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ON DETERMINISTIC AND KOLMOGOROV EXTENSIONS FOR TOPOLOGICAL FLOWS

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Abstract. The concepts of deterministic and Kolmogorov extensions of topological flows are introduced. We show that the class of deterministic extensions contains distal extensions and moreover that for the deterministic extensions the relative topological entropy vanishes and hence they preserve the topological entropy. On the other hand we relate the Kolmogorov extensions to the asymptotic ones and we show that the class of these extensions contains uniquely ergodic u.p.e. extensions and also the class of flows admitting an invariant relative K-measure with full support.

The main tool used to get these results is the relative version of the Rokhlin–Sinai theorem concerning the existence of perfect measurable partitions.

1. Introduction

The investigation of extensions of topological flows is one of the most important purposes of topological dynamics. Classical results concerning the theory of extensions can be found in books of such authors like Auslander, Ellis, Furstenberg, Glasner, de Vries.

This paper is a continuation of our papers [15], [16]. We investigate in them two classes of topological flows, deterministic and Kolmogorov ones. The object

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of the present paper is to study relative versions of these classes of flows, i.e. deterministic and Kolmogorov extensions (K-extensions). Their basic properties are founded on a topological analogue of the relative version of the Rokhlin-Sinai theorem on the existence of perfect partitions (cf. [14]) from ergodic theory. This analogue is presented in our main result (Theorem 3.1). As a consequence, we obtain the existence of relatively extreme relations. We characterize deterministic extensions in terms of the algebra of continuous functions (Proposition 3.6). This result can be regarded as a topological analogue of the determinism considered in the theory of stationary processess. Applying the relative version of Ellis theorem on distal flows we show that distal extensions are deterministic. On the other hand, it appears (Proposition 3.9) that infinite extensions with asymptotic pairs (for example infinite expansive flows) which are identified by the homomorphism defining this extension are not deterministic. Using the above property and the result of Huang, Ye, Zhang on relative topological entropy pairs, one can deduce that the relative topological entropy of deterministic extensions equals 0 and thus deterministic extensions preserve entropy. The K-extensions are closely related to asymptotic extensions. Namely, we show that for any K-extension the set of asymptotic pairs is dense in the appropriate relation. We are unable to decide whether minimal K-extensions are relatively weakly mixing. Some related results are contained in the paper [10] of Glasner. Applying Theorem 3.1 we show that uniquely ergodic extensions having relative uniformly positive entropy and also flows admitting an invariant relative K-measure (i.e. a measure of completely positive relative entropy) with the relative full support are K-extensions. We also get the description of the relative topological Pinsker factor in terms of relative extreme relations.

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2. Preliminaries

Let (X,d) be a compact metric space and let $T\colon X \to X$ be a continuous surjection. The pair (X,T) is said to be a *topological flow*.

For a given $x \in X$, the set $O_T^+(x) = \{T^n x,\ n \geq 0\}$ is called the *positive semiorbit* of x. In the case when T is invertible, the sets $O_T^-(x) = \{T^n x,\ n \leq 0\}$ and $O_T(x) = \{T^n x,\ n \in \mathbb{Z}\}$ are called the *negative semiorbit* and *the orbit* of x, respectively.

A flow (Y,S) is called a *factor flow* of (X,T) if there exists a continuous surjection $\pi\colon X \to Y$ with $\pi \circ T = S \circ \pi$.

We assume in the sequel that T is invertible; of course factors of T need not to be invertible.

By CER(X), we denote the set of all closed equivalence relations in $X \times X$ and by Δ the diagonal relation. A relation $R \in$ CER(X) is said to be *positively invariant* (resp. *invariant*) with respect to T if $(T \times T)(R) \subset R$ (resp. $(T \times T)(R) = R$). The symbol ICER$^+(X)$ (ICER(X)) stands for the set of all positively invariant (invariant) relations in $X \times X$. For a given relation $R \in$ CER(X) invariant under $T \times T$, the factor flow defined by R is denoted by $(X/R, T/R)$. For a subset $F \subset X \times X$, the smallest invariant relation $R \in$ CER(X) containing F is denoted by $\langle F \rangle$. For a family $\{R_i\} \subset$ CER(X), the symbol $\bigvee_i R_i$ means the smallest closed invariant equivalence relation containing all R_i's.

Let $\mathbf{A}(T)$ be the set of all asymptotic pairs for T. Recall that $(x, x') \in \mathbf{A}(T)$ if $\lim_{n \to +\infty} d(T^n x, T^n x') = 0$. Obviously $T\mathbf{A}(T) = \mathbf{A}(T)$.

Let $\mathcal{B}$ be the σ-algebra of Borel sets of X and let $M(X, T)$ be the space of all probability measures on $\mathcal{B}$ invariant under T. In the sequel, we shall consider measurable partitions of the Lebesgue space $(X, \mathcal{B}, \mu)$, $\mu \in M(X, T)$. For the definitions and basic properties of measurable partitions we refer the reader to [21]. We denote by ε the measurable partition of X on single points. For a given measurable partition ξ, the σ-algebra generated by ξ is denoted by $\sigma(\xi)$. On the other hand, for any σ-algebra $\mathcal{A} \subset \mathcal{B}$ the symbol $\xi(\mathcal{A})$ stands for the measurable partition generated by $\mathcal{A}$. It is clear that for any relation $R \in$ CER(X), the partition ξ_R on equivalence classes of R is measurable with respect to any invariant measure. For a given measurable partition ξ of X we put $\Delta_\xi = \{(x, y) \in X \times X,\ y \in \xi(x)\}$, where the symbol $\xi(x)$ means the element of ξ which contains $x \in X$. Let μ be a probability measure and $\mathcal{A} \subset \mathcal{B}$ be a σ-algebra. Define the probability measure $\mu \times_{\mathcal{A}} \mu$ on $(X \times X, \mathcal{B} \otimes \mathcal{B})$ as follows

$$(\mu \times_{\mathcal{A}} \mu)(A \times B) = \int_X \mu(A|\mathcal{A})(x)\mu(B|\mathcal{A})(x)\mu(dx),$$

$A, B \in \mathcal{B}$. If ξ is a measurable partition of X, then $\mu \times_\xi \mu$ stands for $\mu \times_{\sigma(\xi)} \mu$.

Let $h_\mu(T)$ and $\pi_\mu(T)$ be the entropy and the Pinsker partition of T, respectively. If σ is an invariant measurable partition, i.e. $T\sigma = \sigma$, by $h_\mu(T|\sigma)$ and $\pi_\mu(T|\sigma)$ we denote the σ-relative entropy of T and the σ-relative Pinsker partition of T, respectively. For the definition and properties of entropy (resp. relative entropy) we refer the reader to [25] (resp. [13]).

Let us recall that a measurable partition ξ is called *σ-relatively extreme* ([13]) if it satisfies the following conditions

(a) $\sigma \preccurlyeq T^{-1}\xi \preccurlyeq \xi$,
(b) $\bigvee_{n=0}^{\infty} T^n \xi = \varepsilon$,
(c) $\bigwedge_{n=0}^{\infty} T^{-n} \xi = \pi_\mu(T|\sigma)$.

Here $\xi \preccurlyeq \eta$ means that η is finer than ξ. It is known ([14]) that for any T there exists a σ-relative extreme partition.

Let $h(T)$ and $E(X,T)$ be the topological entropy and the set of topological entropy pairs of T, respectively ([2]). The relation $\Pi(T) = \langle E(X,T)\rangle$ is called the *Pinsker relation* of T (cf. [5]). For a given $\mu \in M(X,T)$ and T-invariant measurable partition σ, $E_\mu(X,T)$ (resp. $E_\mu(X,T|\sigma)$) denotes the set of entropy pairs (resp. σ-relative entropy pairs) for T with respect to μ (cf. [3], [19]). The relation $\Pi_\mu(T|\sigma) = \langle E_\mu(X,T|\sigma)\rangle$ is called the *σ-relative Pinsker relation* of T with respect to μ.

Let $\Sigma \in \mathrm{CER}(X)$ be an invariant relation.

We denote by $h(T|\Sigma)$, $E(X,T|\Sigma)$ and $\Pi(T|\Sigma)$ the relative topological entropy, the set of relative topological entropy pairs and the relative Pinsker relation, respectively (cf. [8], [12]). It is clear that if $\Sigma = X\times X$ then $h(T|\Sigma) = h(T)$, $E(X,T|\Sigma) = E(X,T)$ and $\Pi(T|\Sigma) = \Pi(T)$.

By analogy with the concept of a σ-relative extreme measurable partition, we define a Σ-relative extreme relation as follows.

Definition 2.1. We say that a relation $R \in \mathrm{CER}(X)$ is *Σ-relatively extreme with respect to μ* if

(a) $(T\times T)(R) \subset R \subset \Sigma$,
(b) $\bigcap_{n=0}^\infty (T\times T)^n(R) = \Delta$,
(c) $\bigvee_{n=0}^\infty (T\times T)^{-n}(R) = \Pi_\mu(T|\sigma)$,

where σ is the invariant measurable partition associated with Σ.

Proposition 2.2. *For any measure $\mu \in M(X,T)$ and any invariant measurable partition σ, there exists a σ-relative extreme partition ζ with $\Delta_\zeta \subset \mathbf{A}(T)$.*

Proof. Let (ξ_n) be a sequence of finite measurable partitions of X such that $\xi_1 \preccurlyeq \xi_2 \preccurlyeq \ldots$ and $\operatorname{diam}\xi_n \le 1/n$ for $n \ge 1$. Hence, $\xi_n \nearrow \varepsilon$.

Now one constructs a σ-relative extreme partition ζ in the same way as the extreme partition in [23] (see also [22], [20]). Namely, let (n_k) be an arbitrary sequence of natural numbers and

$$\eta_p = \bigvee_{k=1}^{p} T^{-n_k}\xi_k, \quad \eta = \bigvee_{p=1}^{\infty} \eta_p.$$

Now put $\zeta = \eta \vee \eta^- \vee \sigma$. Clearly $\zeta \succeq \sigma$, $T^{-1}\zeta \preceq \zeta$ and $\bigvee_{n=0}^\infty T^n\zeta = \varepsilon$.

We show that $\Delta_\zeta \subset \mathbf{A}(T)$. Let $(x,y) \in \Delta_\zeta$, i.e. x and y belong to the same atom of ζ, therefore to the same atom of η^-. Thus T^ix, T^iy, $i \ge 1$ belong to the same atom of η_p, $p \ge 1$ and so $T^{i+n_p}x$, $T^{i+n_p}y$ belong to the same atom of ξ_p, i.e. T^nx, T^ny belong to the same atom of ξ_p for $n > n_p$. This means that $d(T^nx, T^ny) < 1/p$, $n > n_p$, $p \ge 1$, i.e. $(x,y) \in \mathbf{A}(T)$.

Similarly as in the proof of Theorem 6.11 in [20] one can show that the sequence (n_k) can be especially chosen so that $H_\mu(\eta_p|\eta_p^- \vee \sigma) - H_\mu(\eta_p|\zeta^- \vee \sigma) < 1/p$ for any $p \geq 1$. This forces the equality $\bigwedge_{n=0}^{\infty} T^{-n}\zeta = \pi_\mu(T|\sigma)$. □

Let $\lambda_\mu^\sigma = \mu \times_{\pi_\mu(T|\sigma)} \mu$ and let Λ_μ^σ be the topological support of λ_μ^σ. One could prove the following proposition applying similar methods as those used in [4] (Lemma 7). We give here the proof based on relative versions of some classical results from ergodic theory.

Proposition 2.3. *If $\mu \in M(X,T)$ is ergodic then the measure λ_μ^σ is ergodic.*

Proof. Let $\mathcal{F}$ be the σ-algebra generated by $\pi_\mu(T|\sigma)$. It is well known (cf. [13]) that T is a relative K-automorphism w.r. to $\mathcal{F}$. Hence using classical methods (see the proof of Theorem 1 of [7, p. 283]) one shows that T is relatively K-mixing w.r. to $\mathcal{F}$, i.e.

$$\lim_{n\to\infty} \sup_{A \in T^{-n}\mathcal{A}^- \vee \mathcal{F}} ||\mu(A \cap B|\mathcal{F}) - \mu(A|\mathcal{F})\mu(B|\mathcal{F})||_1 = 0$$

for any finite algebra $\mathcal{A}$ and $B \in \mathcal{B}$, and hence it is relatively weakly mixing w.r. to $\mathcal{F}$. Since μ is ergodic a simple direct reasoning shows that $\mu \times_{\mathcal{F}} \mu$ is ergodic.□

3. Results

For any invariant relation $\Sigma \in \mathrm{CER}(X)$ let σ denotes the invariant measurable partition associated with Σ.

Theorem 3.1. *For any ergodic measure $\mu \in M(X,T)$ and invariant relation $\Sigma \in \mathrm{CER}(X)$, there exists a relation $R = R_\mu \in \mathrm{CER}(X)$ with*

(a) $(T \times T)(R) \subset R \subset \Sigma$,
(b) $\bigcap_{n=0}^{\infty} (T \times T)^n(R) = \Delta$,
(c) $E_\mu(X,T|\sigma) \cup S(\mu) \subset \overline{\bigcup_{n=0}^{\infty} (T \times T)^{-n}(R)} \subset \Pi_\mu(T|\sigma)$,

where $S(\mu) = \{(x,x) \in X \times X;\ x \in \operatorname{Supp}\mu\}$.

Proof. If $h_\mu(T|\sigma) = 0$ we put $R = \Delta$. Let now $\mu \in M(X,T)$ be ergodic with $h_\mu(T|\sigma) > 0$ and let ζ be a σ-relative extreme partition of $(X, \mathcal{B}, \mu, T)$ such that $\Delta_\zeta \subset \mathbf{A}(T)$. First, observe that the relative version of Proposition 4 in [3] holds, i.e. $h_{\mu\circ\varphi^{-1}}(T/\Pi_\mu(T|\sigma)|\sigma) = 0$, where $\varphi\colon X \to X/\Pi_\mu(T|\sigma)$ denotes the quotient map. Therefore $\xi_{\Pi_\mu(T|\sigma)} \preccurlyeq \pi_\mu(T|\sigma)$. Now, since $\zeta \succcurlyeq \pi_\mu(T|\sigma)$ and $\zeta \succcurlyeq \sigma$, there exists a subset $X_0 \subset X$ such that $\mu(X_0) = 1$ and the following inclusions hold

$$\begin{aligned} (3.1)\qquad \Delta_\zeta \cap (X_0 \times X_0) &\subset \Delta_{\pi_\mu(T|\sigma)} \cap (X_0 \times X_0) \\ &\subset \Delta_{\xi_{\Pi_\mu(T|\sigma)}} \cap (X_0 \times X_0) = \Pi_\mu(T|\sigma) \cap (X_0 \times X_0), \end{aligned}$$

$$(3.2)\qquad \Delta_\zeta \cap (X_0 \times X_0) \subset \Delta_\sigma \cap (X_0 \times X_0) = \Sigma \cap (X_0 \times X_0).$$

It follows from Lemma 6 of [4] that $(\mu \times_\zeta \mu)(\Delta_\zeta) = 1$. Since the equality $\mu(X_0) = 1$ implies $(\mu \times_\zeta \mu)(X_0 \times X_0) = 1$, we have

$$(\mu \times_\zeta \mu)(\Delta_\zeta \cap (X_0 \times X_0)) = 1.$$

Similarly as in the proof of Proposition 5 ([4]) one constructs a set $G \subset X \times X$ in the following manner. First, one shows that there exists a sequence (U_k) of open sets such that $\lambda_\mu^\sigma(U_k) > 0$, $k \geq 1$, every point of $X \times X$ belongs to infinitely many U_k, $k \geq 1$ and the diameters of U_k tend to 0 as $k \to \infty$. Next one takes the set V_k of all $x \in X$ belonging to infinitely many sets $(T \times T)^n U_k$, $n \geq 1$, and finally defines

$$G = \bigcap_{k=1}^{\infty} V_k.$$

The set G has the following two properties. For any point $(x, x') \in G$, the semiorbit $O^-_{T\times T}(x, x')$, and so the orbit $O_{T\times T}(x, x')$, is dense in Λ_μ^σ and $(\mu \times_\zeta \mu)(G) = 1$. Thus, we have $(\mu \times_\zeta \mu)(\Delta_\zeta \cap (X_0 \times X_0) \cap G) = 1$. Put $\Delta_\zeta^0 = \Delta_\zeta \setminus \Delta$. Since $h_\mu(T|\sigma) > 0$ we have $\zeta \neq \varepsilon$ and so the measure $\mu \times_\zeta \mu$ is not concentrated on Δ (cf. [4], Lemma 5 (i)). Thus we have $(\mu \times_\zeta \mu)(\Delta_\zeta^0) > 0$ and in consequence $(\mu \times_\zeta \mu)(\Delta_\zeta^0 \cap (X_0 \times X_0) \cap G) > 0$.

Now choose an arbitrary pair $(x, x') \in \Delta_\zeta^0 \cap (X_0 \times X_0) \cap G$ and define a relation R as follows

$$R = O^+_{T\times T}(x, x') \cup O^+_{T\times T}(x', x) \cup \Delta.$$

This relation is, of course, reflexive, symmetric and positively invariant. Since $(x, x') \in \mathbf{A}(T)$, it is closed and the equality (b) is satisfied.

Applying similar methods as in the proof of Theorem 1 ([9]), one can show the following relative version of this theorem:

$$\Lambda_\mu^\sigma = E_\mu(X, T|\sigma) \cup S(\mu).$$

The density of $O_{T\times T}(x, x')$ in Λ_μ^σ implies

$$E_\mu(X, T|\sigma) \cup S(\mu) \subset \overline{\bigcup_{n=0}^{\infty} (T \times T)^{-n}(R)}.$$

The fact that $x \neq x'$ and the assumption $h_\mu(T|\sigma) > 0$ imply that the orbits $O_T(x)$ and $O_T(x')$ are infinite and disjoint. Therefore, R is transitive.

From (3.1), we get $(x, x') \in \Pi_\mu(T|\sigma)$, hence, $R \subset \Pi_\mu(T|\sigma)$. Since $\Pi_\mu(T|\sigma)$ is closed and invariant, we immediately get

$$\overline{\bigcup_{n=0}^{\infty} (T \times T)^{-n}(R)} \subset \Pi_\mu(T|\sigma).$$

Since Σ is closed and invariant, then applying (3.2) we get $R \subset \Sigma$, i.e. R satisfies all desired properties. □

From Theorem 3.1 it follows at once

Corollary 3.2. *For any ergodic measure $\mu \in M(X,T)$, there exists a σ-relative extreme relation with respect to μ.*

Let (X,T) be an extension of the flow (Y,S) via the factor map $\pi\colon X \to Y$. By $\Sigma_\pi = \Sigma$ denote the invariant closed equivalence relation given by π (i.e. $\Sigma_\pi = \{(x,x') \in X \times X :\ \pi(x) = \pi(x')\}$).

Definition 3.3. The homomorphism π is called *deterministic* (or (X,T) is called *a deterministic extension* of (Y,S)) if for every relation $R \in \mathrm{CER}(X)$ such that $(T \times T)(R) \subset R \subset \Sigma_\pi$ we have $(T \times T)(R) = R$.

In the case $\Sigma_\pi = X \times X$, (X,T) is said to be deterministic (cf. [15]). It is easy to show the following.

Proposition 3.4. *A homomorphism $\pi\colon (X,T) \to (Y,S)$ is deterministic if and only if for every factor (Z,U) with $(X,T) \to (Z,U) \to (Y,S)$ the map U is invertible.*

Corollary 3.5. *If $\pi\colon (X,T) \to (Y,S)$ is deterministic and $(X,T) \xrightarrow{\rho} (Z,U) \xrightarrow{\eta} (Y,S)$, where $\pi = \eta \circ \rho$, then so is η.*

In the following proposition we characterize the deterministic extensions by means of subalgebras of the algebra $\mathrm{C}(X)$ of continuous functions on X. The map

$$\mathrm{C}(X/\Sigma) \ni \varphi \mapsto \varphi \circ \pi \in \mathrm{C}(X)$$

embeds the algebra $\mathrm{C}(X/\Sigma)$ in $\mathrm{C}(X)$ as a subalgebra $\mathrm{C}(X,\Sigma)$ of all continuous functions on X constant on equivalence classes of Σ. Let $\mathcal{A}_\Sigma^+(f)$ denote the smallest closed linear algebra containing functions $U_T^m f$ for $m \geq 1$ together with $\mathrm{C}(X,\Sigma)$.

Proposition 3.6. *A homomorphism $\pi\colon (X,T) \to (Y,S)$ is deterministic if and only if $f \in \mathcal{A}_\Sigma^+(f)$ for every function $f \in \mathrm{C}(X)$.*

Proof. ($\Rightarrow$) Let $f \in \mathrm{C}(X)$. It is enough to show that $U_T \mathcal{A}_\Sigma^+(f) = \mathcal{A}_\Sigma^+(f)$. For this observe that if

$$R = \bigcap_{\varphi \in \mathcal{A}_\Sigma^+(f)} R_\varphi$$

then $R \subset \Sigma$ since $\Sigma = \bigcap_{\varphi \in \mathrm{C}(X,\Sigma)} R_\varphi$. Since obviously $(T \times T)R \subset R$, the determinism of π forces $(T \times T)R = R$, hence $U_T \mathrm{C}(X,R) = \mathrm{C}(X,R)$.

From the definition of R it follows that $\mathcal{A}_\Sigma^+(f) \subset \mathrm{C}(X,R)$. By the definition $\mathcal{A}_\Sigma^+(f)$ contains all constant functions. Take $[x_1]_R, [x_2]_R \in X/R$ with

$(x_1, x_2) \notin R$. There exists $\varphi \in \mathcal{A}^+_\Sigma(f)$ such that $\varphi(x_1) \neq \varphi(x_2)$. Treating φ as an element of the space $\mathrm{C}(X/R)$ it means that φ distinguishes points $[x_1]_R$ and $[x_2]_R$. Now applying the Stone–Weierstrass theorem we obtain $\mathcal{A}^+_\Sigma(f) = \mathrm{C}(X,R)$.

($\Leftarrow$) Let $R \in \mathrm{CER}(X)$ be such that $(T\times T)(R) \subset R \subset \Sigma$. It follows that

$$U_T\mathrm{C}(X,R) \subset U_T\mathrm{C}(X,(T\times T)R) = \mathrm{C}(X,R). \tag{3.3}$$

We will show that $U_T^{-1}\mathrm{C}(X,R) \subset \mathrm{C}(X,R)$. Let $f \in \mathrm{C}(X,R)$. By (3.3), $f\circ T^m \in \mathrm{C}(X,R)$ for $m \geq 1$, thus $\mathcal{A}^+_\Sigma(f) \subset \mathrm{C}(X,R)$. Since also $\mathrm{C}(X,\Sigma) \subset \mathrm{C}(X,R)$, we get

$$U_T^{-1}f \in U_T^{-1}\mathcal{A}^+_\Sigma(f) \subset \mathcal{A}^+_\Sigma(f) \subset \mathrm{C}(X,R).$$

We have obtained $U_T\mathrm{C}(X,R) = \mathrm{C}(X,R)$ which means that $(T\times T)(R) = R$. □

Proposition 3.7. *Every distal extension of a minimal flow is deterministic.*

Proof. Let $\pi\colon (X,T) \to (Y,S)$ be the distal extension where (Y,S) is minimal. Using Theorem 6 ([1, p. 141]) we have that $(\Sigma_\pi, T\times T)$ is a disjoint union of minimal sets. Let $R \in \mathrm{ICER}^+(X)$, $R \subset \Sigma_\pi$. We show that $R \in \mathrm{ICER}(X)$. Let $(x,y) \in \Sigma_\pi$ be such that $(Tx,Ty) \in R$. Since (x,y) belongs to some minimal subset of Σ_π, it is positively recurrent, i.e. there exists an increasing sequence (n_k) of positive integers with $(T\times T)^{n_k}(x,y) \to (x,y)$. By the assumption on R we obtain $(x,y) \in R$, i.e. $R \in \mathrm{ICER}(X)$. This means that π is deterministic. □

Remark 3.8. In Glasner–Maon, [11], there is a notion of weak rigidity. A flow (X,T) is *weakly rigid* if the identity homeomorphism $I\colon X \to X$ is a limit point of the collection $\{T^n;\ n \in \mathbb{Z}\}$ in the topology of pointwise convergence; in other words, if I is not an isolated point in the enveloping semigrup $E(X)$. Clearly every infinite minimal distal flow is weakly rigid, but the class of weakly rigid flows is much larger, e.g. it includes the class of rigid (hence also uniformly rigid) flows. In particular it contains some weakly mixing minimal flows.

Following the referee suggestion, one can define *positive weak rigidity* by the condition $I \in \overline{\{T^n; n \geq 1\}}$ and observe that in the absolute case the proof of Proposition 3.7 shows that every positively weakly rigid flow is deterministic. In fact the proof shows that every positively doubly recurrent flow (cf. [26]), i.e. a flow such that every point in the product $X\times X$ is positively recurrent under $T\times T$, is deterministic. In particular it holds for positively weakly rigid flows. Since for a weakly rigid flow either (X,T) or (X,T^{-1}) are positively weakly rigid it follows that for a weakly rigid flow at least one of (X,T) or (X,T^{-1}) is deterministic.

Proposition 3.9. *If $\pi\colon (X,T) \to (Y,S)$ is deterministic, X is infinite and (Y,S) is minimal then $\mathbf{A}(T) \cap \Sigma_\pi = \emptyset$.*

In order to prove the above proposition we will need the following result due to M. Kuczma ([17]). We include its proof to make our exposition more self-contained, as well as, to provide some notations used in later considerations.

Lemma 3.10 ([17], [24, p. 121]). *Let X be a locally compact space and $T\colon X \to X$ be a continuous transformation. Then every finite ω-limit set of (X,T) consists of a single periodic orbit.*

Proof. Assume that a set $\omega_T(x)$ of all limit points of $O_T^+(x)$ is finite. Take $\varepsilon > 0$ such that $\{\overline{B}(y,\varepsilon) : y \in \omega_T(x)\}$ is a finite collection of compact pairwise disjoint neighborhoods. Let $x' \in \omega_T(x)$ and $O_T(x') = \{x', Tx', \dots, T^{l-1}x'\}$. There exist $\varepsilon_l < \varepsilon_1 < \dots < \varepsilon_{l-1} < \varepsilon_0 < \varepsilon$ such that

$$TB(T^j x', \varepsilon_{j+1}) \subset B(T^{j+1}x', \varepsilon_{j+2 (\mathrm{mod}\ l)})$$

for all $j \in \{0,\dots,l-1\}$. Let $\mathbb{E} = \{n \in \mathbb{N} : T^n x \in \bigcup_{j=0}^{l-1} B(T^j x', \varepsilon_{j+1})\}$ and $\mathbb{F} = \{n \in \mathbb{E} : n+1 \notin \mathbb{E}\}$. Observe that if $n \in \mathbb{F}$ then $T^n x \in B(T^{l-2}x', \varepsilon_{l-1})$, hence $T^{n+1}x \in \overline{B}(T^{l-1}x', \varepsilon_0) \setminus B(T^{l-1}x', \varepsilon_l)$. If $\mathbb{F}$ were infinite then the set $\overline{B}(T^{l-1}x', \varepsilon_0) \setminus B(T^{l-1}x', \varepsilon_l)$ would contain a point from $\omega_T(x)$ distinct from $T^{l-1}x'$ that is a contradiction. Therefore $\mathbb{F}$ is finite and thus every n large enough belongs to $\mathbb{E}$ since the latter set is infinite. It follows that $\omega_T(x) = O_T(x')$. $\square$

Remark 3.11. Let us notice that using the above consideration we are able to show a little more. Namely:
If $O_T(x)$ is infinite and $\omega_T(x)$ has l elements then y can be chosen from $\omega_T(x)$ in such a way that $\lim_{n\to\infty} T^{nl+j}x = T^j y$ for $j \in \{0,\dots,l-1\}$, i.e. the flow $(O_T(x)\cup\omega_T(x), T)$ is the l-point compactification of $\mathbb{Z}$, where $\mathbb{N}$ spirals down to $\omega_T(x)$. Obviously $(x,y) \in \mathbf{A}(T)$.

Proof. Let $\omega(x) = \{x', Tx', \dots, T^{l-1}x'\}$. Given $m \in \mathbb{N}\setminus\{0\}$ take $U_j^m = B(T^j x', \varepsilon_{j+1})$ where ε_{j+1}'s are like in the proof of the above lemma with $\varepsilon = 1/m$. Let k_m be the first time the forward orbit of x enters $\bigcup_{j=0}^{l-1} U_j^m$ and stays in this set forever. Passing to a subsequence if necessary we may assume that $T^{k_m}x \in U_{j_0}^m$ for some $j_0 \in \{0,\dots,l-1\}$. Passing again to a subsequence find $i \in \{0,\dots,l-1\}$ with $k_m \equiv i (\mathrm{mod}\ l)$. Put $y = T^{j_0+l-i}x'$. $\square$

Proof of Proposition 3.9. Suppose $\mathbf{A}(T) \cap \Sigma_\pi \neq \emptyset$. We shall show that π is not deterministic. Let $(x,\overline{x}) \in \mathbf{A}(T) \cap \Sigma_\pi$. Hence, at least one of the orbits $O_T(x)$ and $O_T(\overline{x})$ must be infinite. We shall consider the following three cases.

(A) The orbits $O_T(x)$ and $O_T(\overline{x})$ are infinite and disjoint.

In this case, we put like in [15] (or in the proof of Theorem 3.1)

$$R = O^+_{T\times T}(x,\overline{x}) \cup O^+_{T\times T}(\overline{x},x) \cup \Delta$$

and we see that $R \in \mathrm{ICER}^+(X) \setminus \mathrm{ICER}(X)$.

(B) Precisely one of the orbits $O_T(x)$ and $O_T(\overline{x})$ is infinite.

Assume that $O_T(\overline{x}) = \{\overline{x}, T\overline{x}, \ldots, T^{k-1}\overline{x}\}$. We put $O_i = \{T^i\overline{x}, T^{nk+i}x;\ n \geq 0\}$ for $0 \leq i \leq k-1$ and define

$$R = \bigcup_{i=0}^{k-1} (O_i \times O_i) \cup \Delta.$$

Now we check again that $R \in \mathrm{ICER}^+(X) \setminus \mathrm{ICER}(X)$.

Note that we have not used minimality of (Y, S) so far.

(C) The orbits $O_T(x)$ and $O_T(\overline{x})$ are infinite and $O_T(x) \cap O_T(\overline{x}) \neq \emptyset$, hence $O_T(x) = O_T(\overline{x})$.

Assume first that $\omega_T(x)$ is of finite cardinality l. It can be shown that $l = \min\{k > 0 : (x, T^k x) \in \mathbf{A}(T)\}$ but this is not relevant for the proof. We know from Remark 3.11 that $\lim_{n\to\infty} T^{nl}x = y$ for some $y \in \omega_T(x)$, so $(x, y) \in \mathbf{A}(T)$ by Lemma 3.10 (y is l-periodic). Since (X, T) has periodic orbits, by minimality of (Y, S), Y is finite (this is the only place we use minimality of (Y, S)). Therefore there is n_0 such that $(T^{n_0 l}x, y) \in \mathbf{A}(T) \cap \Sigma_\pi$. We have come to the case (B).

If $\omega_T(x)$ is infinite consider a factor $(X, T) \xrightarrow{\rho} (Z, U) \xrightarrow{\eta} (Y, S)$, where

$$\Sigma_\rho = ((\omega_T(x) \times \omega_T(x)) \cup \Delta) \cap \Sigma_\pi.$$

Observe that $\rho|_{X \setminus \omega_T(x)}$ is one-to-one and $O_T^+(x) \cap \omega_T(x) = \emptyset$ since every point of $\omega_T(x)$ is periodic while $O_T(x)$, hence also $O_T^+(x)$, is infinite. It follows that $(\rho(x), \rho(\overline{x})) \in \mathbf{A}(U) \cap \Sigma_\eta$. Therefore η is not deterministic since $\omega_U(\rho(x)) = \rho(\omega_T(x))$ is finite. By Corollary 3.5, π is not deterministic. □

Let us consider the absolute case, i.e. Y is a one point space.

Corollary 3.12. *If (X, T) is deterministic then $\mathbf{A}(T) = \emptyset$.*

In [6] Bryant and Walters have shown that if (X, T) is expansive then $\mathbf{A}(T) \neq \emptyset$. Therefore applying Proposition 3.9 we get at once

Corollary 3.13. *If (X, T) is infinite expansive then (X, T) is not deterministic.*

From the above corollary, we get the proof of Corollary 3 in [15] whose proof has a gap.

It is shown in [12] (Theorem 4.6), that the set $\mathbf{A}(T) \cap \Sigma_\pi$ is dense in the set of the relative topological entropy pairs. Using this result and Proposition 3.9, we get the following

Corollary 3.14. *Every deterministic homomorphism onto a minimal flow has zero relative topological entropy.*

The minimality assumption in the above corollary can be omitted (cf. Remark 3.24).

From this, it follows that deterministic extensions preserve topological entropy (cf. [18]).

DEFINITION 3.15. The homomorphism $\pi\colon (X,T) \to (Y,S)$ is called *Kolmogorov* (or (X,T) is called *a Kolmogorov extension* of (Y,S)) if there exists a relation $R \in \mathrm{CER}(X)$ such that

(a) $(T\times T)(R) \subset R \subset \Sigma_\pi$,
(b) $\bigcap_{n=0}^{\infty}(T\times T)^n(R) = \Delta$,
(c) $\overline{\bigcup_{n=0}^{\infty}(T\times T)^{-n}(R)} = \Sigma_\pi$.

The relation R is called *the relative K-relation of* (X,T). In the case $\Sigma_\pi = X\times X$, (X,T) is said to be a K-flow (cf. [15]).

EXAMPLE 3.16. One easily checks that if (X,T) is a topological K-flow, (Y,S) is an arbitrary flow and the flow (Z,U) is the product flow of (X,T) and (Y,S) then the projection of Z onto Y is a Kolmogorov homomorphism.

The following proposition relates K-extensions to asymptotic extensions, i.e. extensions $\pi\colon (X,T)\to(Y,S)$ such that $\Sigma_\pi \subset \mathbf{A}(T)$.

PROPOSITION 3.17. *If π is a K-extension then $\mathbf{A}(T)\cap\Sigma_\pi$ is dense in Σ_π.*

PROOF. Let R be a relative K-relation of (X,T). First observe that $R\subset \mathbf{A}(T)$. Indeed, if it is not the case, it would exist a pair $(x,y)\in R$ and a sequence (n_i) of positive integers such that $d(T^{n_i}x, T^{n_i}y) \geq \varepsilon$ for some $\varepsilon > 0$. By the compactness of X we see that there exists $(u,v)\in X\times X$ and a subsequence (n_{i_j}) of (n_i) such that $\lim_{j\to\infty} T^{n_{i_j}}x = u$ and $\lim_{j\to\infty} T^{n_{i_j}}y = v$. Applying (a) and (b) of Definition 3.15 it is easy to see that $(u,v)\in\Delta$ and so $\lim_{j\to\infty} d(T^{n_{i_j}}x, T^{n_{i_j}}y) = 0$ what is impossible.

The inclusion $R\subset\mathbf{A}(T)$ and (c) of Definition 3.15 imply

$$\Sigma_\pi = \overline{\bigcup_{n=0}^{\infty}(T\times T)^{-n}(R)} \subset \overline{\mathbf{A}(T)\cap\Sigma_\pi}$$

which gives the desired result. □

It would be interesting to know whether the relative analogue of Theorem 6 of [15] is true, i.e. whether the minimality of (X,T) implies that any K-extension π is relatively weakly mixing, i.e. the dynamical system $(\Sigma_\pi, T\times T)$ is transitive. Proposition 3.17 allows to reduce the above question to the following. Let (X,T) be minimal and let $\mathbf{A}(T)\cap\Sigma_\pi$ be dense in Σ_π. Is it true that π is relatively weakly mixing?

Let us note the following result of E. Glasner ([10]) close to our question: *If (X,T) is minimal, π is open and $Q_\pi^{(n)} = R_\pi^{(n)}$ for every $n\geq 2$ then π is a weakly mixing extension.*

Here

$$R_\pi^{(n)} = \{(x_1, \dots, x_n) \in X^n;\ \pi(x_i) = \pi(x_j),\ 1 \le i, j \le n\},$$
$$Q_\pi^{(n)} = \bigcap \{\overline{TV \cap R_\pi^{(n)}};\ V \text{ a neighborhood of the diagonal in } X^n\}.$$

Remark 3.18. Using Theorem 3.1 and applying the same argument as in the proof of Proposition 3.17 one can easily show that for any extension $\pi\colon (X,T) \to (Y,S)$ the set $\mathbf{A}(T) \cap \Sigma_\pi$ is dense in $E(X,T|\Sigma_\pi)$ which gives the relative version of Proposition 4 of [4].

Definition 3.19. We say that a measure $\mu \in M(X,T)$ has *a relative full support w.r. to an invariant relation* $\Sigma \in \mathrm{CER}(X)$ if Supp $(\mu \times_\sigma \mu) = \Sigma$, where σ is the invariant partition associated with Σ.

Proposition 3.20. *If a flow (X,T) admits an ergodic measure $\mu \in M(X,T)$ with $\pi_\mu(T|\sigma) = \sigma$ for an invariant partition σ which has full support relatively to invariant relation $\Sigma \in \mathrm{CER}(X)$ associated with σ then (X,T) is a Kolmogorov extension of the factor flow determined by Σ.*

Proof. It follows from our assumption and Proposition 1 from [19] that

$$E_\mu(X,T|\sigma) \cup \Delta = \Lambda_\mu^\sigma = \text{Supp } (\mu \times_\sigma \mu) = \Sigma.$$

Applying Theorem 3.1 we see that there exists a relation $R \in \mathrm{CER}(X)$ satisfying the conditions (a)–(c). The last equality implies R is a relative K relation with respect to Σ. □

Definition 3.21 ([12], [19]). A homomorphism $\pi\colon (X,T) \to (Y,S)$ is said to have *relative u.p.e.* (or (X,T) is called *a u.p.e. extension* of (Y,S)) if for any two-set open cover $\alpha = \{U, V\}$ of X such that $\overline{U} \ne \Sigma_\pi(x) \ne \overline{V}$ for some $x \in X$, we have $h(T, \alpha|\Sigma_\pi) > 0$. Here $\Sigma_\pi(x)$ denotes the equivalence class of Σ_π containing x.

Proposition 3.22. *Any uniquely ergodic extension having relative u.p.e. is a Kolmogorov extension.*

Proof. Let μ be the unique invariant measure on (X,T) and let $\pi\colon (X,T) \to (Y,S)$ be a u.p.e. extension. By Theorem 4.1 ([12]), $E_\mu(X,T|\sigma_\pi) = E(X,T|\Sigma_\pi) = \Sigma_\pi$ where σ_π is the partition determined by π. Therefore the relation given by Theorem 3.1 is a relative K-relation of (X,T). □

Using Theorem 3.1, we give the following description of $\Pi(T|\Sigma)$ different from those in [18] and [19].

Proposition 3.23.

$$\Pi(T|\Sigma) = \Big\langle \overline{\bigcup_{\mu} \bigcup_{n=0}^{\infty} (T \times T)^{-n}(R_\mu)} \Big\rangle,$$

where $\mu \in M(X,T)$ runs over all ergodic measures with $h_\mu(T|\sigma) > 0$.

Proof. Proposition 2 of [19] yields $\Pi(T|\Sigma) = \langle \bigcup_\mu E_\mu(X,T|\sigma)\rangle$. Therefore, by (c) of Theorem 3.1, we have

$$\begin{aligned}\Pi(T|\Sigma) &\subset \Big\langle \overline{\bigcup_{\mu} \bigcup_{n=0}^{\infty} (T \times T)^{-n}(R_\mu)} \Big\rangle \subset \Big\langle \bigcup_{\mu} \Pi_\mu(T|\sigma) \Big\rangle \\ &= \Big\langle \bigcup_{\mu} \langle E_\mu(X,T|\sigma)\rangle \Big\rangle = \Big\langle \bigcup_{\mu} E_\mu(X,T|\sigma) \Big\rangle = \Pi(T|\Sigma). \qquad \square\end{aligned}$$

Remark 3.24. Using Proposition 3.23, similarly like in [16], we may give yet another proof of Corollary 3.14 without minimality assumption. Indeed. Let $\pi\colon (X,T) \to (Y,S)$ be a deterministic homomorphism. Then for every ergodic $\mu \in M(X,T)$ with $h_\mu(T|\sigma) > 0$ we have $R_\mu = \Delta$, hence $\Pi(T|\Sigma) = \Delta$ that is equivalent, by Theorem 2 of [8] and Remark 1 of [18], to the fact the relative topological entropy of $\pi\colon (X,T) \to (Y,S)$ is equal to zero.

Remark 3.25. Following Remark 3.8 and applying Remark 3.24 in the absolute case we obtain new proofs of the facts that positively doubly recurrent flows and also weakly rigid flows have zero topological entropy (cf. [26]).

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