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REMARKS TO THE ORIENTATION AND HOMOTOPY IN COINCIDENCE PROBLEMS INVOLVING FREDHOLM OPERATORS OF NONNEGATIVE INDEX

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Abstract. We introduce a notion of orientation of a Fredholm operators of nonnegative index and use it in a generalized homotopy property of the respective coincidence index.

1. Introduction

The coincidence degree theory for perturbations of a linear Fredholm operator of index zero was started by Mawhin (see e.g. [24], [25]) and next developed and applied by many authors (e.g. [11], [26], [29]). Roughly speaking, the problem was the following:

$$L(x) = f(x), \quad (\text{or equivalently } 0 = L(x) - f(x))$$

where L was a Fredholm operator (of index 0) and f a continuous map. The coincidence degree was strictly connected with Leray–Schauder degree. The homotopy property concerned at the beginning only f, while the role of L was similar to the identity in the fixed point problem.

There are several directions of possible generalizations of this problem. The first one is connected with perturbations — now they can be multivalued and not

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necessarily compact (see e.g. [10], [9]). The others concern the linear part: e.g. admitting the homotopy of L, which needs some concept of saving orientation along the homotopy (see e.g. [12], [2], [9]) or considering nonlinear Fredholm operators (see e.g. [12], [28]).

But if we assume that the index of L is nonnegative, then previous methods are not sufficient (comp. [24]). Although there are some ways to deal in such situation (see [1], [28]), the suitable homotopy invariant is due to Kryszewski (see [22]). But while it is defined for a quite large class of perturbations, including noncompact multivalued maps with nonconvex values ([14], [13]), the Fredholm operator has to be fixed.

This paper is a continuation of [14], where the whole construction of the generalized coincidence index was involved. We discuss here more general homotopy property, which admits continuous deformations in both parts of coincidence problem, the linear and nonlinear one. Some ideas concerning orientations are taken from [2] and [4].

The paper is organized as follows. In the next section we introduce some notation and definitions needed in the sequel. Then, in Section 3 we discuss the notion of an orientation for Fredholm operators and compare different possibilities of defining it. The last section is devoted to the main considerations. We start with short recalling a construction of the generalized index and next we pass to results concerning the homotopy property.

2. Preliminaries

All spaces considered in the paper are *metric*. If V is a subset of a space, then we denote the *closure*, the *interior* and the *boundary* of V by $\operatorname{cl} V$, $\operatorname{int} V$, and $\operatorname{bd} V$, respectively. If z belongs to a Banach space E, then $B^E(z,\varepsilon) = \{x \in E \mid \|x - z\| < \varepsilon\}$, $D^E(z,\varepsilon) = \operatorname{cl} B^E(z,\varepsilon)$. For a closed set $A \subset E$, by $\mathcal{O}_\varepsilon(A)$ we denote the set $\{x \in E \mid \inf_{a\in A} \|x - a\| < \varepsilon\}$.

We always assume that single-valued maps are *continuous*. If $g\colon X \to Y$ is a map, A, B are closed subsets of X and Y, respectively, and $g(A) \subset B$, then we write $g\colon (X,A) \to (Y,B)$. We denote by I_X the identity map of the space X.

Recall that by the homotopy between two single-valued maps $f_0, f_1\colon X \to Y$ one understands a map $H\colon X \times [0,1] \to Y$ such that

$$H(\,\cdot\,,0) = f_0 \quad \text{and} \quad H(\,\cdot\,,1) = f_1.$$

Let E, E' be Banach spaces. We denote by $L(E,E')$ the Banach space of bounded linear maps from E to E', and by $\operatorname{Iso}(E,E')$ its open subset of isomorphisms. An operator $L \in L(E,E')$ is called *Fredholm* if dimensions of its kernel $\operatorname{Ker} L$ and cokernel $\operatorname{Coker} L := E'/\operatorname{Im} L$ (where $\operatorname{Im} L$ is the image of L)

are finite. The index of a Fredholm operator L is defined as the integer

$$i(L) = \dim \operatorname{Ker} L - \dim \operatorname{Coker} L.$$

We always assume that $i(L) \geq 0$. The set $\Phi_n(E, E')$ of all Fredholm operators of index n is an open subset of $L(E, E')$ (see [31]). Since $\operatorname{Im} L$ is a closed subspace of E' (see [16, IV.2.6]), both $\operatorname{Ker} L$ and $\operatorname{Im} L$ are direct summands in E and E', respectively. Therefore we may consider continuous linear projections $P: E \to E$ and $Q: E' \to E'$, such that $\operatorname{Ker} L = \operatorname{Im} P$ and $\operatorname{Ker} Q = \operatorname{Im} L$. Clearly E, E' split into (topological) direct sums

$$\operatorname{Ker} P \oplus \operatorname{Ker} L = E, \quad \operatorname{Im} L \oplus \operatorname{Im} Q = E'. \tag{2.1}$$

Moreover, $L|_{\operatorname{Ker} P}$ is a linear homeomorphism onto $\operatorname{Im} L$. By $K_P: \operatorname{Im} L \to \operatorname{Ker} P$ we denote the inverse operator for $L|_{\operatorname{Ker} P}$. Note also that L is proper when restricted to a closed bounded set or, more generally, to a closed set X such that $P(X)$ is bounded.

Remark 2.1. Observe that, if $X \oplus Z \subset Y \oplus Z = E$, and Z is a finite dimensional subspace of E, then there is $X_1 \subset Y$ such that $X \oplus Z = X_1 \oplus Z$. Indeed for any $v = x + z$, where $x \in X$ and $z \in Z$, since $x = x_1 + z_1$ with $x_1 \in Y$ and $z_1 \in Z$, we get $v = x_1 + (z_1 + z)$, so X_1 is generated by the set $\operatorname{pr}(B_X)$, where B_X is the basis of X and pr is the projection of X on the space Y along Z.

Lemma 2.2 (comp. [5, Chapter 12]). *If E, E' are Banach spaces, $T: E \to E'$ is an invertible operator with $\|T^{-1}\| = d$, then any linear operator $S: E \to E'$ such that $\|T - S\| < 1/d$ is invertible, and for any $\varepsilon \in (0, 1/2)$, if $\|T - S\| < \varepsilon/d$, then $\|T^{-1} - S^{-1}\| < 2\varepsilon\|T^{-1}\|$.*

Corollary 2.3. *If the assumptions of Lemma 2.2 are satisfied, then*

$$\|S^{-1}\| \leq \|S^{-1} - T^{-1}\| + \|T^{-1}\| \leq \|T^{-1}\|(1 + 2\varepsilon).$$

Let $X \subset E$. By a multivalued map $\varphi: X \multimap E'$ we understand an *upper semi-continuous* transformation which assigns to a point $x \in X$ a *compact nonempty* set $\varphi(x) \subset E'$. We say that φ is *compact* if $\operatorname{cl} \varphi(X)$ is compact.

Let us remind that a compact space W is *cell-like* if there exists an absolute neighborhood retract Y and an embedding $i: W \to Y$ such that the set $i(W)$ is contractible in any of its neighbourhoods $U \subset Y$. Compact convex or contractible, or R_δ-sets (i.e. the intersections of decreasing families of compact contractible sets) are cell-like. Cell-like sets are acyclic; however there are examples of acyclic sets which are not cell-like.

Definition 2.4. Let Γ be a space and $X \subset E$. A proper surjection $p\colon \Gamma \to X$ is

- a *Vietoris map* if, for each $x \in X$, the fiber $p^{-1}(x)$ is acyclic ([1]),
- a *cell-like map* if, for each $x \in X$, the fiber $p^{-1}(x)$ is cell-like.

We say that a pair of maps (p,q) where $X \xleftarrow{p} \Gamma \xrightarrow{q} E'$, is

- *admissible in the sense of Górniewicz*, if p is a Vietoris map,
- *c-admissible* if p is a cell-like map.

Also the multivalued map $\phi\colon X \multimap E'$ determined by (p,q), i.e. such that $\phi(x) = q(p^{-1}(x))$, is called *admissible in the sense of Górniewicz* or, respectively, c-admissible.

Observe that a multivalued map φ determined by the pair (p,q) is compact if and only if q is compact. In such situation we will also say that (p,q) is a compact pair of maps.

It is known that the Vietoris map induces an isomorphism between respective cohomology groups (see the Vietoris-Begle theorem in e.g. [30]), that allows to define the fixed point index for maps admissible in the sense of Górniewicz (see[18], [17], [7], [6] and [19]). But here we lead with a generalized coincidence index suitable for more general problem $L(x) \in \phi(x)$, where Fredholm operator L admits a "dimensional defect" (see [22], [14]). We need, instead of the classical Vietoris-Begle theorem, its cohomotopy version due to Kryszewski (see [20], [21], [22]). Therefore we have to consider one of the following types of pairs or multivalued maps (all are admissible in the sense of Górniewicz):

(i) Admissible in the sense of Górniewicz and such that

$$\sup_{x\in X} \dim p^{-1}(x) < \infty$$

(see [22]),

or

(ii) c-admissible.

All results of this paper can be stated for each of these classes. Therefore we fix one of them, and from now, by the *admissible pair* (or map) we will understand the element of the chosen class.

Definition 2.5. We say that admissible pairs $X \xleftarrow{p_k} \Gamma_k \xrightarrow{q_k} E'$, $k = 0,1$ (or maps determined by them) are *homotopic*, if there exists an admissible pair $X \times [0,1] \xleftarrow{R} \Gamma \xrightarrow{S} Y$ and homeomorphical embeddings $j_k\colon \Gamma_k \to \Gamma$, $k = 0,1$ ([2])

([1]) A compact space A is *acyclic* with respect to the Čech cohomology H^* with integer coefficients, if $H^*(A) = H^*(pt)$, where pt is a one-point space.

([2]) i.e. a map $j_k\colon \Gamma_k \to j_k(\Gamma_k)$ is a homeomorphism.

such that the following diagram commutes:

$$\begin{array}{ccccc} X & \xleftarrow{p_0} & \Gamma_1 & & \\ {\scriptstyle i_0}\downarrow & & {\scriptstyle j_0}\downarrow & \searrow^{q_0} & \\ X\times[0,1] & \xleftarrow{R} & \Gamma & \xrightarrow{S} & E' \\ {\scriptstyle i_1}\uparrow & & {\scriptstyle j_1}\uparrow & \nearrow_{q_1} & \\ X & \xleftarrow[p_1]{} & \Gamma_1 & & \end{array}$$

where $i_k(x) = (x, k)$ for $k = 0, 1$ and $x \in X$. The pair (R, S) (or the map H_ϕ determined by it) is called a *homotopy* between pairs (p_0, q_0) and (p_1, q_1) (or between maps determined by them). If additionally (R, S) is a compact pair, then we call it a compact homotopy.

Remark 2.6. Observe that:

(a) If $\varphi: X \multimap E'$ is an admissible map, then its restriction $\varphi|_A$ to $A \subset X$ is admissible.

(b) If $f: Y \to E'$ is a single-valued map, then it is admissible (since determined by (I_Y, f)). Moreover, the single-valued homotopy $H: Y \times [0, 1] \to E'$ may be treated as a homotopy in the sense of the above definition. Namely: $\Gamma \equiv Y \times [0, 1]$, $R \equiv I_{Y\times[0,1]}$, $S \equiv H$.

(c) If (R, S) is a homotopy between admissible pairs (p_0, q_0) and (p_1, q_1), then for any $\lambda \in [0, 1]$, (p_0, q_0) is homotopic to (p_λ, q_λ), where

$$\begin{aligned} p_\lambda &= \pi \circ R|_{R^{-1}(X\times\{\lambda\})}: R^{-1}(X \times \{\lambda\}) \to X, \\ q_\lambda &= S|_{R^{-1}(X\times\{\lambda\})}: R^{-1}(X \times \{\lambda\}) \to E' \end{aligned}$$

and $\pi: X \times \{\lambda\} \to X$ is a projection. Indeed, the homotopy is determined by the pair $X \times [0, 1] \xleftarrow{P} \widetilde{\Gamma} \xrightarrow{Q} E'$, where $\widetilde{\Gamma} = R^{-1}(X \times [0, \lambda])$, $Q(\gamma) = S(\gamma)$, $P(\gamma) = z \circ R(\gamma)$ and $z(x, t) = (x, t/\lambda)$.

(d) A set-valued map $\varphi: X \multimap Y_1 \times Y_2$ given, for $x \in X$, by $\varphi(x) = \varphi_1(x) \times \varphi_2(x)$, where $\varphi_i: X \multimap Y_i$ $(i = 1, 2)$ are admissible maps, is also an admissible one. Similarly, if $Y = Y_1 = Y_2$, the map $\psi: X \multimap Y$, defined by $\psi(x) := \varphi_1(x) + \varphi_2(x) = \{y_1 + y_2 \mid y_1 \in \varphi_1(x),\, y_2 \in \varphi_2(x)\}$, $x \in X$, is admissible. In particular, if an admissible pair (p, q) determines a map $\varphi: X \multimap Y$, and $f: X \to Y$, then the map $f + \varphi: X \multimap Y$, given by $(f + \varphi)(x) = \{f(x) + y \mid y \in \varphi(x)\}$ for $x \in X$, is admissible (it is determined by the pair $(p, f \circ p + q)$).

3. Orientation of Fredholm operators

3.1. Main notions. We always assume that the orientation in $\mathbb{R}^n$ is a canonical one, i.e. represented by the ordered basis $(e_1, \ldots, e_n)$, where $e_j = (\delta_{ij})_{i=1}^n$.

If W is a real n-dimensional linear space, then by an orientation in W we understand one of two equivalence classes of linear isomorphisms of the form $\eta\colon W \to \mathbb{R}^n$, where $\eta_1 \sim \eta_2$ if and only if $\det(\eta_1 \circ \eta_2^{-1}) > 0$. Of course, in fact, each η determines a choice of an ordered basis $(a_1, \dots, a_n)$ in W (by $a_i = \eta^{-1}(e_i)$) and conversely, if the ordered basis $(a_1, \dots, a_n)$ is given, then it appoints an isomorphism η by $\eta(a_i) = e_i$. Let η_W and η_V represent orientations in W and V, respectively. The orientation of the pair of spaces (W, V) can be considered as the orientation in $W \times V$ determined by $(\eta_W, \eta_V)\colon W \times V \to \mathbb{R}^{\dim W} \times \mathbb{R}^{\dim V}$. Clearly if we change the orientation in W *or* in V, then automatically the same happens in $W \times V$, but if we change it in *both* W and V, then the orientation in $W \times V$ stays the same.

Definition 3.1. By an *oriented Fredholm operator* we understand a pair $(L, [(\eta, \eta')])$ where $L \in \Phi_n(E, E')$ and $[(\eta, \eta')]$ is a fixed orientation in $\operatorname{Ker} L \times \operatorname{Coker} L$, determined by isomorphisms

$$\eta\colon \operatorname{Ker} L \to \mathbb{R}^{\dim \operatorname{Ker} L}, \qquad \eta'\colon \operatorname{Coker} L \to \mathbb{R}^{\dim \operatorname{Coker} L}.$$

Let now L be oriented in the sense of Definition 3.1 and P, Q be respective projections in E and E'. Then $[\eta']$ determines an orientation $[\eta'_Q]$ in $\operatorname{Im} Q$ as follows: $\eta'_Q = \eta' \circ z \circ i$, where $i\colon \operatorname{Im} Q \to E'$ is the inclusion and $z\colon E' \to \operatorname{Coker} L$ is the quotient map. Observe that if $(\eta_1, \eta'_1) \in [(\eta, \eta')]$ and $\eta'_1 \notin [\eta']$, then also $\eta_1 \notin [\eta]$, but still $(\eta_1, (\eta_Q)'_1) \in [(\eta, \eta'_Q)]$, where $(\eta_Q)'_1 = \eta'_1 \circ z \circ i$.

Observe that if $\dim E, \dim E' < \infty$, then there is an one-to-one correspondence between the orientation of L and the orientation of the pair of spaces (E, E'). Indeed, let (ξ, ξ') represents the orientation in $E \times E'$, η' in $\operatorname{Coker} L$ and η'_1 in $\operatorname{Im} L$. We take in $\operatorname{Im} Q$ the orientation $[\eta'_Q]$ determined, like earlier, by η' and in $\operatorname{Ker} P$ the orientation represented by $\eta_1 := \eta'_1 \circ L|_{\operatorname{Ker} P}$. At last we select the isomorphism $\eta\colon \operatorname{Ker} L \to \mathbb{R}^{\dim \operatorname{Ker} L}$ such that $((\eta_1, \eta), (\eta'_1, \eta'_Q)) \in [(\xi, \xi')]$. Then $\operatorname{Ker} L \times \operatorname{Coker}$ is oriented by (η, η'). The inverse process is analogous.

Definition 3.1 allows of course $\operatorname{Coker} L \equiv \{0\}$. Then the orientation of L depends only on the orientation of $\operatorname{Ker} L$, but is still equivalent to the orientation of $E \times E'$. Then, if L is not an isomorphism, we can determine its orientation by orientation in $E \times E'$.

But if L is an isomorphism (then $i(L) = 0$), also $\operatorname{Ker} L \equiv 0$, so L does not have orientation in the sense of the above definition. Nevertheless we can consider two orientations of L as determined by orientations of $E \times E'$ (the positive one, if L saves an orientation in $E \times E'$ and the negative one, if not).

If $\dim E, \dim E' = \infty$, then in the similar way, the orientation of $L \in \Phi_n(E, E')$ is equivalent to the orientation in $E_1 \times E'_1$, where E_1, E'_1 are fixed finite dimensional subspaces of E and E' respectively, such that $\operatorname{Im} Q \subset E'_1$

and $E_1 = L^{-1}(E_1')$. Indeed, observe that, there is a finite dimensional subspace W of E_1' such that $W \oplus \operatorname{Im} Q = E_1'$. Hence $E_1 = K_P(W) \oplus \operatorname{Ker} L$ and $L|_{E_1} \in \Phi_n(E_1, E_1')$ with $\operatorname{Ker} L = \operatorname{Ker} L|_{E_1}$ and $\operatorname{Coker} L = \operatorname{Coker} L|_{E_1}$.

Another notion of orientation for Fredholm operators was proposed by Benevieri and Furi in [3]. They generalized the earlier ideas of Mawhin (see [24]). Below we introduce their definition and compare it with Definition 3.1. But since in [3] most considerations concern Fredholm operators of index 0, while in this paper the ones of nonnegative index, we have to adapt and develop some notions.

At the beginning we remind that if the linear operator $T: E \to E$ is of the form $T = I_E - K$, where $\dim \operatorname{Im} K = k < \infty$, then for any finite dimensional subspace E_1 containing $\operatorname{Im} K$, $T(E_1) \subset E_1$. The determinant $\det T$ (equal to $\det(\eta \circ T|_{E_1} \eta^{-1})$, where η determines the orientation in E_1), does not depend on a choice of E_1 (and an orientation $[\eta]$).Thus, by $\det T$ we understand $\det T|_{E_1}$, where E_1 is an arbitrary space satisfying the above conditions.

Let $L \in \Phi_n(E, E')$. Denote by $z(L)$ the map acting between E and $E' \times \mathbb{R}^n$, given by $z(L)(x) = (L(x), 0)$. Clearly $z(L) \in \Phi_0(E, E' \times \mathbb{R}^n)$. We say that a linear operator $A: E \to E' \times \mathbb{R}^n$ with finite dimensional range is a *corrector* of L, provided $z(L) + A$ is an isomorphism. We consider the following equivalence relation in the set $C(L)$ of all correctors of L. Observe that if $A, B \in C(L)$, then

$$\begin{aligned}(z(L) + B)^{-1} \circ (z(L) + A) &= (z(L) + B)^{-1} \circ (z(L) + B + A - B)\\ &= I_E - (z(L) + B)^{-1} \circ (A - B)\end{aligned}$$

and $(z(L) + B)^{-1} \circ (A - B)$ has a finite dimensional range. Thus

$$\det((z(L) + B)^{-1} \circ (z(L) + A))$$

is well defined. We say that A is *L-equivalent* to B if

$$\det(z(L) + B)^{-1} \circ (z(L) + A) > 0$$

(see [2] or [3] for details).

Definition 3.2 (see [2], [3], compare also [29]). An orientation of a Fredholm operator L is one of two equivalence classes of $C(L)$. L is oriented, when the orientation is chosen.

The following notion will help us to compare Definitions 3.2 and 3.1.

Definition 3.3. A corrector A of a Fredholm operator $L \in \Phi_n(E, E')$ is called *canonical* with respect to the projections P, Q and the orientation $[(\eta, \eta')]$, if $A|_{\operatorname{Ker} P} \equiv 0$ and $A|_{\operatorname{Ker} L}: \operatorname{Ker} L \to \operatorname{Im} Q \times \mathbb{R}^n$ is an isomorphism which saves the respective orientations, i.e. such that

$$\det((\eta_Q', I_{\mathbb{R}^n}) \circ A|_{\operatorname{Ker} L} \circ \eta^{-1}) > 0,$$

where η'_Q determines the respective orientation in $\operatorname{Im} Q$ ([3])

Obviously, since $\operatorname{Ker} L$ and $\operatorname{Im} Q$ are finite dimensional spaces, a canonical corrector exists. If $i(L) = 0$, then $z(L) = L$ and the canonical corrector is an isomorphism between $\operatorname{Ker} L$ and $\operatorname{Im} Q$, if $\dim \operatorname{Ker} L > 0$, or is equal to 0, if L is an isomorphism. If $i(L) > 0$, then any corrector must be a nontrivial operator.

Proposition 3.4. *Let $L \in \Phi_n(E, E')$ and P, Q, $[(\eta, \eta')]$ be fixed. Then all canonical correctors of L are L-equivalent.*

Proof. Let A and B be canonical correctors of L. Observe that

$$\begin{aligned}
&\det((z(L) + A)^{-1} \circ (z(L) + B)) \\
&\quad = \det(\eta \circ ((z(L) + A)|_{\operatorname{Ker} L})^{-1} \circ (z(L) + B)|_{\operatorname{Ker} L} \circ \eta^{-1}) \\
&\quad = \det(\eta \circ ((A|_{\operatorname{Ker} L})^{-1} \circ B|_{\operatorname{Ker} L} \circ \eta^{-1}) \\
&\quad = \det(\eta \circ (A|_{\operatorname{Ker} L})^{-1} \circ (\eta'_Q, I_{\mathbb{R}^n})^{-1} \circ (\eta'_Q, I_{\mathbb{R}^n}) \circ B|_{\operatorname{Ker} L} \circ \eta^{-1}) \\
&\quad = \det[((\eta'_Q, I_{\mathbb{R}^n}) \circ A|_{\operatorname{Ker} L} \circ \eta^{-1})^{-1}] \det((\eta'_Q, I_{\mathbb{R}^n}) \circ B|_{\operatorname{Ker} L} \circ \eta^{-1}) > 0, \ \square
\end{aligned}$$

Now it is obvious that if (η, η') determines the orientation of L, then the corresponding equivalence class of correctors contains the canonical ones. Conversely, if we choose the orientation $[B]$ of L in the sense of Definition 3.2, then there always exists $A \in [B]$ such that $A|_{\operatorname{Ker} P} \equiv 0$ and $A|_{\operatorname{Ker} L}\colon \operatorname{Ker} L \to \operatorname{Im} Q \times \mathbb{R}^n$ is an isomorphism. Indeed, we take an arbitrary isomorphism A between respective spaces, if A does not belong to $[B]$, then $\overline{A}$ does, where $\overline{A}$ is given by $\overline{A}(\eta^{-1}(e_1)) = -A(\eta^{-1}(e_1))$, $\overline{A}(\eta^{-1}(e_i)) = A(\eta^{-1}(e_i))$ for $i \neq 1$. One can take any orientation η' in $\operatorname{Coker} L$ and select an orientation η of $\operatorname{Ker} L$ such that A will be a canonical corrector.

Further we will prove that this correspondence does not depend on the choice of projections P and Q.

The following proposition is an obvious consequence of the definition. Let $L \in \Phi_n(E, E')$ be oriented in the sense of Definition 3.1 by $[(\eta, \eta')]$.

Proposition 3.5. *Let $V = W \oplus \operatorname{Im} Q$, $W \subset \operatorname{Im} L$ and $\dim V < \infty$. Choose the orientation in W represented by $\eta'_W\colon W \to \mathbb{R}^s$. In $K_P(W)$ the orientation is carried back by L from W, i.e. determined by $\eta_W := \eta'_W \circ L|_{K_P(W)}$. Then for any canonical corrector B of L, the operator*

$$(z(L) + B)|_{K_P(W) \oplus \operatorname{Ker} L}\colon K_P(W) \oplus \operatorname{Ker} L \to W \oplus \operatorname{Im} Q \times \mathbb{R}^n$$

saves the orientation, i.e.

$$\det((\eta'_W, \eta'_Q, I_{\mathbb{R}^n}) \circ (z(L) + B)|_{K_P(W) \oplus \operatorname{Ker} L} \circ (\eta_W, \eta)^{-1}) > 0.$$

([3]) i.e. $\eta'_Q = \eta' \circ z \circ i$ and $i\colon \operatorname{Im} Q \to E'$ is the inclusion, $z\colon E' \to \operatorname{Coker} L$ is the quotient map.

Proposition 3.6. *Let P_A, Q_A and P_B, Q_B be two pairs of projections satisfying* (2.1) *and A, B be a canonical correctors of L defined for P_A, Q_A and P_B, Q_B, respectively. Then A and B are L-equivalent.*

Proof. Assume for the moment that $P = P_A = P_B$ and define $V := \operatorname{Im} Q_A + \operatorname{Im} Q_B$, $W := V \cap \operatorname{Im} L$. Observe that $W \oplus \operatorname{Im} Q_A = V = W \oplus \operatorname{Im} Q_B$, hence, if we fix the orientation $[\eta'_W]$ in W, then the orientation in V can be determined by (η'_W, η'_{Q_A}) or by (η'_W, η'_{Q_B}). We shall prove that in fact they are the same, i.e. $\det((\eta'_W, \eta'_{Q_B}) \circ I_V \circ (\eta'_W, \eta'_{Q_A})^{-1}) > 0$. Indeed, if $V \ni y = y_1 + y_A$ and $y_A = y_2 + y_B$, where $y_1, y_2 \in W$, $y_A \in \operatorname{Im} Q_A$, $y_B \in \operatorname{Im} Q_B$, then $I_V(y) = y_1 + (y_A - y_B) + y_B$ and $y_A - y_B \in W$. Let $s\colon \operatorname{Im} Q_A \to \operatorname{Im} Q_B$ be such that $s(y_A) = y_B$ and $w\colon V \to W$ be given by $w(y) = y_1 + (y_A - y_B)$. Then

$$I_V = \begin{bmatrix} I_W & w|_{\operatorname{Im} Q_A} \\ 0 & s \end{bmatrix}$$

But $\eta'_{Q_B} \circ s \circ (\eta'_{Q_A})^{-1} = (\eta' \circ z \circ i_B) \circ s \circ (\eta' \circ z \circ i_A)^{-1} = \eta' \circ z \circ i_B \circ s \circ (z \circ i_A)^{-1} \circ (\eta')^{-1}$ ([4]), and since $z(y_A) = z(y_B)$, we know that $z \circ i_B \circ s \circ (z \circ i_A)^{-1} = I_{\operatorname{Coker} L}$, therefore $\det(\eta'_{Q_B} \circ s \circ (\eta'_{Q_A})^{-1}) = \det(\eta' \circ I_{\operatorname{Coker} L} \circ (\eta'^{-1})) = 1 > 0$. At last we get $\det((\eta'_W, \eta'_{Q_B}) \circ I_V \circ (\eta'_W, \eta'_{Q_A})^{-1}) = 1 > 0$.

Denote by $\xi = (\eta_W, \eta)$ the map determining an orientation in $E_1 := K_P(W) \oplus \operatorname{Ker} L$ (comp. Proposition 3.5). Since $\operatorname{Im}(A - B) \subset V \times \mathbb{R}^n$, and consequently $\operatorname{Im}((z(L) + B)^{-1} \circ (z(L) + A)) = \operatorname{Im}(z(L) + B)^{-1} \circ (A - B) \subset E_1$, we get

$$\begin{aligned}
\det((z(L)&+B)^{-1} \circ (z(L) + A)) \\
&= \det(\xi \circ (z(L) + B)^{-1} \circ (z(L) + A)|_{E_1} \circ \xi^{-1}) \\
&= \det(\xi \circ ((z(L) + B)|_{E_1})^{-1} \circ (\eta'_W, \eta'_{Q_B}, I_{\mathbb{R}^n})^{-1} \\
&\qquad \circ (I_V, I_{\mathbb{R}^n}) \circ (\eta'_W, \eta'_{Q_A}, I_{\mathbb{R}^n}) \circ (z(L) + A)|_{E_1} \circ \xi^{-1}) \\
&= \det(\xi \circ ((z(L) + B)|_{E_1})^{-1} \circ (\eta'_W, \eta'_{Q_B}, I_{\mathbb{R}^n})^{-1}) \\
&\qquad \cdot \det((\eta'_W, \eta'_{Q_A}, I_{\mathbb{R}^n}) \circ (z(L) + A)|_{E_1} \circ \xi^{-1}) \\
&= \det(((\eta'_W, \eta'_{Q_B}, I_{\mathbb{R}^n}) \circ (z(L) + B)|_{E_1} \circ \xi^{-1})^{-1}) \\
&\qquad \cdot \det((\eta'_W, \eta'_{Q_A}, I_{\mathbb{R}^n}) \circ (z(L) + A)|_{E_1} \circ \xi^{-1}) > 0.
\end{aligned}$$

If now $P_A \neq P_B$, then still $E_1 = K_{P_A}(W) \oplus \operatorname{Ker} L = K_{P_B}(W) \oplus \operatorname{Ker} L$ with orientations determined by

$$\xi_A := (\eta'_W \circ L|_{K_{P_A}(W)}, \eta) \quad \text{and} \quad \xi_B := (\eta'_W \circ L|_{K_{P_B}(W)}, \eta),$$

respectively. Since

$$\eta'_W \circ L|_{K_{P_B}(W)} = \eta'_W \circ L|_{K_{P_A}(W)} \circ K_{P_A} \circ L|_{K_{P_B}(W)},$$

([4]) Recall that $z\colon E' \to \operatorname{Coker} L$ is a quotient map; $i_A\colon \operatorname{Im} Q_A \to E'$ and $i_B\colon \operatorname{Im} B \to E'$ are the inclusions then considering the identity map I_{E_1}, we have

$$I_{E_1}(x_B + x_1) = K_{P_A} \circ L|_{K_{P_B}(W)}(x_B) + x_1 + (x_B - K_{P_A} \circ L|_{K_{P_B}(W)}(x_B)),$$

where $x_B \in K_{P_B}(W)$, $x_1 \in \operatorname{Ker} L$, and using arguments similar to the above, one can prove that ξ_A and ξ_B determines the same orientation in E_1. Proposition 3.5 completes the proof. □

Remark 3.7. We have just proved that there is a one-to-one correspondence between the orientations of L defined in Definition 3.2 and in Definition 3.1, and that it does not depend on a choice of P and Q. Thus, from now we shall identify both orientations. By a *positive* corrector of an oriented Fredholm operator L we will understand any one belonging to the chosen orientation (in the sense of Definition 3.2). It means that it is L-equivalent to any (then, by Propositions 3.4 and 3.5 to all) canonical corrector for the orientation described in Definition 3.1. Observe that all positive correctors of L are L-equivalent and any corrector being L-equivalent to a positive one is also positive.

Below we introduce a few technical results concerning correctors which will be used in the further considerations.

Remark 3.8 (comp. [3]). Let E'_1 be a finite dimensional subspace of E' such that $E' = \operatorname{Im} L + E'_1$ and let $E_1 = L^{-1}(E'_1)$. Then $L_1 \colon E_1 \to E'_1$ defined by $L_1(x) = L(x)$ is a Fredholm operator and $i(L_1) = i(L)$. Since E_1 is finite dimensional, one can split E in a topological direct sum $E = E_0 \oplus E_1$. Moreover, also $E' = L(E_0) \oplus E_1$, because $L_0 \colon = L|_{E_0} \colon E_0 \to E'_0$ is an isomorphism. Thus L and $z(L)$ can be represented by

$$L = \begin{bmatrix} L_0 & 0 \\ 0 & L_1 \end{bmatrix} \quad \text{and} \quad z(L) = \begin{bmatrix} L_0 & 0 \\ 0 & z(L_1) \end{bmatrix}.$$

A linear operator $A \colon E \to E' \times \mathbb{R}^{i(L)}$, represented by

$$A = \begin{bmatrix} 0 & 0 \\ 0 & A_1 \end{bmatrix},$$

is a corrector of L if and only if A_1 is a corrector of L_1. One can easily check, that two correctors of L_1 are L_1-equivalent if and only if the corresponding correctors of L are L-equivalent.

Proposition 3.9. *Let $L \in \Phi_n(E, E')$ be oriented by $[(\eta, \eta')]$. If $V = W \oplus \operatorname{Im} Q \subset E'$ and $\dim V < \infty$, then there is a positive corrector B of L such that $\operatorname{Im} B = V \times \mathbb{R}^n$.*

Proof. Without loosing the generality we can assume that $W \subset \operatorname{Im} L$ (see Remark 2.1) and, since W is finite dimensional, that $\operatorname{Im} L = W_0 \oplus W$. Let take a canonical corrector A and put $B|_{\operatorname{Ker} L} \equiv A$, $B|_{K_P(W)}(x) = (1/2)L(x)$ and

$B|_{K_P(W_0)} \equiv 0$. One can easily check that B is L-equivalent to A, thus it is a positive corrector of L. □

Corollary 3.10. *Proposition* 3.5 *is also true for any positive corrector A such that* $\operatorname{Im} A \subset V \times \mathbb{R}^n$.

3.2. The continuity of the orientation. Some results of this subsection may be compared with [3], where Fredholm operators of index 0 are considered.

Proposition 3.11. *If A and B are L-equivalent correctors of $L \in \Phi_n(E, E')$, then there is an open neighborhood $V \subset \Phi_n(E, E')$ of L such that for any $L_1 \in V$, A and B are L_1-equivalent correctors of L_1.*

Proof. At first observe that, since $\operatorname{Iso}(E, E' \times \mathbb{R}^n)$ is an open subset of $L(E, E' \times \mathbb{R}^n)$, there is $\varepsilon_A > 0$ such that for any $Z \in L(E, E' \times \mathbb{R}^n)$,

$$\text{if } \|L + A - Z\| < \varepsilon_A, \text{ then } Z \in \operatorname{Iso}(E, E' \times \mathbb{R}^n).$$

Thus for any $L_1 \in \Phi_n(E, E')$ such that $\|L - L_1\| < \varepsilon_A$, also

$$\|z(L) + A - (z(L_1) + A)\| < \varepsilon_A,$$

what implies that $z(L_1) + A$ is an isomorphism and consequently, that A is a corrector of L_1.

In the same way we get ε_B for B. Let $\varepsilon < \min(\varepsilon_A, \varepsilon_B)$. For $L_1 \in \Phi_n(E, E')$, $\|L - L_1\| < \varepsilon$, A and B are correctors of L_1. We have to prove that A and B are L_1-equivalent.

Observe that for a finite dimensional space

$$E_1 = \operatorname{Im}((z(L) + B)^{-1} \circ (A - B)) + \operatorname{Im}((z(L_1) + B)^{-1} \circ (A - B)),$$

since $((z(L)+B)^{-1} \circ (z(L)+A))(E_1) \subset E_1$ and $((z(L_1)+B)^{-1} \circ (z(L_1)+A))(E_1) \subset E_1$, we get for any isomorphism $\xi \colon E_1 \to \mathbb{R}^k$ determining the orientation of E_1,

$$\det((z(L) + B)^{-1} \circ (z(L) + A)) = \det(\xi \circ (z(L) + B)^{-1} \circ (z(L) + A)|_{E_1} \circ \xi^{-1}) > 0$$

and

$$\det((z(L_1) + B)^{-1} \circ (z(L_1) + A)) = \det(\xi \circ (z(L_1) + B)^{-1} \circ (z(L_1) + A)|_{E_1} \circ \xi^{-1}).$$

Of course, if $\dim E < \infty$, one can take simply $E_1 = E$. But $\det\colon \operatorname{Iso}(\mathbb{R}^k, \mathbb{R}^k) \to \mathbb{R}$ is a continuous function, so if $\xi \circ (z(L) + B)^{-1} \circ (z(L) + A)|_{E_1} \circ \xi^{-1}$ and $\xi \circ (z(L_1) + B)^{-1} \circ (z(L_1) + A)|_{E_1} \circ \xi^{-1}$ are "sufficiently close", then also $\det(\xi \circ (z(L_1) + B)^{-1} \circ (z(L_1) + A)|_{E_1} \circ \xi^{-1}) > 0$.

Observe that for any $\varepsilon > 0$ and $L_1 \in \Phi_n(E, E')$ such that $\|L - L_1\| < \varepsilon / \|z(L) + B\|$ also $\|z(L)+B-(z(L_1)+B)\| < \varepsilon / \|z(L) + B\|$. Then by Lemma 2.2

and Corollary 2.3,

$$\begin{aligned}
&\|(z(L) + B)^{-1} \circ (z(L) + A) - (z(L_1) + B)^{-1} \circ (z(L_1) + A)\| \\
&\quad = \|(z(L) + B)^{-1} \circ (z(L) + A) - I_{E_1} + I_{E_1} - (z(L_1) + B)^{-1} \circ (z(L_1) + A)\| \\
&\quad = \|(z(L) + B)^{-1} \circ (z(L) + A) - (z(L) + B)^{-1} \circ (z(L) + B) \\
&\qquad + (z(L_1) + B)^{-1} \circ (z(L_1) + B) - (z(L_1) + B)^{-1} \circ (z(L_1) + A)\| \\
&\quad = \|(z(L) + B)^{-1} \circ (z(L) + A - z(L) - B) \\
&\qquad + (z(L_1) + B)^{-1} \circ (z(L_1) + A - z(L_1) - B)\| \\
&\quad = \|(z(L) + B)^{-1} \circ (A - B) + (z(L_1) + B)^{-1} \circ (A - B)\| \\
&\quad = \|((z(L) + B)^{-1} - (z(L_1) + B)^{-1}) \circ (A - B)\| \\
&\quad \leq \|(z(L) + B)^{-1} - (z(L_1) + B)^{-1}\| \cdot \|A - B\| \\
&\quad < 2\varepsilon\|(z(L) + B)^{-1}\| \cdot \|A - B\|.
\end{aligned}$$

Therefore we can choose $\varepsilon > 0$ so small that the distance between

$$(z(L) + B)^{-1} \circ (z(L) + A)|_{E_1} \quad \text{and} \quad (z(L_1) + B)^{-1} \circ (z(L_1) + A)|_{E_1}$$

is sufficient for $\det(\xi \circ (z(L_1) + B)^{-1} \circ (z(L_1) + A)|_{E_1} \circ \xi^{-1}) > 0$. □

Thus, if $L \in \Phi_n(E, E')$ is an oriented Fredholm operator, then it determines the choice of "consistent" orientations of maps in some its neighborhood in $\Phi_n(E, E')$. Of course the neighborhood depends on the choice of a corrector. However, it is worth stressing, that the sets of positive correctors may differ even for close operators, as one can see in the example below. Here, and in the next examples we describe the orientations of the finite dimensional spaces by the ordered basis (see remarks at the beginning of Section 3).

Example 3.12. Let $E = \mathbb{R}^3$, $E' = \mathbb{R}^2$, $L_t(x, y, z) = (x, ty)$. Observe that $A: \mathbb{R}^3 \to \mathbb{R}^2 \times \mathbb{R}$ given by $A(x, y, z) = ((0, y), z)$ is a corrector of L_t, if e.g. $t \in [-1/2, 1/2]$, while $C: \mathbb{R}^3 \to \mathbb{R}^2 \times \mathbb{R}$, $C(x, y, z) = ((0, 0), z)$ is a corrector of L_t for any $t \neq 0$, but not of L_0. Also $C(L_0) \not\subset C(L_t)$, since $B_t: \mathbb{R}^3 \to \mathbb{R}^2 \times \mathbb{R}$ given by $B(x, y, z) = ((x, -ty), -z)$ is a corrector of L_0 but not of L_t.

Moreover, A determines an orientation of the pair of spaces

$$\operatorname{Ker} L_0 = \operatorname{Lin}((0, 1, 0), (0, 0, 1)) \quad \text{and} \quad \operatorname{Coker} L_0 \equiv \operatorname{Lin}((0, 1))$$

as follows: we choose the orientation $[(0, 1)]$ of $\operatorname{Coker} L_0$, it determines the orientation $[(0, 1)]$ in $\operatorname{Im} Q$ for $Q(x, y) := (0, y)$, the orientation $[(0,1,0),(0,0,1)]$ in $\operatorname{Ker} L_0$ is transposed by the canonical corrector A from $\operatorname{Im} Q \times \mathbb{R}$, since $A(0, 1, 0) = ((0, 1), 0)$ and $A(0, 0, 1) = ((0, 0), 1)$.

For $t \neq 0$, $\operatorname{Ker} L_t = \operatorname{Lin}((0,0,1))$ and $\operatorname{Coker} L_t = \{0\}$, thus the orientation depends only on those of $\operatorname{Ker} L_t$. Observe that $C_t\colon \mathbb{R}^3 \to \mathbb{R}^2 \times \mathbb{R}$,

$$C_t(x,y,z) = \begin{cases} ((0,0),z) & \text{for } t > 0, \\ ((0,0),-z) & \text{for } t < 0, \end{cases}$$

is a corrector of L_t, L_t-equivalent to A. Thus, the "consistent" orientation of $\operatorname{Ker} L_t$ is transposed by C_t from $\{0\} \times \mathbb{R}$, namely it is $[(0,0,1)]$ for $t > 0$ and $[(0,0,-1)]$ for $t < 0$.

We will describe the notion of the orientability for continuous maps of the form $H_L\colon \Lambda \to \Phi_n(E,E')$, where Λ is a topological space.

Let $\widehat{\Phi}_n(E,E') = \{(L,\omega) \mid L \in \Phi_n(E,E'),\ \omega$ be a class of L-equivalent correctors of $L\}$. Observe that the sets $O_{W,A} := \{(L,\omega) \in \widehat{\Phi}_n(E,E') \mid L \in W,\ A \in \omega\}$, where W is an open subset of $\Phi_n(E,E')$, $A\colon E \to E' \times \mathbb{R}^n$ is bounded linear and $\dim \operatorname{Im} A < \infty$, form a basis for a topology on $\widehat{\Phi}_n(E,E')$. Moreover, the projection $r\colon (L,\omega) \mapsto L$ is a double covering of $\Phi_n(E,E')$.

Definition 3.13. An *orientation* of the map H_L defined above is a continuous map $\widehat{H_L}\colon \Lambda \to \widehat{\Phi}_n(E,E')$, such that $r \circ \widehat{H_L} = H_L$, i.e. $\widehat{H_L}$ is a lifting of H_L. The map H_L is *orientable* when it admits an orientation and *oriented* if an orientation is chosen. The subset $\mathcal{W}$ of $\Phi_n(E,E')$ is orientable (oriented) if so is the inclusion map $i\colon \mathcal{W} \to \Phi_n(E,E')$.

Remark 3.14. Let $H_L\colon \Lambda \to \Phi_n(E,E')$, then:

(a) Any restriction of H_L is orientable provided so is H_L.

(b) If Λ_1 is a topological space, $s\colon \Lambda_1 \to \Lambda$ is a continuous map and H_L is orientable, then $H_L \circ s$ is orientable.

(c) If H_L is orientable, then $z(H_L)\colon \Lambda \to \Phi_0(E, E' \times \mathbb{R}^n)$, given by $z(H_L)(\lambda) = (H_L(\lambda), 0)$ is orientable in the sense of definition from [3] (i.e. Definition 3.13 for $n = 0$).

(d) H_L is orientable, if $H_L(\Lambda)$ is contained in an orientable subset of the set $\Phi_n(E,E')$.

(e) If Λ is path connected, then $H_L\colon \Lambda \to \Phi_n(E,E')$ is orientable, and then the choice of orientation for $H_L(\lambda)$ in one $\lambda \in \Lambda$ determines the orientations for the others.

Moreover,

(f) by Proposition 3.11, $\Phi_n(E,E')$ is locally orientable, and therefore any continuous $H_L\colon \Lambda \to \Phi_n(E,E')$ is locally orientable. If $\dim E, \dim E' < \infty$, then $\Phi_n(E,E')$ is simply orientable.

(g) In particular, the homotopy $H_L\colon [0,1] \to \Phi_n(E,E')$ is orientable and the orientation may be determined by orientation of e.g. $H_L(0)$. But if we have two homotopies H_L, H'_L such that $H_L(0) = H'_L(0) = L_0$, $H_L(1) = H'_L(1) = L_1$

and for both we take the orientation determined by the same orientation of L_0, then orientation of $H_L(1)$ can be different than that of $H'_L(1)$ (see e.g. [3, Theorem 3.15]). It may happen when $\mathrm{Iso}\,(E,E')$ is nonempty and connected i.e. in some infinitely dimensional spaces (see e.g. [23], [27]).

It is also worth mentioning that another interesting notion of the orientation for maps of the form $H_L\colon \Lambda \to \Phi_0(E,E')$ (and some more general, but still with Fredholm index equal to 0) was earlier proposed by Fitzpatrick, Pejsachowicz and Rabier (see e.g. [11], [12]). One can find the comparison in [3].

4. The homotopy property of the coincidence index

As we have mentioned in Introduction, the main aim of this paper is to generalize the homotopy property of the coincidence index, which is used in problems of the form

$$L(x) \in \phi(x),$$

where $L \in \Phi_n(E,E')$, and $\phi\colon X \multimap E'$, $(X \subset E)$ is a multivalued map. Till now the homotopy property has concerned the situation with fixed L, i.e. $\mathrm{Ind}_L(\phi_1, X) = \mathrm{Ind}_L(\phi_2, X)$, provided ϕ_1 and ϕ_2 are homotopic (see [22], [14]). But it is quite natural to admit also a continuous deformation of L (see [12], [2], [9] for Fredholm operators of index 0).

In this section we first remind briefly a definition of the generalized coincidence index and then we pass to the main results concerning the homotopy property.

4.1. Coincidence index. The coincidence index $\mathrm{Ind}_L((p,q),X)$ of a compact admissible pair $X \overset{p}{\longleftarrow} \Gamma \overset{q}{\longrightarrow} E'$ is defined in [22] or, in a bit different way, in [14]. We do not repeat the whole construction, but only discuss some parts of it, strictly connected with the homotopy property. All other properties of Ind, i.e. existence, localization, additivity, restriction, as well as the very definition, stay the same if we admit the continuous deformation of L. But we would like to stress that, because of possible "dimensional defect", this coincidence index is an element of the respective stable homotopy group. However, if there is no "dimensional defect", i.e. $i(L) = 0$, then the coincidence index is in fact equivalent to the fixed point index for admissible maps, when $L = id$ (comp. [22]), or to the index due to Mawhin in a general case (see [26], [25]).

Assume for a moment that $E = \mathbb{R}^m$, $E' = \mathbb{R}^n$, $m \geq n$, U is an open bounded subset of $\mathbb{R}^m$ and $\mathrm{cl}\,U \overset{p}{\longleftarrow} \Gamma \overset{q}{\longrightarrow} E'$ is an admissible pair such that $q(p^{-1}(\mathrm{bd}\,U)) \subset \mathbb{R}^n \setminus \{0\}$ and then of course $q(p^{-1}(\mathrm{bd}\,U)) \subset \mathbb{R}^n \setminus B^n(0,\rho)$ for some $\rho > 0$. We consider the problem

$$0 \in q(p^{-1}(x))$$

with dimensional defect equal to $m-n$. In the following sequence of maps:

$$(\mathbb{R}^n, \mathbb{R}^n \setminus B^n(0,\rho)) \xleftarrow{q} (\Gamma, \Gamma') \xrightarrow{p} (\operatorname{cl} U, \operatorname{bd} U) \xrightarrow{i_1} (\mathbb{R}^m, \mathbb{R}^m \setminus U) \xleftarrow{i_2} (\mathbb{R}^m, \mathbb{R}^m \setminus B^m(0,r)),$$

$\Gamma' := p^{-1}(\operatorname{bd} U)$, i_1, i_2 are the inclusions and $r > 0$ is such that $\operatorname{cl} U \subset B(0,r)$. All these maps induce maps between respective cohomotopy sets. Moreover, since p and i_1 induce bijections (see [22] or [14] for details), the map

$$\mathcal{K} := i_2^{\#} \circ (i_1^{\#})^{-1} \circ (p^{\#})^{-1} \circ q^{\#} : \pi^n(S^n) \cong \pi^n(\mathbb{R}^n, \mathbb{R}^n \setminus B^n(0,\rho)) \to \pi^n(\mathbb{R}^m, \mathbb{R}^m \setminus B^m(0,\varepsilon)) \cong \pi^n(S^m).$$

is well defined.

Definition 4.1. By the generalized degree of the admissible pair (p,q) on the set U in 0 we understand the element

$$\deg((p,q), U, 0) := \mathcal{K}(\mathbf{1}) \in \pi^n(S^m),$$

where $\mathbf{1}$ is the homotopy class of the identity map $\mathrm{id}\colon S^n \to S^n$ in $\pi^n(S^n) \cong \mathbb{Z}$.

Let now T, V be finite dimensional Banach spaces with $\dim T = m$, $\dim V = n$ and U be an open bounded subset of T. The isomorphisms $\eta_T\colon T \to \mathbb{R}^m$ and $\eta_V\colon V \to \mathbb{R}^n$ determine the orientations in T and V. Assume that

$$(\operatorname{cl} U, \operatorname{bd} U) \xleftarrow{p} (p^{-1}(\operatorname{cl} U), p^{-1}(\operatorname{bd} U)) \xrightarrow{q} (V, V \setminus B^V(0,\rho))$$

is an admissible pair.

Definition 4.2. By the generalized degree of (p,q) given above on the set U in 0 we understand the element

$$\deg((p,q), U, 0) := \deg((\eta_T \circ p, \eta_V \circ q), \eta_T(U), 0).$$

Of course the degree (in particular, its "sign") defined above depends on the choice of orientations in T and V.

As one can see, the generalized degree deg defined above depends on dimensions m and n. There is also a possibility to define so-called stable degree Deg which depends only on the difference $m-n$ and belongs to the stable homotopy group of spheres Π_{m-n}, when one uses deg and respective suspension operators (see [22], [15]). We only mention that for $m < 2n-1$, deg and Deg are equal.

This generalized degree is a homotopy invariant with usual properties: existence, localization, additivity (see [22], [14] or [13]). We explain here only the homotopy property, since it is strictly connected with our considerations.

Theorem 4.3. *Let* $\operatorname{cl} U \times [0,1] \xleftarrow{R} \Gamma \xrightarrow{S} \mathbb{R}^n$ *be a homotopy between two admissible pairs* (p_0, q_0) *and* (p_1, q_1) *such that* $0 \notin S(R^{-1}(x,t))$ *for* $x \in \operatorname{bd} U$, $t \in [0,1]$. *Then* $\deg((p_0,q_0),U,0) = \deg((p_1,q_1),U,0)$.

If we consider the coincidence problem

$$L(x) \in q(p^{-1}(x)),$$

where $L\colon T \to V$ is a Fredholm operator, then it is of course equivalent to the following

$$0 \in L(x) - q(p^{-1}(x)).$$

Observe that the map on the right hand side is admissible and determined by a pair $(p, L \circ p - q)$. Therefore the following definition of the coincidence index is correct:

$$\operatorname{Ind}_L((p,q),U) := \operatorname{Deg}((p, L \circ p - q), U, 0).$$

Let now E, E' be infinite dimensional Banach spaces, $L\colon E \to E'$ be an oriented (see Definition 3.1) Fredholm operator of index $i(L) = k \geq 0$. Like earlier we denote by P, Q the respective projections, by K_P the map inverse to $L|_{\operatorname{Ker} P}$ and by η, η', η'_Q the isomorphisms representing orientations in $\operatorname{Ker} L$, $\operatorname{Coker} L$ and in $\operatorname{Im} Q$, respectively (η'_Q is determined by η', comp. Section 3).

Assume that V is a finite dimensional subspace of E' such that $V = W \oplus \operatorname{Im} Q$ and η'_W represents a fixed orientation in W. Then (η'_Q, η'_W) represents an orientation in V. In $L^{-1}(V) = K_P(W) \oplus \operatorname{Ker} L$ we choose the orientation given by $(\eta_{K_P(W)}, \eta)$, where $\eta_{K_P(W)} = \eta'_W \circ L|_{K_P(W)}$.

Remark 4.4. Observe that, if we take another projection $P_1\colon E \to E$ satisfying respective conditions, then the composition $K_P(W) \xrightarrow{L} W \xrightarrow{K_{P_1}} K_{P_1}(W)$ saves the orientation, i.e. $\eta_{K_{P_1}(W)} \circ K_{P_1} \circ L|_{K_P(W)} = \eta'_W \circ L|_{K_{P_1}(W)} \circ K_{P_1} \circ L|_{K_P(W)} = \eta'_W \circ L|_{K_P(W)} = \eta_{K_P(W)}$.

Assume that an admissible pair $\operatorname{cl} U \xleftarrow{p} \Gamma \xleftarrow{q} E'$ is compact and that the set of coincidence points $C = \{x \in \operatorname{cl} U \mid L(x) \in q(p^{-1}(x))\}$ is compact and contained in U (recall that U is an open bounded subset of E). Since $L|_{\operatorname{cl} U}$ is a proper map, $D := \{y \in E' \mid y \in L(x) - q(p^{-1}(x)),\ x \in \operatorname{bd} U\}$ is a closed set and $0 \notin D$. Therefore there exists $\varepsilon_0 > 0$ such that $D \cap B^{E'}(0, 2\varepsilon_0) = \emptyset$.

Take $\varepsilon \in (0, \varepsilon_0]$. Let $l_\varepsilon\colon \operatorname{cl} q(p^{-1}(U)) \to E'$ be a Schauder projection (comp. §II in [8]) of the compact set $\operatorname{cl} q(p^{-1}(U))$ on the finite dimensional subspace $Z \in E'$ such that $\|l_\varepsilon(y) - y\|_{E'} < \varepsilon$ for any $y \in \operatorname{cl} q(p^{-1}(U))$.

One can find a finite dimensional subspace W of $\operatorname{Im} L$ such that $Z \subset V := W \oplus \operatorname{Im} Q$ and $U_V := U \cap L^{-1}(V) \neq \emptyset$. Observe that the closure and the boundary of U_V in $L^{-1}(V)$ are contained in $\operatorname{cl} U \cap L^{-1}(V)$ and in $\operatorname{bd} U \cap L^{-1}(V)$, respectively.

Moreover, the pair (p_V, q_V), where $p_V = p|_{p^{-1}(\mathrm{cl}\, U_V)}: p^{-1}(\mathrm{cl}\, U_V) \to \mathrm{cl}\, U_V$, $q_V = l_\varepsilon \circ q|_{p^{-1}(\mathrm{cl}\, U_V)}: p^{-1}(\mathrm{cl}\, U_V) \to V$ is admissible and $L_V := L|_{L^{-1}(V)}: L^{-1}(V) \to V$ is a Fredholm operator with $i(L_V) = i(L) = k$.

Without loosing the generality one can assume that $\dim V := n \geq k+2$. Then, since $m := \dim L^{-1}(V) = n + k$, we get $m > n$ and $m < 2n - 1$. For such m and n the following definition is correct, i.e. does not depend on the choice of ε, l_ε, W, P and Q (see [22], [14]).

Definition 4.5. By the *generalized coincidence index* of the compact admissible pair (p, q) satisfying the above assumptions we understand the element

$$\mathrm{Ind}_L((p,q),U) := \deg((p_V, L_V \circ p_V - q_V), U_V, 0) \in \pi^n(S^m) \cong \Pi_k.$$

Remark 4.6. This generalized index has also usual properties like existence, additivity, localization and homotopy (see [22], [14]). But it of course depends on the (fixed) operator L and the orientation of the space $\mathrm{Ker}\, L \times \mathrm{Coker}\, L$. The homotopy property is the following:

- Let $\mathrm{cl}\, U \times [0,1] \xleftarrow{R} \Gamma \xrightarrow{S} E'$ be the compact homotopy between two admissible pairs (p_0, q_0) and (p_1, q_1) such that $L(x) \notin S(R^{-1}(x,t))$ for $x \in \mathrm{bd}\, U$, $t \in [0,1]$. Then $\mathrm{Ind}_L((p_0,q_0),U) = \mathrm{Ind}_L((p_1,q_1),U)$.

This property follows from the respective one for the finite dimensional case if one uses reduction similar to that in the very definition.

As we have mentioned earlier, in this property the homotopy concerns only admissible pairs, while the Fredholm operator is not being changed.

4.2. The general homotopy property. Let us introduce at the beginning the homotopy which will be considered and its simple properties. Like earlier E, E' are real Banach spaces and $L \in \Phi_n(E, E')$.

Definition 4.7. Let L_0, $L_1 \in \Phi_n(E,E')$ and ϕ_0, ϕ_1 be admissible maps. We say that pairs (L_0, ϕ_0) and (L_1, ϕ_1) are homotopic (and write $(L_0, \phi_0) \sim (L_1, \phi_1)$), if there is a pair of maps (H_L, H_ϕ) such that $H_L: [0,1] \to \Phi_n(E,E')$ is continuous and H_ϕ is a homotopy of admissible maps in the sense of Definition 2.5.

Remark 4.8. The map $H: E \times [0,1] \to E'$ defined by $H(x,t) = H_L(t)(x)$ (H_L is as in the above definition) is a homotopy between two single-valued continuous maps L_0 and L_1. We will often use the map H denoting it also by H_L.

Conversely, if such homotopy H is given, and one knows that it is uniformly continuous on $\mathrm{cl}\, B^E(0,1)$ and for any $t \in [0,1]$, $H(\,\cdot\,,t) \in \Phi_n(E,E')$, then the map $[0,1] \ni t \mapsto H(\,\cdot\,,t) \in \Phi_n(E,E')$ is continuous. Observe that, if e.g. $\dim E$, $\dim E' < \infty$, then a linear homotopy between two Fredholm operators satisfies this conditions.

Proposition 4.9. *If (H_L, H_ϕ) is a homotopy between (L_0, ϕ_0) and (L_1, ϕ_1), where ϕ_i is determined by (p_i, q_i), then there is a homotopy $\widetilde{H_\phi}$ between admissible pairs $(p_0, L_0 \circ p_0 - q_0)$ and $(p_1, L_1 \circ p_1 - q_1)$. Moreover, if H_L and H_ϕ have not a coincidence point in some set $W \subset E$ (i.e. for any $t \in [0,1]$ and $x \in W$, $H_L(t)(x) \notin H_\phi(x,t)$), then $0 \notin \widetilde{H_\phi}(x,t)$ for any $x \in W$, $t \in [0,1]$.*

Proof. If (R, S) determines H_ϕ, then $\widetilde{H_\phi}$ is determined by $(R, \widetilde{S})$, where $\widetilde{S}\colon \Gamma \to E'$ and $\widetilde{S}(\gamma) = H_L(R(\gamma)) - S(\gamma)$.

Assume now that $0 \in \widetilde{S}(R^{-1}(x,t))$. It means that

$$0 \in (H_L \circ R - S)(\{\gamma \in \Gamma \mid R(\gamma) = (x,t)\})$$

and, consequently, that

$$0 \in \{H_L \circ R(\gamma) - S(\gamma) \mid \gamma \in \Gamma, R(\gamma) = (x,t)\}.$$

Then $0 \in \{H_L(x,t) - S(R^{-1}(x,t)) \mid (x,t) \in X \times [0,1]\}$ and hence $H_L(x,t) \in S(R^{-1}(x,t))$, what implies that $x \notin W$. □

Now we want to consider a homotopy from Definition 4.7 and prove the general homotopy property. The main idea of reducing the problem to a finite dimensional situation is in fact the same, but tools are quite different then earlier. They are partially taken from [4], but here we use them for another purpose.

We start from preparing some additional maps. Let $H_L\colon [0,1] \to \Phi_r(E, E')$ be an oriented homotopy. For simplicity we denote $H_L(t)$ by L_t. Take any $\lambda \in [0,1]$ and a positive corrector A of L_λ. There is $\delta_1 > 0$ such that for any $t \in (\lambda - \delta_1, \lambda + \delta_1) \cap [0,1]$, A is a positive corrector of L_t (see Proposition 3.11 and Remark 3.14). Moreover, $\operatorname{Im} A \subset E' \times \mathbb{R}^r$ and $\dim \operatorname{Im} A < \infty$. Therefore there exists a closed subspace E_0' of E' such that $(E_0' \times \{0\}) \oplus \operatorname{Im} A = E' \times \mathbb{R}^r$ (see Remark 2.1). Of course also

$$\operatorname{Im} z(L_t) + \operatorname{Im} A = E' \times \mathbb{R}^r \quad \text{for } t \in (\lambda - \delta_1, \lambda + \delta_1) \cap [0,1].$$

Take $G\colon E' \to E'$ being a linear projection onto E_0' with $G(y) = 0$ for any $(y, 0) \in \operatorname{Im} A$. Then $\operatorname{Ker} G \times \mathbb{R}^r = \operatorname{Im} A$ and G is a Fredholm operator of index $i(G) = 0$. Define the map $\overline{G}\colon E' \times \mathbb{R}^r \to E' \times \mathbb{R}^r$ by $\overline{G}(x,s) = (G(x), 0)$. Consider the composition $G \circ L_t$, for any $t \in (\lambda - \delta_1, \lambda + \delta_1) \cap [0,1]$. Observe that $\operatorname{Im} \overline{G} \circ (z(L_t) + A) = E' \times \{0\}$, since $z(L_t) + A$ is an isomorphism, and that for $x \in E$

$$\overline{G} \circ (z(L_t) + A)(x) = \overline{G} \circ z(L_t)(x) + \overline{G} \circ A(x) = (G \circ L_t(x), 0).$$

Therefore $\operatorname{Im} G \circ L_t = E_0'$. Moreover,

$$\begin{aligned} \dim \operatorname{Ker} (G \circ L_t) &= i(G \circ L_t) + \dim \operatorname{Coker} (G \circ L_t) \\ &= i(G) + i(L_t) + \dim \operatorname{Im} A - r = 0 + r + \dim \operatorname{Im} A - r = \dim \operatorname{Im} A. \end{aligned}$$

Let $E_0 = (z(L_\lambda) + A)^{-1}(E'_0 \times \{0\})$. Observe that

$$E_0 \oplus \operatorname{Ker} G \circ L_\lambda = E, \tag{4.1}$$

and $G \circ L_\lambda|_{E_0}\colon E_0 \to E'_0$ is a linear isomorphism. Therefore, since $\operatorname{Iso}(E_0, E'_0)$ is an open subset of $\Phi(E_0, E'_0)$, there exists $\delta_2 > 0$ such that $\delta_2 \le \delta_1$ and for any $t \in (\lambda - \delta_2, \lambda + \delta_2) \cap [0,1]$ the map $G \circ L_t|_{E_0}\colon E_0 \to E'_0$ is also an isomorphism.

Consider a family of maps (comp. [4]) $B_t\colon E \to E$ for $t \in (\lambda-\delta_2, \lambda+\delta_2) \cap [0,1]$ defined by

$$B_\lambda = \mathrm{id}_E,$$
$$B_t(x) = x - (G \circ L_t|_{E_0})^{-1} \circ (G \circ L_t)(x_1),$$

where $x = x_0 + x_1$ and $x_0 \in E_0$, $x_1 \in \operatorname{Ker} G \circ L_\lambda$ (see (4.1) above). Observe that $B_t(\operatorname{Ker} G \circ L_\lambda) \subset \operatorname{Ker} G \circ L_t$, since

$$\begin{aligned}(G \circ L_t)(B_t(x_1)) &= (G \circ L_t)(x_1 - (G \circ L_t|_{E_0})^{-1} \circ (G \circ L_t)(x_1)) \\ &= G \circ L_t(x_1) - G \circ L_t(x_1) = 0.\end{aligned}$$

Moreover, $\operatorname{Ker} G \circ L_t \subset B_t(\operatorname{Ker}(G \circ L_\lambda))$. Indeed, if $x = x_0 + x_1 \in \operatorname{Ker} G \circ L_t$, where, like earlier, $x_0 \in E_0$, $x_1 \in \operatorname{Ker} G \circ L_\lambda$, then $G \circ L_t(x_0 + x_1) = 0$, i.e. $G \circ L_t(x_0) = -G \circ L_t(x_1)$. Then $B_t(x_1) = x_1 - (G \circ L_t|_{E_0})^{-1} \circ G \circ L_t(x_1) = x_1 - (G \circ L_t|_{E_0})^{-1} \circ (-G \circ L_t(x_0)) = x_1 + x_0 = x$, what means that $x = B_t(x_1)$. Hence

$$B_t(\operatorname{Ker} G \circ L_\lambda) = \operatorname{Ker} G \circ L_t.$$

Lemma 4.10. *B_t defined above is an isomorphism for any $t \in (\lambda - \delta_2, \lambda + \delta_2) \cap [0,1]$.*

Proof. If $B_t(x_0+x_1) = 0$, then $x_0+x_1-(G\circ L_t|_{E_0})^{-1}\circ(G\circ L_t)(x_1) = 0$. But, since $(G\circ L_t|_{E_0})^{-1}\circ(G\circ L_t)(x_1) \in E_0$, we get $x_0-(G\circ L_t|_{E_0})^{-1}\circ(G\circ L_t)(x_1) = 0$ and $x_1 = 0$, what implies $(G\circ L_t|_{E_0})^{-1}\circ(G\circ L_t)(x_1) = 0$ and consequently $x_0 = 0$, what proves that B_t is a monomorphism.

Take any $v \in E$, $v = v_0 + v_1$, where $v_0 \in E_0$ and $v_1 \in \operatorname{Ker} G \circ L_\lambda$. Let $y_0 = G \circ L_t(v_0)$ and $y_1 = G \circ L_t(v_1)$. Of course both y_0 and y_1 belong to E'_0. For $x_0 = (G \circ L_t|_{E_0})^{-1}(y_0 + y_1) \in E_0$ and $x_1 = v_1$,

$$\begin{aligned}B_t(x_0 + x_1) &= x_0 + x_1 - (G \circ L_t|_{E_0})^{-1} \circ (G \circ L_t)(x_1) \\ &= (G \circ L_t|_{E_0})^{-1}(y_0 + y_1) + v_1 - (G \circ L_t|_{E_0})^{-1} \circ (G \circ L_t)(v_1) \\ &= (G \circ L_t|_{E_0})^{-1}(y_0) + (G \circ L_t|_{E_0})^{-1}(y_1) + v_1 - (G \circ L_t|_{E_0})^{-1}(y_1) \\ &= v_0 + v_1 = v,\end{aligned}$$

what implies that B_t is an epimorphism and ends the proof. □

Lemma 4.11. *The map* $(\lambda - \delta_2, \lambda + \delta_2) \cap [0,1] \ni t \mapsto B_t \in \Phi(E,E)$ *is continuous.*

Proof. Take any

$$t_0 \in (\lambda - \delta_2, \lambda + \delta_2) \cap [0,1] \quad \text{and} \quad \varepsilon \in (0, \|L_{t_0}\| \cdot \|(G \circ L_{t_0}|_{E_0})^{-1}\|).$$

Since H_L is continuous, there is $\delta > 0$ such that,

$$\|L_t - L_{t_0}\| < \varepsilon/(4 \cdot \|(G \circ L_{t_0}|_{E_0})^{-1}\|^2 \cdot \|L_{t_0}\|)$$

for $t \in (t_0 - \delta, t_0 + \delta) \subset (\lambda - \delta_2, \lambda + \delta_2) \cap [0,1]$.

Observe that, for any $x = x_0 + x_1$, where $x_0 \in E_0$ and $x_1 \in \operatorname{Ker} G \circ L_\lambda$,

$$\begin{aligned}
\|B_{t_0}(x) \ &- B_t(x)\| \\
&= \|x - (G \circ L_{t_0}|_{E_0})^{-1} \circ (G \circ L_{t_0}(x_1) - x + (G \circ L_t|_{E_0})^{-1} \circ (G \circ L_t)(x_1)\| \\
&= \|(G \circ L_{t_0}|_{E_0})^{-1} \circ (G \circ L_{t_0})(x_1) - (G \circ L_t|_{E_0})^{-1} \circ (G \circ L_{t_0})(x_1) \\
&\quad + (G \circ L_t|_{E_0})^{-1} \circ (G \circ L_{t_0})(x_1) - (G \circ L_t|_{E_0})^{-1} \circ (G \circ L_t)(x_1)\| \\
&\le \|(G \circ L_{t_0}|_{E_0})^{-1} - (G \circ L_t|_{E_0})^{-1}\| \cdot \|G\| \cdot \|L_{t_0}(x_1)\| + \\
&\quad + \|(G \circ L_t|_{E_0})^{-1}\| \cdot \|G\| \cdot \|L_{t_0}(x_1) - L_t(x_1)\| \\
&= \|(G \circ L_{t_0}|_{E_0})^{-1} - (G \circ L_t|_{E_0})^{-1}\| \cdot \|L_{t_0}(x_1)\| \\
&\quad + \|(G \circ L_t|_{E_0})^{-1}\| \cdot \|L_{t_0}(x_1) - L_t(x_1)\|.
\end{aligned}$$

Hence

$$\|B_{t_0} - B_t\| \le \|(G \circ L_{t_0}|_{E_0})^{-1} - (G \circ L_t|_{E_0})^{-1}\| \cdot \|L_{t_0}\| + \|(G \circ L_t|_{E_0})^{-1}\| \cdot \|L_{t_0} - L_t\|.$$

But since

$$\|(G \circ L_{t_0}|_{E_0}) - (G \circ L_t|_{E_0})\| \le \|L_{t_0} - L_t\|,$$

i.e.

$$\|(G \circ L_{t_0}|_{E_0}) - (G \circ L_t|_{E_0})\| \le \frac{\varepsilon'}{\|(G \circ L_{t_0}|_{E_0})^{-1}\|},$$

where $\varepsilon' = \varepsilon/(4\|L_{t_0}\| \cdot \|(G \circ L_{t_0}|_{E_0})^{-1}\|)$ and $\varepsilon' \le 1/4$, Lemma 2.2 implies

$$\begin{aligned}
\|(G \ &\circ L_{t_0}|_{E_0})^{-1} - (G \circ L_t|_{E_0})^{-1}\| \cdot \|L_{t_0}\| + \|(G \circ L_t|_{E_0})^{-1}\| \cdot \|L_{t_0} - L_t\| \\
&\le 2\varepsilon' \cdot \|(G \circ L_{t_0}|_{E_0})^{-1}\| \cdot \|L_{t_0}\| \\
&\quad + (1 + 2\varepsilon') \cdot \|(G \circ L_{t_0}|_{E_0})^{-1}\| \cdot \frac{\varepsilon}{4 \cdot \|(G \circ L_{t_0}|_{E_0})^{-1}\|^2 \cdot \|L_{t_0}\|} \\
&\le \frac{\varepsilon}{2} + 2 \cdot \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}$$

□

Let $V = W \oplus \operatorname{Im} Q_\lambda$ be a finite dimensional subspace of E' (for example containing the image of the Schauder projection of some compact set $\operatorname{cl} \phi(U)$ for compact map ϕ), where Q_λ is a respective projection for L_λ. Without loosing the generality we assume that $W \subset \operatorname{Im} L_\lambda$ (see Remark 2.1). There exists a positive corrector A of L_λ such that $\operatorname{Im} A = V \times \mathbb{R}^r$ (see Proposition 3.9). Let E_0', E_0, G,

B_t be defined like above for this A. Observe that then $L_t^{-1}(V) = \operatorname{Ker} G \circ L_t = (z(L_t)+A)^{-1}(\operatorname{Im} A)$ for any $t \in (\lambda-\delta_2, \lambda+\delta_2) \cap [0,1]$.

If we fix the orientation $[\xi]$ in V, then we automatically get orientations $[(\xi, I_{\mathbb{R}^n})]$ in $\operatorname{Im} A$ and $[(\xi, I_{\mathbb{R}^n}) \circ (z(L_t)+A)|_{\operatorname{Ker}(G\circ L_t)}]$ in $\operatorname{Ker} G \circ L_t$ for $t \in (\lambda-\delta_2, \lambda+\delta_2) \cap [0,1]$.

Lemma 4.12. *There is $\delta_3 > 0$, such that for $t \in (\lambda-\delta_3, \lambda+\delta_3) \cap [0,1]$ the map $B_t|_{\operatorname{Ker} G\circ L_\lambda} : \operatorname{Ker} G\circ L_\lambda \to \operatorname{Ker} G \circ L_t$ saves the orientations described above.*

Proof. Observe that $(L_t \circ B_t(\operatorname{Ker} G\circ L_\lambda)) \times \{0\} \subset \operatorname{Im} A$ for $t \in (\lambda-\delta_2, \lambda+\delta_2) \cap [0,1]$. Indeed, if $x_1 \in \operatorname{Ker} G \circ L_\lambda$ and $L_t(x_1) = y_0 + y_1$, where $y_0 \in E_0'$ and $(y_1, 0) \in \operatorname{Im} A$, then $G \circ L_t(x_1) = y_0$ and, consequently, $(G \circ L_t|_{E_0})^{-1}(y_0) = \overline{x} \in E_0$, what means that $L_t(\overline{x}) = y_0 + \overline{y_1}$ with $(\overline{y_1}, 0) \in \operatorname{Im} A$. Therefore

$$\begin{aligned} L_t \circ B_t(x_1) &= L_t(x_1 - (G\circ L_t|_{E_0})^{-1} \circ G \circ L_t(x_1)) \\ &= L_t(x_1) - L_t(\overline{x}) = y_0 + y_1 - y_0 - \overline{y_1} = y_1 - \overline{y_1}, \end{aligned}$$

and $(y_1 - \overline{y_1}, 0) \in \operatorname{Im} A$.

Since

$$\det((\xi, I_{\mathbb{R}^r}) \circ (z(L_\lambda)+A)|_{\operatorname{Ker} G\circ L_\lambda} \circ [(\xi, I_{\mathbb{R}^r}) \circ (z(L_\lambda)+A)|_{\operatorname{Ker}(G\circ L_\lambda)}]^{-1}) > 0,$$

and the following maps $t \mapsto L_t$, $t \mapsto B_t$, det are continuous, there is $\delta_3 \in (0, \delta_2)$ such that for $t \in (\lambda-\delta_3, \lambda+\delta_3) \cap [0,1]$, A is also a corrector of $L_t \circ B_t$ ([5]) and

$$\det((\xi, I_{\mathbb{R}^r}) \circ (z(L_t\circ B_t)+A)|_{\operatorname{Ker} G\circ L_\lambda} \circ [(\xi, I_{\mathbb{R}^r}) \circ (z(L_\lambda)+A)|_{\operatorname{Ker}(G\circ L_\lambda)}]^{-1}) > 0.$$

But observe that $A \circ B_t|_{\operatorname{Ker} G\circ L_\lambda}(x) = A(x - (G\circ L_t|_{E_0})^{-1} \circ G \circ L_t(x)) = A(x)$, because $(G\circ L_t|_{E_0})^{-1} \circ G \circ L_t(x) \in E_0$. Then $(z(L_t \circ B_t)+A)|_{\operatorname{Ker} G\circ L_\lambda} = (z(L_t \circ B_t) + A \circ B_t)|_{\operatorname{Ker} G\circ L_\lambda} = (z(L_t + A) \circ B_t)|_{\operatorname{Ker} G\circ L_\lambda}$ and hence

$$\begin{aligned} 0 &< \det((\xi, I_{\mathbb{R}^r}) \circ (z(L_t\circ B_t)+A)|_{\operatorname{Ker} G\circ L_\lambda} \circ [(\xi, I_{\mathbb{R}^r}) \\ &\quad \circ (z(L_\lambda)+A)|_{\operatorname{Ker}(G\circ L_\lambda)}]^{-1}) \\ &= \det((\xi, I_{\mathbb{R}^r}) \circ (z(L_t)+A) \circ B_t|_{\operatorname{Ker} G\circ L_\lambda}) \circ [(\xi, I_{\mathbb{R}^r}) \\ &\quad \circ (z(L_\lambda)+A)|_{\operatorname{Ker}(G\circ L_\lambda)}]^{-1}) \\ &= \det((\xi, I_{\mathbb{R}^r}) \circ (z(L_t)+A)|_{\operatorname{Ker} G\circ L_t} \circ [(\xi, I_{\mathbb{R}^r}) \circ (z(L_t)+A)|_{\operatorname{Ker}(G\circ L_t)}]^{-1} \\ &\quad \circ [(\xi, I_{\mathbb{R}^r}) \circ (z(L_t)+A)|_{\operatorname{Ker}(G\circ L_t)}] \\ &\quad \circ B_t|_{\operatorname{Ker} G\circ L_\lambda} \circ [(\xi, I_{\mathbb{R}^r}) \circ (z(L_\lambda)+A)|_{\operatorname{Ker}(G\circ L_\lambda)}]^{-1}) \\ &= \det((\xi, I_{\mathbb{R}^r}) \circ (z(L_t)+A)|_{\operatorname{Ker} G\circ L_t} \circ [(\xi, I_{\mathbb{R}^r}) \circ (z(L_t)+A)|_{\operatorname{Ker}(G\circ L_t)}]^{-1}) \\ &\quad \cdot \det([(\xi, I_{\mathbb{R}^r}) \circ (z(L_t)+A)|_{\operatorname{Ker}(G\circ L_t)}] \\ &\quad \circ B_t|_{\operatorname{Ker} G\circ L_\lambda} \circ [(\xi, I_{\mathbb{R}^r}) \circ (z(L_\lambda)+A)|_{\operatorname{Ker}(G\circ L_\lambda)}]^{-1}), \end{aligned}$$

([5]) see Proposition 3.11, $L_t \circ B_t$ is sufficiently close to L_λ

that implies

$$\det([(\xi, I_{\mathbb{R}^r}) \circ (z(L_t) + A)|_{\operatorname{Ker}(G\circ L_t)}] \circ B_t|_{\operatorname{Ker} G\circ L_\lambda} \circ [(\xi, I_{\mathbb{R}^r}) \circ (z(L_\lambda) + A)|_{\operatorname{Ker}(G\circ L_\lambda)}]^{-1}) > 0,$$

i.e. B_t saves the respective orientations. □

Now assume additionally that $t \in (\lambda - \delta_3, \lambda + \delta_3) \cap [0,1]$ and U is an open subset of E.

Lemma 4.13. *The set* $\widetilde{U} := \{(x,t) \mid x \in U \cap (L_t \circ B_t)^{-1}(V)\}$ *is open in* $\operatorname{Ker}(G \circ L_\lambda) \times ((\lambda - \delta_3, \lambda + \delta_3) \cap [0,1])$.

Proof. Take any $(x,t) \in \widetilde{U}$. It means that $x \in U$ and $L_t \circ B_t(x) \in W$. Let $\varepsilon_1 > 0$ be such that $B^E(x, \varepsilon_1) \in U$ and $\varepsilon_2 > 0$ such that $(t - \varepsilon_2, t + \varepsilon_2) \subset (\lambda - \delta_3, \lambda + \delta_3) \cap [0,1]$. We shall prove that $B^{E\times((\lambda-\delta_3,\lambda+\delta_3)\cap[0,1])}((x,t),\varepsilon) \cap (\operatorname{Ker} G \circ L_\lambda \times ((\lambda - \delta_3, \lambda + \delta_3) \cap [0,1])) \subset \widetilde{U}$ for $\varepsilon := \min(\varepsilon_1, \varepsilon_2)$.

If $(x', t') \in B^{E\times((\lambda-\delta_3,\lambda+\delta_3)\cap[0,1])}((x,t),\varepsilon) \cap \operatorname{Ker} G\circ L_\lambda \times ((\lambda-\delta_3, \lambda+\delta_3)\cap[0,1])$, then $x' \in B^{E\times((\lambda-\delta_3,\lambda+\delta_3)\cap[0,1])}((x,t),\varepsilon)$ and $x' \in \operatorname{Ker} G \circ L_\lambda$ and $t' \in (t - \varepsilon, t + \varepsilon) \cap ((\lambda - \delta_3, \lambda + \delta_3) \cap [0,1])$. But condition $x' \in \operatorname{Ker} G \circ L_\lambda$ implies that $B_{t'}(x') \in \operatorname{Ker} G \circ L_{t'}$, what means that $G \circ L_{t'} \circ B_{t'}(x') = 0$, and then $L_{t'} \circ B_{t'}(x') \in \operatorname{Ker} G = V$. We have just proved that $x' \in U \cap (L_{t'} \circ B_{t'})^{-1}(V)$ and, consequently, that $(x', t') \in \widetilde{U}$. □

Below we give two examples illustrating notions described above.

Example 4.14. As in Example 3.12, let $E = \mathbb{R}^3$, $E' = \mathbb{R}^2$ and $L_t(x,y,z) = (x, ty)$. Like earlier $A\colon \mathbb{R}^3 \to \mathbb{R}^2 \times \mathbb{R}$ given by $A(x,y,z) = ((0,y),z)$ is a positive corrector of L_0 (and consequently for L_t if $t \in [-1/2, 1/2]$). Observe that $\operatorname{Im} A = \operatorname{Lin}(((0,1),0),((0,0),1))$, so we define $E'_0 = \operatorname{Lin}((1,0)) \subset \mathbb{R}^2$ and $G\colon \mathbb{R}^2 \to \mathbb{R}^2$, $G(u,v) = (u,0)$. Consequently, $\operatorname{Ker} G \circ L_0 = \operatorname{Lin}((0,1,0),(0,0,1)) = \operatorname{Ker} G \circ L_t$ and $E_0 = \operatorname{Lin}((1,0,0))$. Then

$$\begin{aligned} B_t(x,y,z) &= (x,y,z) - (G \circ L_t|E_0)^{-1} \circ G \circ L_t(0,y,z) \\ &= (x,y,z) - (G \circ L_t|E_0)^{-1}(0,0) = (x,y,z) - (0,0,0) = (x,y,z), \end{aligned}$$

i.e. $B_t = I_{\mathbb{R}^3}$.

In $\operatorname{Im} A$ the orientation is represented by $[((0,1),0),((0,0),1)]$ and then in $\operatorname{Ker} G \circ L_0 = \operatorname{Ker} L_0$ we get the orientation $[(0,1,0),(0,0,1)]$ (since A is a canonical corrector of L_0), while in $\operatorname{Ker} G \circ L_t$ the induced orientation is represented by $[(0, 1/(1+t), 0), (0,0,1)]$, because $(z(L_t) + A)((0, 1/(1+t), 0) = ((0,1),0)$ and $(z(L_t) + A)((0,0,1)) = ((0,0),1)$. The determinant of B_t in these orientations is positive. Indeed,

$$\det B_t = \begin{vmatrix} 1+t & 0 \\ 0 & 1 \end{vmatrix} = 1 + t > 0.$$

Example 4.15. Let $E = E' = l^2$, $t \in [-1/2, 1/2]$ and $L_t((x_1, x_2, \dots)) = (((1-t)(x_1+x_2), t^2(x_3-x_1), x_4 - tx_2, \dots, x_{n+1} - x_{n-1}, \dots))$. Observe that $\operatorname{Ker} L_0 = \operatorname{Lin}((1,-1,0,0,\dots),(0,0,1,0,\dots))$, $\operatorname{Coker} L_0 \equiv \operatorname{Lin}((0,1,0,0,\dots))$ and for $t \neq 0$, $\operatorname{Ker} L_t = \operatorname{Lin}((1,-1,1,-t,t,-t^2,t^2,-t^3,\dots))$, $\operatorname{Coker} L_t \equiv 0$. Then $i(L_t) = 1$. For the finite dimensional subspace $V = \operatorname{Lin}((1,0,0,\dots),(0,1,0,\dots),(0,0,1,0,\dots))$, the positive corrector $A\colon l^2 \to l^2 \times \mathbb{R}$ of L_0 such that $\operatorname{Im} A = V \times \mathbb{R}$ can be given by $A((x_1,x_2,\dots)) = ((x_2,x_3,x_4,0,\dots),-x_1)$ (the orientation in $\operatorname{Ker} L_0$ and $\operatorname{Coker} L_0$ is determined by specified elements in given order).

We get $E'_0 = \operatorname{Lin}((0,0,0,1,0,\dots),(0,0,0,0,1,0,\dots),\dots)$ and the projection $G\colon l^2 \to E'_0$, i.e. $G((y_1,y_2,\dots)) = (0,0,0,y_4,y_5,\dots)$. Then

$$\begin{aligned} G \circ L_0((x_1,x_2,\dots)) &= (0,0,0,x_5,x_6,\dots),\\ G \circ L_t((x_1,x_2,\dots)) &= (0,0,0,x_5 - tx_3, x_6 - tx_4,\dots) \end{aligned}$$

and, consequently,

$$\begin{aligned} \operatorname{Ker} G \circ L_0 &= \operatorname{Lin}((1,0,0,\dots),(0,1,0,\dots),(0,0,1,0,\dots),(0,0,0,1,0,\dots)),\\ E_0 &= \operatorname{Lin}((0,0,0,0,1,0,\dots),(0,0,0,0,0,1,0,\dots),\dots),\\ \operatorname{Ker} G \circ L_t &= \operatorname{Lin}((1,0,0,\dots),(0,1,0,\dots),(0,0,1,0,t,0,t^2,0,t^3,\dots),\\ &\qquad (0,0,0,1,0,t,0,t^2,\dots)). \end{aligned}$$

Observe that $\dim \operatorname{Ker} G \circ L_0 = \dim \operatorname{Ker} G \circ L_t = 4$. We fix the orientation in $\operatorname{Im} A$:

$$[((1,0,0,\dots),0),((0,1,0,\dots),0),((0,0,1,0,\dots),0),((0,0,\dots),1)]$$

and carry it back to $\operatorname{Ker} G \circ L_t$ by $(z(L_t)+A)^{-1}$. Namely we get the following orientations:

$$\left[\left(0,\frac{1}{2},0,\dots\right),(0,0,1,0,\dots),\left(0,0,0,\frac{1}{2},0,\dots\right),\left(-1,\frac{1}{2},0,\dots\right)\right]$$

in $\operatorname{Ker} G \circ L_0$ and

$$\begin{aligned} &\bigg[\left(0,\frac{1}{2-t},0,\frac{t}{4-2t},0,\frac{t^2}{4-2t},\dots\right),\left(0,0,\frac{1}{1+t^2},0,\frac{t}{1+t^2},\dots\right),\\ &\left(0,0,0,\frac{1}{2},0,\frac{t}{2},\dots\right),\left(-1,\frac{1-t}{2-t},\frac{-t^2}{1+t^2},\frac{t(1-t)}{4-2t},\frac{-t^3}{1+t^2},\frac{t^2(1-t)}{4-2t},\dots\right)\bigg] \end{aligned}$$

in $\operatorname{Ker} G \circ L_t$ for $t \neq 0$.

The maps $B_t\colon l^2 \to l^2$ are given by

$$B_t((x_1,x_2,\dots)) = (x_1,x_2,x_3,x_4,x_5 - tx_3, x_6 - tx_4, x_7 - t^2x_3, x_8 - t^2x_4,\dots).$$

For any $t \in [-1/2,1/2]$, $B_t|_{\operatorname{Ker} G\circ L_0}\colon \operatorname{Ker} G \circ L_0 \to \operatorname{Ker} G \circ L_t$ saves the orientation described above.

We are going to prove that if (H_L, H_ϕ) is the homotopy between $(L_0,(p_0,q_0))$ and $(L_1,(p_1,q_1))$ in the sense of Definition 4.7, such that H_L is orientable and H_ϕ is compact, then

$$\mathrm{Ind}_{L_0}((p_0,q_0),U) = \mathrm{Ind}_{L_1}((p_1,q_1),U).$$

In a finite dimensional situation it is an obvious consequence of the definition and some properties (see Proposition 4.9, Theorem 4.3 and remarks after). But we have to generalize earlier considerations before applying in the infinite dimensional case. Namely we need an open set $\mathcal{U} \subset E\times[0,1]$ instead of $U\times[0,1]$ as a domain of the homotopy.

Lemma 4.16. *Let $\mathcal{U}$ be an open subset of $E\times[0,1]$ such that $U_\alpha := \{x \in E \mid (x,\alpha)\in\mathcal{U}\} \neq \emptyset$ for any $\alpha\in[0,1]$ and Z be a compact subset of $\mathcal{U}$. Denote: $Z_\alpha := \{x\in E \mid (x,\alpha)\in Z\}$, $\widetilde{U} := \bigcup_{\alpha\in[0,1]} U_\alpha$. For any $\alpha\in[0,1]$ there are $\delta>0$, $\rho>0$, such that*

$$(\widetilde{U}\times((\alpha-\delta,\alpha+\delta)\cap[0,1]))\cap \mathrm{cl}\,\mathcal{O}_\delta^{E\times[0,1]}(Z) \subset \mathrm{cl}\,\mathcal{O}_\rho^{E}(Z_\alpha)\times((\alpha-\delta,\alpha+\delta)\cap[0,1]) \subset \mathcal{U}.$$

Proof. Let $\rho = 1/2\mathrm{dist}\,(Z,\mathrm{bd}\,\mathcal{U}) > 0$ (6). Then it is easy to see that for any $\delta\le\rho/2$, the last inclusion is true, i.e. $\mathcal{O}_\rho^E(Z_\alpha)\times((\alpha-\delta,\alpha+\delta)\cap[0,1])\subset\mathcal{U}$.

We prove that there is $\delta\le\rho/2$ such that $\bigcup_{\beta\in(\alpha-\delta,\alpha+\delta)\cap[0,1]} Z_\beta\times\{\beta\} \subset \mathcal{O}_{\rho/2}(Z_\alpha\times\{\alpha\})$. Suppose for a moment that it is not true. Then, for any $n\in\mathbb{N}$, there is $(x_n,t_n)\in\bigcup_{\beta\in(\alpha-1/n,\alpha+1/n)\cap[0,1]} Z_\beta\times\{\beta\}$ and $(x_n,t_n)\notin\mathcal{O}_{\rho/2}(Z_\alpha\times\{\alpha\})$. It means that $x_n\in Z_{t_n}$ and, since $t_n\in(\alpha-1/n,\alpha+1/n)\cap[0,1]$, $\lim_{n\to\infty} t_n=\alpha$. But Z is a compact set, so (passing to the subsequence, if necessary) we get $\lim_{n\to\infty}(x_n,t_n) = (x_0,\alpha)\in Z$, and, consequently, $x_0\in Z_\alpha$. Therefore, there is $n_0\in\mathbb{N}$, such that $(x_n,t_n)\in\mathcal{O}_{\rho/2}(Z_\alpha\times\{\alpha\})$ for $n>n_0$, and this contradiction proves the hypothesis.

Let now $(x,t)\in(\widetilde{U}\times((\alpha-\delta,\alpha+\delta)\cap[0,1]))\cap\mathrm{cl}\,\mathcal{O}_\delta(Z)$. Then $t\in(\alpha-\delta,\alpha+\delta)$ and there is $(x_z,t_z)\in Z$ such that $\|(x,t)-(x_z,t_z)\|\le\delta$. But $(x_z,t_z)\in\bigcup_{\beta\in(\alpha-\delta,\alpha+\delta)\cap[0,1]} Z_\beta\times\{\beta\}$. Therefore $(x_z,t_z)\in\mathcal{O}_{\rho/2}(Z_\alpha\times\{\alpha\})$, i.e. there is $x_\alpha\in Z_\alpha$ such that $\|(x_\alpha,\alpha)-(x_z,t_z)\|<\rho/2$. Hence $\|(x,t)-(x_\alpha,\alpha)\|<\delta+\rho/2\le\rho$, what implies that $(x,t)\in\mathrm{cl}\,\mathcal{O}_\rho^E(Z_\alpha)\times((\alpha-\delta,\alpha+\delta)\cap[0,1])$ and the proof is complete. □

Observe that the family of sets $\{O_\alpha := \mathcal{O}_\rho^E(Z_\alpha)\times((\alpha-\delta_\alpha,\alpha+\delta_\alpha)\cap[0,1])\}_{\alpha\in[0,1]}$ composes an open covering of a compact set Z, so one can choose a finite sequence $(\alpha_1,\dots\alpha_k)$ such that $Z\subset\bigcup_{i\in\{1,\dots,k\}} O_{\alpha_i}$. Without loosing the generality we assume that $[0,1]\subset\bigcup_{i\in\{1,\dots,k\}}(\alpha_i-\delta_{\alpha_i},\alpha_i+\delta_{\alpha_i})$ (since the respective construction is valid also for $Z_\alpha=\emptyset$).

(6) Recall that for a compact set Z and a closed set A, $\mathrm{dist}\,(Z,A) := \sup_{z\in Z}\inf_{v\in A}\|z-v\|$.

Consider a compact admissible pair of maps $\mathrm{cl}\,\mathcal{U} \xleftarrow{R} \Gamma \xrightarrow{S} \mathbb{R}^n$, where $\mathcal{U}$ is an open subset of $\mathbb{R}^m \times [0,1]$ and assume that $Z = \{x \in \mathrm{cl}\,\mathcal{U} \mid 0 \in S(R^{-1}(x))\}$ is a compact set contained in $\mathcal{U}$.

Let $p_\alpha\colon R^{-1}(U_\alpha \times \{\alpha\}) \to U_\alpha$ and $q_\alpha\colon R^{-1}(U_\alpha \times \{\alpha\}) \to \mathbb{R}^n$ be given by $(p_\alpha(\gamma), \alpha) = R(\gamma)$ and $q_\alpha(\gamma) = S(\gamma)$, where U_α is as in Lemma 4.16.

Lemma 4.17. *Under the above assumptions and notation,*

$$\deg((p_0, q_0), U_0, 0) = \deg((p_1, q_1), U_1, 0).$$

Proof. Let $Z \subset \bigcup_{i\in\{1,\dots,k\}} O_{\alpha_i}$ where $O_{\alpha_i} = \mathcal{O}^E_\rho(Z_{\alpha_i}) \times ((\alpha_i - \delta_i, \alpha_i + \delta_i) \cap [0,1])$. Observe that, by the localization property of the generalized degree deg, for any $t \in (\alpha_i - \delta_i, \alpha_i + \delta_i)$,

$$\deg((p_t, q_t), U_t, 0) = \deg((p_t, q_t), \mathcal{O}^E_\rho(Z_{\alpha_i}), 0),$$

and, by Remark 2.6(c) and the homotopy property of deg,

$$\deg((p_t, q_t), (\mathcal{O}^E_\rho(Z_{\alpha_i}), 0) = \deg((p_{\alpha_i}, q_{\alpha_i}), \mathcal{O}^E_\rho(Z_{\alpha_i}), 0).$$

Once more by the localization property we get

$$\deg((p_{\alpha_i}, q_{\alpha_i}), (\mathcal{O}^E_\rho(Z_{\alpha_i}), 0) = \deg((p_{\alpha_i}, q_{\alpha_i}), U_{\alpha_i}, 0).$$

It means that the map $t \mapsto \deg((p_t, q_t), U_t, 0)$ is constant in $(\alpha_i - \delta_i, \alpha_i + \delta_i) \cap [0,1]$. Therefore, since $\{O_{\alpha_i}\}_{i\in\{1,\dots,k\}}$ is a finite covering of Z, and $[0,1] \subset \bigcup_{i\in 1,\dots,k\}} (\alpha_i - \delta_{\alpha_i}, \alpha_i + \delta_{\alpha_i})$,

$$\deg((p_0, q_0), U_0, 0) = \deg((p_1, q_1), U_1, 0). \qquad \square$$

Corollary 4.18. *Let V, T be Banach spaces,* $\dim V = n$, $\dim T = m$ *and η_V, η_T represent their orientations, respectively. If $\mathcal{W}$ is an open subset of $T \times [0,1]$ and $\mathrm{cl}\,\mathcal{W} \xleftarrow{r} \Gamma \xrightarrow{s} \mathbb{R}^n$ is an admissible pair such that $\mathcal{Z} = \{x \in \mathrm{cl}\,\mathcal{W} \mid 0 \in s(r^{-1}(x))\}$ is a compact set contained in $\mathcal{W}$, then* $\deg((r_0, s_0), W_0, 0) = \deg((r_1, s_1), W_1, 0)$, *where $W_\alpha := \{x \in V \mid (x, \alpha) \in \mathcal{W}\}$ and $r_\alpha : r^{-1}(W_\alpha \times \{\alpha\}) \to W_\alpha$, $s_\alpha : r^{-1}(W_\alpha \times \{\alpha\}) \to T$ are given by $(r_\alpha(\gamma), \alpha) = r(\gamma)$ and $s_\alpha(\gamma) = s(\gamma)$.*

Proof. It is enough to define $\mathcal{U} := \eta_V(\mathcal{W})$, $R := \eta_V \circ r$, $S := \eta_T \circ s$, and then apply Lemma 4.17 and Definition 4.2. $\square$

Observe that the above lemma is in fact the mentioned generalization of the homotopy property for deg. We will use it in the last lemma.

Lemma 4.19. *Let* $F\colon\mathbb{R}^m\to\mathbb{R}^m$ *be a linear isomorphism,* $H\colon[0,1]\to\operatorname{Iso}(\mathbb{R}^m,\mathbb{R}^m)$ *be a continuous map such that* $H(0)\equiv I_{\mathbb{R}^n}$*,* $H(1)\equiv F$ *and* $\det H(\alpha)>0$ *for any* $\alpha\in[0,1]$*. Then*

$$\deg((F^{-1}\circ p,q),F^{-1}(U),0)=\deg((p,q),U,0)$$

for any admissible pair

$$(\operatorname{cl}U,\operatorname{bd}U)\times[0,1]\xleftarrow{p}(p^{-1}(\operatorname{cl}U),p^{-1}(\operatorname{bd}U))\xrightarrow{q}(\mathbb{R}^n,\mathbb{R}^n\setminus B^n(0,\rho)).$$

Proof. Let $R\colon\Gamma\times[0,1]\to\bigcup_{\alpha\in[0,1]}((H(\alpha))^{-1}(U)\times\{\alpha\})$ and $S\colon\Gamma\times[0,1]\to\mathbb{R}^n$ be given by $R(\gamma,\alpha)=((H(\alpha))^{-1}\circ p(\gamma),\alpha)$ and $S(\gamma,\alpha)=q(\gamma)$. We shall prove that all assumptions of Lemma 4.17 are satisfied.

Observe that the map $\widetilde{H}\colon\mathbb{R}^m\times[0,1]\to\mathbb{R}^m\times[0,1]$ given by $\widetilde{H}(x,t)=(H(t)(x),t)$ is a continuous map. Therefore $\mathcal{U}:=\bigcup_{\alpha\in[0,1]}((H(\alpha))^{-1}(U)\times\{\alpha\})$ is open in $\mathbb{R}^m\times[0,1]$, since it is equal to the preimage $\widetilde{H}^{-1}(U\times[0,1])$ of an open set $U\times[0,1]$.

It is easy to see that the map S is continuous, since so is q. Examine the map R. Let $(\gamma,t)\in\Gamma\times[0,1]$, and $\varepsilon\in(0,2\|(H(t))^{-1}\|\cdot\|p(\gamma)\|)$.

There are $\delta_1>0$ and $\delta_2>0$, such that, if $\|\gamma'-\gamma\|<\delta_1$, then

$$\|p(\gamma')-p(\gamma)\|<\frac{\varepsilon}{4\|(H(t))^{-1}\|},$$

and, if $|t'-t|<\delta_2$, then

$$\|H(t')-H(t)\|<\frac{\varepsilon}{4\|p(\gamma)\|\cdot\|(H(t))^{-1}\|}.$$

This implies that

$$\|(H(t))^{-1}-(H(t'))^{-1}\|\le 2\frac{\varepsilon}{4\|p(\gamma)\|}$$

(see Lemma 2.2). Observe that, if $\|(\gamma',t')-(\gamma,t)\|<\delta:=\min(\delta_1,\delta_2)$, then

$$\begin{aligned}\|R(\gamma,t)\ -R(\gamma',t')\|&=\|(H(t))^{-1}\circ p(\gamma)-(H(t'))^{-1}\circ p(\gamma')\|\\&\le\|(H(t))^{-1}\circ p(\gamma)-(H(t))^{-1}\circ p(\gamma')\|\\&\quad+\|(H(t))^{-1}\circ p(\gamma')-(H(t'))^{-1}\circ p(\gamma')\|\\&\le\|(H(t))^{-1}\|\cdot\|p(\gamma)-p(\gamma')\|+\|(H(t))^{-1}-(H(t'))^{-1}\|\cdot\|p(\gamma')\|\\&\le\|(H(t))^{-1}\|\cdot\frac{\varepsilon}{4\|(H(t))^{-1}\|}+\frac{\varepsilon}{2\|p(\gamma)\|}\cdot(\|p(\gamma')-p(\gamma)\|+\|p(\gamma)\|)\\&\le\frac{\varepsilon}{4}+\frac{\varepsilon}{2\|p(\gamma)\|}\cdot\frac{\varepsilon}{4\|(H(t))^{-1}\|}+\frac{\varepsilon}{2}<\varepsilon,\end{aligned}$$

what implies that R is continuous. Moreover, since

$$R^{-1}(x,t)=(((H(t))^{-1}\circ p)^{-1}(x),t)=(p^{-1}(H(t)(x)),t),$$

and (p,q) is an admissible pair, (R,S) is also admissible. Then, by Lemma 4.17, the proof is complete. □

Now we are ready to prove the main result of this paper, i.e. the homotopy property of the coincidence index Ind, which admits a continuous change of Fredholm operator.

Assume that H_L is an oriented homotopy between L_0 and L_1 and that the admissible pair $\operatorname{cl} U \times [0,1] \xleftarrow{r} \Gamma \xrightarrow{s} E'$ determines the compact homotopy between (p_0, q_0) and (p_1, q_1), where U is an open subset of E. As earlier we will write $L_t := H_L(t)$, $q_t := s|_{r^{-1}(U \times \{t\})}$ and p_t when $(p_t(\gamma), t) = r(\gamma)$. Moreover, let $Z = \{x \in E \mid L_t(x) \in q_t(p_t^{-1}(x)) \text{ for } t \in [0,1]\}$ be compact and contained in U.

Theorem 4.20. *Under the above assumptions*

$$\operatorname{Ind}_{L_0}((p_0, q_0), U) = \operatorname{Ind}_{L_1}((p_1, q_1), U).$$

Proof. Take any $\lambda \in [0,1]$ and a finite dimensional subspace $V = W \oplus \operatorname{Im} Q_\lambda$ of E' sufficiently close to $s(r^{-1}(\operatorname{cl} U \times [0,1])$ (i.e. containing $l_\varepsilon(s(r^{-1}(\operatorname{cl} U \times [0,1]))$, where l_ε is the respective Schauder projection). We have proved that there is $\delta_3 > 0$ such that the family of isomorphisms $\{B_t\}_{t \in (\lambda - \delta_3, \lambda + \delta_3)}$ has properties described in Lemmas 4.10–4.13. We will prove that $\operatorname{Ind}_{L_t}((p_t, q_t), U)$ is constant in $(\lambda - \delta_3, \lambda + \delta_3)$.

Let $t \in (\lambda - \delta_3, \lambda + \delta_3)$. Observe that, by the definition,

$$\operatorname{Ind}_{L_\lambda}((p_\lambda, q_\lambda), U) = \deg((\overline{p_\lambda}, (L_\lambda)_V \circ \overline{p_\lambda} - \overline{q_\lambda}), \mathcal{W}_\lambda, 0),$$

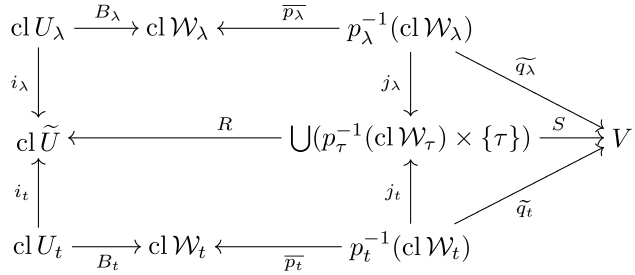
and

$$\operatorname{Ind}_{L_t}((p_t, q_t, U) = \deg((\overline{p_t}, (L_t)_V \circ \overline{p_t} - \overline{q_t}), \mathcal{W}_t, 0),$$

where $\mathcal{W}_\lambda = U \cap L_\lambda^{-1}(V)$, $\mathcal{W}_t = B_t(U) \cap L_t^{-1}(V)$, $\overline{p_t} = p_t|_{p_t-1(\operatorname{cl} \mathcal{W}_t)}$, $\overline{p_\lambda} = p_\lambda|_{p_\lambda^{-1}(\operatorname{cl} \mathcal{W}_\lambda)}$, $\overline{q_t} = l_\varepsilon \circ q_t|_{p_t-1(\operatorname{cl} \mathcal{W}_t)}$, $\overline{q_\lambda} = l_\varepsilon \circ q_\lambda|_{p_\lambda^{-1}(\operatorname{cl} \mathcal{W}_\lambda)}$, $(L_\lambda)_V = L_\lambda|_{L_\lambda^{-1}(V)}$ and $(L_t)_V = L_t|_{L_t^{-1}(V)}$. But of course $L_\lambda^{-1}(V)$ and $L_t^{-1}(V)$ may be different and, consequently, one can not at once compare the right hand sides of the above equalities.

To simplify the notations, here and to the end of this proof, by B_t we understand $B_t|_{\operatorname{Ker} G \circ L_\lambda}$, and by (r, s) the pair determining the homotopy between (p_λ, q_λ) and (p_t, q_t) (in fact (r, s) should be changed like in Remark 2.6(c).

Consider the following commutative diagram:



where $\widetilde{U}$ is as in Lemma 4.13, $U_\lambda = \mathcal{W}_\lambda$, $U_t = B_t^{-1}(\mathcal{W}_t) = B_t^{-1}(L_t^{-1}(V) \cap U)$, $\widetilde{q_t} = (L_t)_V \circ \overline{p_t} - \overline{q_t}$, $\widetilde{q_\lambda} = (L_\lambda)_V \circ \overline{p_\lambda} - \overline{q_\lambda}$ and $p_\tau = r|_{r^{-1}(U_\tau \times \{\tau\})}$.

Observe that R and S given by $R(\gamma,\tau) = (B_\tau^{-1} \circ \overline{p_\tau}(\gamma), \tau)$ and $S(\gamma,\tau) = (L_\tau)_V \circ \overline{p_\tau}(\gamma) - \overline{q_\tau}(\gamma)$ (7) are both continuous. Indeed, all maps composing S are the respective restrictions of continuous maps while R is a composition of an admissible map r and a map $(x,\tau) \mapsto (B_\tau^{-1}(x), \tau)$. The last one is continuous since, by Lemmas 2.2 and 4.11, one can always find $\delta > 0$ such that for $|\tau - \tau'| < \delta$, B_τ^{-1} and $B_{\tau'}^{-1}$ are as close as it is needed. Moreover, (R,S) is admissible, so it is the homotopy between $(B_\lambda^{-1} \circ \overline{p_\lambda}, \overline{q_\lambda})$ and $(B_t^{-1} \circ \overline{p_t}, \overline{q_t})$.

By Lemma 4.17 and Corollary 4.18, since B_λ is the identity map,

$$\begin{aligned}(4.2)\quad \deg((\overline{p_\lambda}, (L_\lambda)_V \circ \overline{p_\lambda} - \overline{q_\lambda}), \mathcal{W}_\lambda, 0) &= \deg((B_t^{-1} \circ \overline{p_t}, (L_t)_V \circ \overline{p_t} - \overline{q_t}), U_t, 0)\\ &= \deg((B_t^{-1} \circ \overline{p_t}, (L_t)_V \circ B_t \circ (B_t)^{-1} \circ \overline{p_t} - \overline{q_t}), U_t, 0).\end{aligned}$$

Denote by ξ the isomorphism representing the fixed orientation in V and by χ_t the isomorphisms determining suitable orientations in $\operatorname{Ker} G \circ L_t$ for any $t \in (\lambda - \delta_3, \lambda + \delta_3)$, i.e. $\chi_t = (\xi, I_{\mathbb{R}^r}) \circ (z(L_t) + A)|_{\operatorname{Ker}(G \circ L_t)}$. We have proved in Lemma 4.12 that $\det(\chi_t \circ B_t \circ \chi_\lambda^{-1}) > 0$. Moreover, by the definition (see Definition 4.2), in fact

$$(4.3)\quad \deg((\overline{p_t}, (L_t)_V \circ \overline{p_t} - \overline{q_t}), \mathcal{W}_t, 0) = \deg((\chi_t \circ \overline{p_t}, \xi \circ ((L_t)_V \circ \overline{p_t} - \overline{q_t})), \chi_t(\mathcal{W}_t), 0),$$

and, by Lemma 4.19 for $F = \chi_t \circ B_t \circ \chi_\lambda^{-1}$,

$$\begin{aligned}(4.4)\quad &\deg((\chi_t \circ \overline{p_t}, \xi \circ ((L_t)_V \circ \overline{p_t} - \overline{q_t})), \chi_t(\mathcal{W}_t), 0)\\ &= \deg(((\chi_t \circ B_t \circ \chi_\lambda^{-1})^{-1} \chi_t \circ \overline{p_t}, \xi \circ ((L_t)_V \circ \overline{p_t} - \overline{q_t})),\\ &\qquad (\chi_t \circ B_t \circ \chi_\lambda^{-1})^{-1}(\chi_t(\mathcal{W}_t)), 0)\\ &= \deg((\chi_\lambda \circ (B_t)^{-1} \circ \overline{p_t}, \xi \circ ((L_t)_V \circ \overline{p_t} - \overline{q_t})), \chi_\lambda \circ (B_t)^{-1}(\mathcal{W}_t), 0)\\ &= \deg((\chi_\lambda \circ (B_t)^{-1} \circ \overline{p_t}, \xi \circ ((L_t)_V \circ B_t \circ (B_t)^{-1} \circ \overline{p_t} - \overline{q_t})), \chi_\lambda(U_t), 0).\end{aligned}$$

Once more, by Definition 4.2,

$$\begin{aligned}&\deg((\chi_\lambda \circ (B_t)^{-1} \circ \overline{p_t}, \xi \circ ((L_t)_V \circ B_t \circ (B_t)^{-1} \circ \overline{p_t} - \overline{q_t})), \chi_\lambda(U_t), 0)\\ &\qquad = \deg(((B_t)^{-1} \circ \overline{p_t}, (L_t)_V \circ B_t \circ (B_t)^{-1} \circ \overline{p_t} - \overline{q_t}), U_t, 0).\end{aligned}$$

Hence by (4.3) and (4.4),

$$\begin{aligned}&\deg((\overline{p_t}, (L_t)_V \circ \overline{p_t} - \overline{q_t}), \mathcal{W}_t, 0)\\ &\qquad = \deg(((B_t)^{-1} \circ \overline{p_t}, (L_t)_V \circ B_t \circ (B_t)^{-1} \circ \overline{p_t} - \overline{q_t}), U_t, 0),\end{aligned}$$

and finally, by (4.2),

$$\deg((\overline{p_\lambda}, (L_\lambda)_V \circ \overline{p_\lambda} - \overline{q_\lambda}), \mathcal{W}_\lambda, 0) = \deg((\overline{p_t}, (L_t)_V \circ \overline{p_t} - \overline{q_t}), \mathcal{W}_t, 0),$$

(7) By $\overline{q_\tau}$ we understand the map $l_\varepsilon \circ s|_{r^{-1}(U_\tau \times \{\tau\})}$

what implies that

$$\operatorname{Ind}_{L_\lambda}((p_\lambda, q_\lambda), U) = \operatorname{Ind}_{L_t}((p_t, q_t), U).$$

It means that the map $[0,1] \ni t \mapsto \operatorname{Ind}_{L_t}((p_t, q_t), U)$ is locally constant. But since $[0,1]$ is a compact connected set, it is also globally constant and the proof is complete. □

At the end observe that quite often it is sufficient to know whether the homotopy invariant is not trivial. As a simple corollary of our consideration we get the following form of the homotopy property:

COROLLARY 4.21. *If $(L_0, (p_0, q_0))$ and $(L_1, (p_1, q_1))$ are homotopic in the sense of Definition* 4.7*, then*

$$\operatorname{Ind}_{L_0}((p_0, q_0), U) = 0 \quad \textit{if and only if} \quad \operatorname{Ind}_{L_1}((p_1, q_1), U) = 0.$$

One can find similar results for Fredholm operators of index 0 e.g. in [9].

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