$u_{\mathrm{HPM}}(x,t) = u_{\mathrm{VIM}}(x,t).$$

6.2.4. Generalized two-dimensional differential transform method. Suppose that the solution $u(x,t)$ of equation (6.8) can be represented as a product of two single-variable functions. Applying the generalized two-dimensional differential to both sides of equation (6.8), we get the following recurrence relation

$$\begin{aligned}
&\frac{\Gamma(\alpha h+\gamma+1)}{\Gamma(\alpha h+1)}U_{1/2,\beta}(k,h+m_1)\\
&\quad= (k+1)\sum_{r=0}^{k+1}\sum_{s=0}^{h}\frac{\Gamma(\beta(k-r+2)+1)}{\Gamma(\beta(k-r+1)+1)}U_{1/2,\beta}(r,h-s)U_{1/2,\beta}(k-r+2,s).
\end{aligned} \tag{6.12}$$

The generalized two-dimensional differential transform of the initial conditions (6.9) are given by

$$U_{1/2,\beta}(k,0) = \delta(k-2), \qquad U_{1/2,\beta}(k,1) = -2\delta(k-2). \tag{6.13}$$

Utilizing the recurrence relation (6.12) and the transformed initial condition (6.13), we get

$$\begin{aligned}
u(x,t) = \Bigg[1 - 2t &+ \frac{3\Gamma(2\beta+1)}{2\Gamma(\beta+1)}t^2 - 2\frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)}t^3\\
&+ \frac{\Gamma(2\beta+1)}{4\Gamma(\beta+1)}\left(\frac{3\Gamma(2\beta+1)}{\Gamma(\beta+1)}+4\right)t^4 - \frac{3\Gamma(2\beta+1)^2}{2\Gamma(\beta+1)^2}t^5\Bigg]x^{2\beta}.
\end{aligned}$$

Table 2 shows the approximate solutions for equation (5.8) obtained for different values of β using the generalized differential transform method and the homotopy perturbation method. It is to be noted that the third-term of the homotopy perturbation solution is the same solution obtained using the decomposition method and the variational iteration method. From the numerical results in Table 2, we observe that the error decreases as β becomes closer to 1. Its evident that the efficiency of these approaches can be enhanced by computing further components of the solution $u(x,t)$ when HPM and GDTM are used.

		$\beta = 0.5$		$\beta = 0.75$		$\beta = 1.0$	
t	x	u_{GDTM}	u_{HPM}	u_{GDTM}	u_{HPM}	u_{GDTM}	u_{HPM}
0.2	0.25	0.163093	0.163011	0.083433	0.083841	0.043380	0.043330
	0.50	0.326185	0.318340	0.235984	0.233123	0.173520	0.173320
	0.75	0.489278	0.472428	0.433530	0.424526	0.390420	0.389971
	1.00	0.652371	0.625875	0.667463	0.649852	0.694080	0.693281
0.4	0.25	0.090039	0.090231	0.050882	0.052307	0.030660	0.030945
	0.50	0.180077	0.156309	0.143917	0.135310	0.122640	0.123779
	0.75	0.270116	0.218646	0.264393	0.236846	0.275940	0.278502
	1.00	0.360154	0.279047	0.407060	0.352949	0.490560	0.495114
0.6	0.25	0.010838	0.020666	0.012875	0.0229310	0.012340	0.020403
	0.50	0.021676	-0.001571	0.036415	0.042394	0.049360	0.081612
	0.75	0.032513	-0.030267	0.066898	0.057095	0.111060	0.183627
	1.00	0.043351	-0.062308	0.102996	0.067218	0.197440	0.326448

TABLE 2. Numerical values when $\beta = 0.5$, 0.75 and 1.0 for equation (6.8)

7. Concluding remarks

In this work, GDTM, HPM, ADM, and VIM have been successfully applied to nonlinear partial differential equations with space- and time-fractional derivatives. The main advantage of the four methods over mesh points methods is the fact that they do not require discretization of the variables, i.e. time and space, and thus they are not affected by computation round off errors and one is not faced with necessity of large computer memory and time.

There are four important points to make here. First, the four methods provide the solutions in terms of convergent series with easily computable components. Second, the HPM produces the same solution as the ADM and VIM with the proper choice of the constructing homotopy and this solution is valid not only for small parameters, but also for large parameters. Third, it seems that the approximate solutions in Examples 6.1 and 6.2 using the GDTM converges faster to the approximate solutions obtained when using the other three methods as the order of fractional derivative approaches one. So, the accuracy of the GDTM depends on the order of partial differential equations.

Finally, the recent appearance of fractional differential equations as models in some fields of applied mathematics makes it necessary to investigate methods of solution for such equations (analytical and numerical) and we hope that this work is a step in this direction.

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Manuscript received September 6, 2007

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