Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 31, 2008, 111–138

OSCILLATION AND CONCENTRATION EFFECTS DESCRIBED BY YOUNG MEASURES WHICH CONTROL DISCONTINUOUS FUNCTIONS

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Abstract. We study oscillation and concentration effects for sequences of compositions $\{f(u^\nu)\}_{\nu\in\mathbb{N}}$ of μ-measurable functions $u^\nu:\Omega\to\mathbb{R}^m$ where Ω is the compact subset of $\mathbb{R}^n$ and f is the (possibly) discontinuous function. The limits are described in terms of Young measures which can control discontinuous functions recently introduced in [14].

1. Introduction

The celebrated theorem by Young [29] asserts that given an arbitrary sequence of measurable functions $\{u^\nu\}_{\nu\in\mathbb{N}}$ defined on a measurable bounded set $\Omega\subseteq\mathbb{R}^n$ with values in $\mathbb{R}^m$ and an arbitrary continuous function $f:\mathbb{R}^m\to\mathbb{R}$ which vanishes at infinity one can extract the subsequence of $\{u^\nu\}_{\nu\in\mathbb{N}}$ such that $\{f(u^\nu)\}_{\nu\in\mathbb{N}}$ converges weakly $*$ in L^∞ to the function $\overline{f}$ described by an integral formulae

$$\Omega\ni x\mapsto\overline{f}(x)=\int_{\mathbb{R}^m}f(\lambda)\mu_x(d\lambda)\tag{1.1}$$

where $\{\mu_x\}_{x\in\Omega}$ is the family of parametrized measures on $\mathbb{R}^m$ independent on the choice of f. Moreover, if K is a closed subset of $\mathbb{R}^m$ and $u^\nu\to K$ in a measure,

2000 *Mathematics Subject Classification.* 49J10, 49J45.
Key words and phrases. Young measures, DiPerna–Majda measures, weak convergence, discontinuous functions.
This research was done while the author was visiting Institute of Mathematics of the Polish Academy of Sciences at Warsaw. I would like to thank IM PAN for hospitality.

then an integral over $\mathbb{R}^m$ in (1.1) can be changed into an integral over K (see e.g. [4], [12], [25], [27], [29]).

Young's discovery became widely applicable in many disciplines of analysis, e.g. in calculus of variations, partial differential equations, optimal control theory, game theory, numerical analysis, see e.g. [3], [9], [10], [23], [25], [27] and their references.

Nowadays many various generalizations of Young's theorem are known (see e.g. [27]). Of our particular interest are those which allow to deal with such sequences of compositions $\{f(u^\nu)\}_{\nu\in\mathbb{N}}$ that the function f may be discontinuous with respect to the Euclidean topology in $\mathbb{R}^m$. For deep abstract results in this direction we refer to works by Balder [3], Chentsov [6] and Fattorini [10]. In their approaches the space $\mathbb{R}^m$ can be substituted by an abstract metric space M and one deals with sequences $\{u^\nu\}_{\nu\in\mathbb{N}}$ with values in M. Assuming that the sequence $\{f(u^\nu)\}_{\nu\in\mathbb{N}}$ is weakly compact in $L^1(\Omega)$ one describes limits of its weakly convergent subsequences in the form (1.1) with M playing the role of $\mathbb{R}^m$ and μ_x defined on M. In particular oscillations effects for compositions like $\{f(u^\nu)\}_{\nu\in\mathbb{N}}$ can be studied within the large class of f, but the concentration effects cannot be explained by this theory.

The assumption that $\{f(u^\nu)\}_{\nu\in\mathbb{N}}$ is weakly compact in $L^1(\Omega)$ seems to be often too strong in the applications. Therefore DiPerna and Majda introduced such an approach where one can assume that the sequence $\{f(u^\nu)\}_{\nu\in\mathbb{N}}$ is only bounded in $L^1(\Omega)$ ([7], see also [1], [17], [18]). Namely, assume that $f\colon\mathbb{R}^m\to\mathbb{R}^m$ is continuous of growth p and $\{u^\nu\}_{\nu\in\mathbb{N}}$ is an L^p-bounded sequence. DiPerna and Majda showed that one can extract the subsequence of the $\{u^\nu\}_{\nu\in\mathbb{N}}$ (denoted by the same expression) such that the weak $*$ limit of the sequence of Radon measures $\{f(u^\nu)dx\}_{\nu\in\mathbb{N}}$ on a compact set Ω is given by the formulae

$$\left(\int_{\gamma\mathbb{R}^m} f_0(\lambda)\hat{\nu}_x(d\lambda)\right)\sigma(dx),$$

where $\gamma\mathbb{R}^m$ is the given compactification of $\mathbb{R}^m$, $f_0\in C(\gamma\mathbb{R}^m)$ is the given transform of f (independent on $\{u^\nu\}_{\nu\in\mathbb{N}}$), σ is the Radon measure on Ω, $\hat{\nu}_x$ are probability measures on $\gamma\mathbb{R}^m$ defined for $x\in\Omega$ and both: σ and $\hat{\nu}_x$ are independent on f but only on the sequence.

In particular not only oscillation effects but also concentrations of the sequence $\{f(u^\nu)\}_{\nu\in\mathbb{N}}$ can be explained by DiPerna and Majda theory. On the contrary to the mentioned abstract approaches the assumptions on f are less general: it is defined on $\mathbb{R}^m$ and continuous with respect to the Euclidean topology of $\mathbb{R}^m$.

The author found the unified approach which allows to deal with discontinuous functions and study the oscillations of sequences $\{f(u^\nu)\}_{\nu\in\mathbb{N}}$ as well, [14], [15]. Abbreviating, the result obtained in [14] can be summarized as follows.

Assume that μ is a Radon measure on a compact set Ω, $g\colon \mathbb{R}^m \to \mathbb{R}$ is positive and $\{u^\nu\}_{\nu\in\mathbb{N}}$ is a given sequence of μ-measurable $L^g_\mu(\Omega)$-bounded functions with values in $\mathbb{R}^m$, i.e.

$$\sup_\nu \int_\Omega g(u^\nu(x))\mu(dx) < \infty.$$

Let further $A_1,\dots,A_k$ are Borel subsets of $\mathbb{R}^m$ called bricks forming the partition of $\mathbb{R}^m$ and $f\colon \mathbb{R}^m \to \mathbb{R}$ be such that f/g has good properties, in particular it is bounded and continuous on each A_i. Let γA_i be the compactification of A_i and $\gamma A_i \setminus A_i$ be the reminder. The main theorem in [14] asserts that after extracting the subsequence of $\{u^\nu\}_{\nu\in\mathbb{N}}$ (which we denote by the same expression, it is independent on f) the sequence of Radon measures $\{f(u^\nu)\mu\}_{\nu\in\mathbb{N}}$ converges weakly $*$ in the space of measures to the measure represented by

$$(1.2)\qquad \sum_{i=1}^k \biggl(\int_{\operatorname{int} A_i} f(\lambda)\mu_x(d\lambda)\mu(dx) + \int_{\partial A_i\cap A_i} f(\lambda)\overline{\nu}^i_x(d\lambda)\overline{m}^i(dx) + \int_{\gamma A_i\setminus A_i} \widetilde{f}_i(\lambda)\nu^i_x(d\lambda)m^i(dx)\biggr),$$

where $\{\mu_x\}_{x\in\Omega}$ are classical Young measures generated by $\{u^\nu\}_{\nu\in\mathbb{N}}$ (for μ being the Lebesgue measure they are the same as that in (1.1)), $\overline{\nu}^i_x$ and ν^i_x are probability measures defined on $\partial A_i \cap A_i$ and $\gamma A_i \setminus A_i$ respectively, $\overline{m}^i$, m^i are Radon measures on Ω, while $\widetilde{f}_i$ is certain transform of f defined on the compactification γA_i of A_i. The quantities: μ_x, $\overline{\nu}^i_x$, ν^i_x and also $\overline{m}^i$, m^i are independent on f while $\widetilde{f}_i$ is independent on the sequence $\{u^\nu\}_{\nu\in\mathbb{N}}$. In the case when μ is the Lebesgue measure, $g(\lambda) = 1 + |\lambda|^p$ and we deal with one brick $A = \mathbb{R}^m$ only, the theorem reduces to the theorem by DiPerna and Majda.

Our goal now is to study more precisely the oscillation and concentration effects for sequences of compositions $\{f(u^\nu)\}_{\nu\in\mathbb{N}}$ within the same class of functions as in [14]. This issue mainly consists of three results.

At first we extend some of our previous results from [15] and study more precisely the behaviour of supports of measures μ_x, $\overline{\nu}^i_x$ and ν^i_x in the representation formulae (1.2), under some additional information on the behaviour of the sequence. This results in three Theorems 3.1–3.3 presented in Section 3.

In Section 4 we find the necessary and sufficient condition for weak compactness in $L^1_\mu(\Omega)$ of the sequence $\{f(u^\nu)\}_{\nu\in\mathbb{N}}$, which is described in terms of the measures in (1.2) (Theorem 4.1).

Next we apply previous results to obtain the variant of the Young theorem where one studies the weak $L^1_\mu(\Omega)$-limit of the sequence of compositions $\{f(u^\nu)\}_{\nu\in\mathbb{N}}$, under an additional information that $u^\nu \to L$ in the measure μ, where L is the closed subset of $\mathbb{R}^m$. This results in Theorem 5.1 in Section 5. We call it the generalization of Young theorem because here the sequence $\{f(u^\nu)\}_{\nu\in\mathbb{N}}$ is nonconcentrating like in the classical theorem of Young. Let me mention that such a theorem is the special case of the known abstract results, but it is specialized to functions defined on $\mathbb{R}^m$ with the prescribed discontinuities. Moreover, the condition $u^\nu \to L$ in a measure means that for an arbitrary $V \supseteq L$, where V open in the Euclidean topology we have

$$\lim_{\nu\to\infty} \mu(\{x \in \Omega : u^\nu(x) \notin V\}) = 0. \tag{1.3}$$

In particular (1.3) uses the Euclidean topology of $\mathbb{R}^m$, while f may be discontinuous with respect to the Euclidean topology. Therefore Young theorem obtained here does not follow directly from abstract mentioned approaches.

Our motivations to study such a generalization of Young's theorem are two folded.

The first one comes from applications to mathematical physics and is dictated by many physical models where one deals with PDE's with discontinuous constraints (see e.g. [11], [13], [20]–[22], [24], [26] or Chapter 4, pages 137–138 in [5]). Our measures are more sensitive as they can detect from which side the sequence approaches the point. Therefore we believe that our generalization of Young's theorem can be applied to such PDE's similarly as the classical theorem of Young is applied to the more regular ones. See also [15] for the more detailed explanation and references.

Our second motivation is linked with Convergence Theorem in set-valued analysis (see e.g. [2]). For simplicity let μ be the Lebesgue measure, f be the single-valued function which is continuous on each brick A_i in decomposition $\mathbb{R}^m = \bigcup_{i=1}^k A_i$. Convergence Theorem reduced to such a case asserts that if $u^\nu \to u$ almost everywhere and $f(u^\nu) \rightharpoonup w$ as $\nu \to \infty$ weakly in $L^1(\Omega)$ then for almost all $x \in \Omega$ the point $w(x)$ belongs to the convex hull of accumulation points of f at $u(x)$. Here we explain the case when the sequence $\{u^\nu\}_{\nu\in\mathbb{N}}$ does not converge almost everywhere or the sequence $\{f(u^\nu)\}_{\nu\in\mathbb{N}}$ does not converge weakly in $L^1(\Omega)$. Further extensions of this approach are in progress [16].

The last section is devoted to illustrations of the presented results.

For more illustrations and examples we refer to papers [14] and [15].

2. Preliminaries and notation

Let A be an arbitrary subset of $\mathbb{R}^m$. By $C(A)$ we denote the space of continuous functions on A. The symbol χ_A denotes the characteristic function of A. If the scalar function f is defined on A or its neighborhood then by $f\chi_A$ we mean the function which is equal to f on A and 0 outside A. By $C_0(\mathbb{R}^m)$ we denote the space of all continuous functions on $\mathbb{R}^m$ which vanish at infinity. We denote: $Y^c = \mathbb{R}^m \setminus Y$, the complement of $Y \subseteq \mathbb{R}^m$ in $\mathbb{R}^m$. The closure of a set $S \subseteq \mathbb{R}^K$ is denoted by $\overline{S}$. If K_1, K_2 are two closed subsets of $\mathbb{R}^N$ such that $K_2 \subseteq K_1$, we denote

$$\partial_{K_1} K_2 = \{x \in K_2 : B(x,r) \cap (K_1 \setminus K_2) \neq \emptyset \text{ for all } r > 0\}. \tag{2.1}$$

Let S be the Borel subset of the Euclidean space $\mathbb{R}^k$. By $\mathcal{M}(S)$ we denote the space of Radon measures on S, while $\mathcal{P}(S)$ is its subset consisting of probability measures. If $\mu \in \mathcal{M}(S)$ and f is μ-measurable, we denote $(f, \mu) := \int_S f(\lambda)\mu(d\lambda)$. By $\operatorname{supp} \mu$ we denote the support of μ.

Arrows $\rightarrow$, $\rightharpoonup$, $\overset{*}{\rightharpoonup}$ are used to denote the strong, weak, and weak $*$ convergence respectively in the given topology.

Let Ω be a Borel subset of $\mathbb{R}^n$, $\mu \in \mathcal{M}(\Omega)$ and $L \subseteq \mathbb{R}^m$ be a closed subset. The sequence $\{u^\nu\}_{\nu\in\mathbb{N}}$ where $u^\nu : \Omega \to \mathbb{R}^m$ are μ-measurable is said to converge to L in a measure if $\mu(\{x : \operatorname{dist}(u^\nu(x), L) > \varepsilon\}) \to 0$ as $\nu \to \infty$, whenever $\varepsilon > 0$.

Recall that the compact topological space γA is the compactification of the topological space A if there is the dense homeomorphical embedding $\Phi : A \to \Phi(A) \subseteq \gamma A$ (see e.g. [19]). If not causing a misunderstanding we will also write A instead of $\Phi(A)$ and $\gamma A \setminus A$ instead of $\gamma A \setminus \Phi(A)$.

If S is a Borel subset of $\mathbb{R}^k$, by $L^\infty_{w*}(\Omega, \mathcal{M}(S), \mu)$ we denote the set of families $\{\mu_x\}_{x\in\Omega}$ of Radon measures on S which are weakly $*$ μ-measurable in the sense of Pettis i.e. for every $f \in C(S)$ the mapping $x \mapsto \int_S f(\lambda)\mu_x(d\lambda)$ is μ-measurable (see e.g. Definition 1 of Section V.4 in [28]). The symbol $\mathcal{P}(\Omega, S, \mu)$ stands for such families of nonnegative measures $\{\mu_x\}_{x\in\Omega} \in L^\infty_{w*}(\Omega, \mathcal{M}(S), \mu)$, which satisfy $\|\mu_x\|_{\mathcal{M}(S)} = 1$ for μ-almost all x.

We recall one version of the classical theorem of Young (see e.g. [1], [3, Lemma 4.11 and Corollary 5.4], [4], [12]).

Theorem (Young's Theorem). *Let Ω be an open bounded subset of $\mathbb{R}^n$ and $\{u^\nu\}_{\nu\in\mathbb{N}}$ be a sequence of μ-measurable functions, $u^\nu : \Omega \to \mathbb{R}^m$. Then there exists a subsequence of $\{u^\nu\}_{\nu\in\mathbb{N}}$ still denoted by the same expression and a family of measures $\{\mu_x\}_{x\in\Omega} \in L^\infty_{w^*}(\Omega, \mathcal{M}(\mathbb{R}^m), \mu)$ such that $\|\mu_x\|_{\mathcal{M}(\mathbb{R}^m)} \leq 1$ for μ almost all x and for every function $f \in C_0(\mathbb{R}^m)$ we have*

$$f(u^\nu) \overset{*}{\rightharpoonup} \overline{f}(x) = (f, \mu_x) \quad \text{in } L^\infty(\Omega, \mu), \text{ as } \nu \to \infty.$$

If additionally the sequence $\{u^\nu\}_{\nu\in\mathbb{N}}$ satisfies the tightness condition:

$$\limsup_{\nu\in\mathbb{N}} \mu(\{x \in \Omega : |u^\nu(x)| \geq r\}) \xrightarrow{r\to\infty} 0.$$

Then $\|\mu_x\|_{\mathcal{M}(\mathbb{R}^m)} = 1$ for μ almost all x.

Definition 2.2. The sequence $\{u^\nu\}_{\nu\in\mathbb{N}}$ where $u^\nu : \Omega \to \mathbb{R}^m$ is said to generate the Young measure $\{\mu_x\}_{x\in\Omega} \in L^\infty_{w^*}(\Omega, \mathcal{M}(\mathbb{R}^m), \mu)$ if for every $f \in C_0(\mathbb{R}^m)$ the sequence $\{f(u^\nu)\}_{\nu\in\mathbb{N}}$ converges weakly $*$ in $L^\infty(\Omega, \mu)$ to the function $\overline{f}(x) := \int_{\mathbb{R}^m} f(\lambda)\mu_x(d\lambda)$.

In the sequel we will use the following general assumptions.

ASSUMPTION A1.

(a) There exist disjoint Borel subsets: $A_1, \dots, A_k$, called *bricks*, such that $\mathbb{R}^m = \bigcup_{i=1}^k A_i$.

(b) Each A_i is compactified by some $\gamma A_i \subseteq \mathbb{R}^{N_i}$ where $N_i \in \mathbb{N}$, with the help of the dense homeomorphic embedding $\Phi_i\colon A_i \to \Phi_i(A_i) \subseteq \gamma A_i$. We additionally assume that each mapping Φ_i has the following property:

$$\left\{ \begin{array}{l} \text{if } R > 0 \text{ is arbitrary and } \{z_r\}_{r\in\mathbb{N}},\ \{y_r\}_{r\in\mathbb{N}} \subseteq A_i \cap B(R) \\ \text{then the condition } \operatorname{dist}(\Phi_i(z_r), \Phi_i(y_r)) \xrightarrow{r\to\infty} 0 \text{ in } \gamma A_i \\ \text{implies } \operatorname{dist}(z_r, y_r) \xrightarrow{r\to\infty} 0 \end{array} \right. \tag{2.2}$$

and the reminder $\gamma A_i \setminus A_i$ is a closed subset of γA_i.

(c) The space $\mathbb{R}^m$ is equipped with *density function* $g\colon \mathbb{R}^m \to (0,\infty)$ such that $g|_{A_i} \in C(A_i)$, $g(\lambda) \geq \alpha$ for every $\lambda \in A_i \cap \partial A_i$ and some $\alpha > 0$ and $\lim_{\lambda\to\infty} g(\lambda) = \infty$.

(d) We deal with the following Banach space of *admissible functions*:

$$\mathcal{F} := \{f\colon \mathbb{R}^m \to \mathbb{R} : \widetilde{f}_i := (f/g) \circ \Phi_i^{-1} \in C(\gamma A_i) \text{ for } i = 1, \dots, k\}, \tag{2.3}$$

equipped with the norm $\|f\|_{\mathcal{F}} = \sum_{i=1}^k \|\widetilde{f}_i\|_{L^\infty(\gamma A_i)}$.

The notation $\widetilde{f}_i \in C(\gamma A_i)$ means that the function $(f/g) \circ \Phi_i^{-1}\colon \Phi_i(A_i) \to \mathbb{R}$ is the restriction of some continuous function defined on γA_i. As this function is uniquely defined we will denote it by the same expression: $\widetilde{f}_i$.

We will also deal with the following condition which is stronger than Assumption A1.

ASSUMPTION A2.

(a) Assumption A1 is satisfied.

(b) The constant mapping $h \equiv 1$ belongs to $\mathcal{F}$ and for every $i \in \{1, \dots, k\}$ we define:

$$R_i^\infty = G_i^{-1}(0), \quad R_i^b = (\gamma A_i \setminus A_i) \setminus R_i^\infty,$$

where G_i is the unique continuous extension of the function

$$\frac{1}{g(\Phi_i^{-1}(\lambda))}\colon \Phi_i(A_i) \to \mathbb{R}$$

to γA_i. In particular R_i^∞ is the closed subset of $\gamma A_i \setminus A_i$.

REMARK 2.3. Note that by definition of $\mathcal{F}$ the mapping $G_i = (1/g) \circ \Phi_i^{-1}\colon \Phi_i(A_i) \to \mathbb{R}$ has a continuous extension to γA_i if and only if $\chi_{A_i} \in \mathcal{F}$.

In [14] we have obtained the following result (which is a particular case of the more general statement, see Remark 2.1 in [15]).

Theorem 2.4. *Suppose that* $\Omega \subseteq \mathbb{R}^n$ *is the compact set equipped with the Radon measure* μ *and that Assumption* A1 *with bricks* $\{A_i\}$ *and density function* $g: \mathbb{R}^m \to (0,\infty)$ *is satisfied. Assume further that there is given the sequence* $\{u^\nu\}_{\nu\in\mathbb{N}}$ *of* μ*-measurable functions,* $u^\nu: \Omega \to \mathbb{R}^m$*, which satisfies the condition*

$$\sup_\nu \int_{\mathbb{R}^m} g(u^\nu)\mu(dx) < \infty. \tag{2.4}$$

Then there exist

(a) *a subsequence of* $\{u^\nu\}$ *denoted by the same expression,*
(b) *measures* $\overline{m}^i, m^i \in \mathcal{M}(\Omega)$*, such that* $\overline{m}^i$ *is absolutely continuous with respect to* μ *and* $\operatorname{supp} m^i \subseteq \operatorname{supp}\mu$ *for* $i \in \{1,\dots,k\}$*,*
(c) *families of probability measures* $\{\mu_x\}_{x\in\Omega} \in \mathcal{P}(\Omega,\mathbb{R}^m,\mu)$*,* $\{\overline{\nu}^i_x\}_{x\in\Omega} \in \mathcal{P}(\Omega,\partial A_i \cap A_i,\mu)$ *and* $\{\nu^i_x\}_{x\in\Omega} \in \mathcal{P}(\Omega,\gamma A_i \setminus A_i, m^i)$ *where* $i \in \{1,\dots,k\}$

such that for an arbitrary $f \in \mathcal{F}$ *the subsequence* $\{f(u^\nu(x))\mu(dx)\}_{\nu\in\mathbb{N}}$ *converges weakly* $*$ *in the space of measures to the measure represented by*

$$\sum_{i=1}^k \biggl(\int_{\operatorname{int} A_i} f(\lambda)\mu_x(d\lambda)\mu(dx) + \int_{\partial A_i\cap A_i} f(\lambda)\overline{\nu}^i_x(d\lambda)\overline{m}^i(dx) + \int_{\gamma A_i\setminus A_i} \widetilde{f}_i(\lambda)\nu^i_x(d\lambda)m^i(dx)\biggr), \tag{2.5}$$

where $\widetilde{f}_i$ *is defined by* (2.3)*. Moreover, measures* $\{\mu_x\}_{x\in\Omega}$ *are the classical Young measures generated by the sequence* $\{u^\nu\}$*.*

Definition 2.5. If the sequence of μ-measurable functions is such that for an arbitrary $f \in \mathcal{F}$ the subsequence $\{f(u^\nu(x))\mu(dx)\}_{\nu\in\mathbb{N}}$ converges weakly $*$ in the space of measures to the measure represented by (2.5) then we will say that measures: $\{\mu_x\}_{x\in\Omega}$, $(\overline{m}^i, \{\overline{\nu}^i_x\}_{x\in\Omega})_{i=1,\dots,k}$, $(m^i, \{\nu^i_x\}_{x\in\Omega})_{i=1,\dots,k}$ are generated by $\{u^\nu\}_{\nu\in\mathbb{N}}$.

The following results were obtained in [15] (we refer there for discussion).

Theorem 2.6. *Let the assumptions of Theorem* 2.4 *be satisfied and let the sequence* $\{u^\nu\}_{\nu\in\mathbb{N}}$ *and measures*

$$\{\mu_x\}_{x\in\Omega}, \quad (\overline{m}^i, \{\overline{\nu}^i_x\}_{x\in\Omega})_{i=1,\dots,k}, \quad (m^i, \{\nu^i_x\}_{x\in\Omega})_{i=1,\dots,k}$$

be the same as in its statement. Assume further that for a given $i \in \{1,\dots,k\}$ *we have*

$$A^i_r := \limsup_{\nu\to\infty} \int_{\{x: u^\nu(x)\in A_i\cap V^r\}} g(u^\nu(x))\mu(dx) \xrightarrow{r\to\infty} 0, \tag{2.6}$$

for a given sequence $\{V^r\}_{r\in\mathbb{N}}$ *of open nonempty subsets of* $\mathbb{R}^n$ *such that* $\overline{V}^{r+1} \subseteq V^r$*. Define* $P = \cap_{r\in\mathbb{N}}\overline{V}_r$*. Then we have*

(a) $\mu_x(P \cap \operatorname{int} A_i) = 0$ *for* μ*-almost all* x*,*

(b) $\overline{\nu}^i_x(P\cap\partial A_i\cap A_i)=0$ *for* $\overline{m}^i$- *almost all* x,

(c) $\nu^i_x(K^0_i\cap(\gamma A_i\setminus A_i))=0$ *for* m^i-*almost all* x, *where*

$$K^0_i=\bigcup_{R\in\mathbb{N}}\bigcap_{r\in\mathbb{N}}\overline{\Phi_i(V^r_R\cap A_i)}\quad\textit{and}\quad V^r_R=V^r\cap\{x:|x|<R+1/r\}.$$

THEOREM 2.7. *Let the assumption of Theorem* 2.6 *is satisfied and assume additionally that for every* $r\in\mathbb{N}$ *we have*

$$\overline{\Phi_i(V^{r+1}\cap A_i)}\cap\partial_{\gamma A i}\overline{\Phi_i(V^r\cap A_i)}=\emptyset. \tag{2.7}$$

Then $\nu^i_x(K_i\cap(\gamma A_i\setminus A_i))=0$ *for* m^i-*almost all* x, *where* $K_i=\bigcap_{r\in\mathbb{N}}\overline{\Phi_i(V^r\cap A_i)}$.

THEOREM 2.8. *Let the assumptions of Theorem* 2.4 *and Assumption* A2 *be satisfied and let the sequence* $\{u^\nu\}_{\nu\in\mathbb{N}}$ *and measures*

$$\{\mu_x\}_{x\in\Omega},\quad(\overline{m}^i,\{\overline{\nu}^i_x\}_{x\in\Omega})_{i=1,\dots,k},\quad(m^i,\{\nu^i_x\}_{x\in\Omega})_{i=1,\dots,k}$$

be the same as in the statement of Theorem 2.4. *Then we have*

(a) $\nu^i_x(R^\infty_i)=1$ *for* m^i_s-*almost all* x, *where* m^i_s *is the singular part of* m^i *with respect to* μ *in the Lebesgue–Nikodym decomposition.*

(b) *If additionally*

$$A^i_r:=\lim_{\nu\to\infty}\int_{\{x:u^\nu(x)\in A_i\cap V^r\}}g(u^\nu(x))\mu(dx)\xrightarrow{r\to\infty}0,$$

for a given sequence $\{V^r\}_{r\in\mathbb{N}}$ *of open nonempty subsets of* $\mathbb{R}^n$ *such that* $\overline{V}^{r+1}\subseteq V^r$, *then*

$$\nu^i_x(K_i\cap R^b_i)=0\quad\textit{for } m^i\textit{-almost all } x, \tag{2.8}$$

where $K_i=\bigcap_{r\in\mathbb{N}}\overline{\Phi_i(V^r\cap A_i)}$.

REMARK 2.9. Obviously $K^0_i\subseteq K_i$ but K_i can be essentially bigger (see Remark 3.1 in [15]). Example 3.1 in [15] shows that in general the set K^0_i in the statement (c) of Theorem 2.6 cannot be substituted by K_i. Moreover, (2.8) is the consequence of the inclusion $K_i\setminus K^0_i\subseteq R^\infty_i$ and part (c) of Theorem 2.6 (see Remark 4.2 in [15]).

3. On supports of the parametrized measures

Our goal now is to obtain the following generalizations of Theorems 2.6, 2.7 and of part (b) in Theorem 2.8.

THEOREM 3.1. *Let the assumptions of Theorem* 2.4 *be satisfied and let the sequence* $\{u^\nu\}_{\nu\in\mathbb{N}}$ *and measures*

$$\{\mu_x\}_{x\in\Omega}, \quad (\overline{m}^i, \{\overline{\nu}^i_x\}_{x\in\Omega})_{i=1,\dots,k}, \quad (m^i, \{\nu^i_x\}_{x\in\Omega})_{i=1,\dots,k}$$

be the same as in its statement. Assume further that $f \in \mathcal{F}$ *and for a given* $i \in \{1,\dots,k\}$ *we have*

$$B^i_r := \lim_{\nu\to\infty} \int_{\{x:u^\nu(x)\in A_i\cap V^r\}} |f(u^\nu(x))|\mu(dx) \xrightarrow{r\to\infty} 0, \tag{3.1}$$

for a given sequence $\{V^r\}_{r\in\mathbb{N}}$ *of open nonempty subsets of* $\mathbb{R}^n$ *such that* $\overline{V}^{r+1} \subseteq V^r$. *Define* $P = \bigcap_{r\in\mathbb{N}} \overline{V}_r$. *Then we have*

(a) $\int_{P\cap \operatorname{int} A_i} |f(\lambda)|\mu_x(d\lambda) = 0$ *for* μ*-almost all* x,
(b) $\int_{P\cap\partial A_i\cap A_i} |f(\lambda)|\overline{\nu}^i_x(d\lambda) = 0$ *for* $\overline{m}^i$*-almost all* x,
(c) $\int_{K^0_i\cap(\gamma A_i\setminus A_i)} \widetilde{|f|}_i(\lambda)\nu^i_x(d\lambda) = 0$ *for* m^i*-almost all* x, *where*

$$K^0_i = \bigcup_{R\in\mathbb{N}} \bigcap_{r\in\mathbb{N}} \overline{\Phi_i(V^r_R\cap A_i)} \quad \text{and} \quad V^r_R = V^r \cap \{x : |x| < R + 1/r\}.$$

THEOREM 3.2. *Let the assumption of Theorem* 3.1 *is satisfied and assume additionally that for every* $r \in \mathbb{N}$ *we have*

$$\overline{\Phi_i(V^{r+1}\cap A_i)} \cap \partial_{\gamma A i}\overline{\Phi_i(V^r\cap A_i)} = \emptyset. \tag{3.2}$$

Then

$$\int_{K_i\cap(\gamma A_i\setminus A_i)} \widetilde{|f|}_i(\lambda)\nu^i_x(d\lambda) = 0$$

for m^i*-almost all* x, *where* $K_i = \bigcap_{r\in\mathbb{N}} \overline{\Phi_i(V^r\cap A_i)}$.

THEOREM 3.3. *Let the assumptions of Theorem* 3.1 *and Assumption* A2 *be satisfied. Then for* m^i*-almost all* x

$$\int_{K_i\cap R^i_b} \widetilde{|f|}_i(\lambda)\nu^i_x(d\lambda) = 0, \quad \textit{where } K_i = \bigcap_{r\in\mathbb{N}} \overline{\Phi_i(V^r\cap A_i)}.$$

REMARK 3.4. Note that Theorem 3.1 implies Theorem 2.6 when we substitute $f = g$ in its assumption and use the fact that g is positive. By the same argument Theorem 3.2 implies Theorem 2.7 and Theorem 3.3 implies part (b) of Theorem 2.8. On the other hand, we could not just plug $g = |f|$ in Theorems 2.6 and 2.8 to obtain Theorems 3.1 and 3.3 as $|f|$ may not satisfy the assumptions required in Assumption A1 (for example it may not be strictly positive).

PROOF OF THEOREM 3.1. We use similar techniques as in Section 4 of [15]. Therefore we will refer to [15] for some of the details. We start with the following assumption.

Assumption B. The following two conditions are satisfied:

(a) $\Omega \subseteq \mathbb{R}^n$ is the compact set equipped with the Radon measure μ, A is the Borel subset of $\mathbb{R}^m$ embedded homeomorphically and densely in the compact set $\gamma A \subseteq \mathbb{R}^N$, with the help of homeomorphism $\Phi: A \to \Phi(A) \subseteq \gamma A$, $g \in C(A)$ is the nonnegative function such that $g \geq \alpha$ on $\partial A \cap A$ for some $\alpha > 0$ and

$$\mathcal{F}_A := \{f: \mathbb{R}^m \to \mathbb{R} : \widetilde{f} := (f/g) \circ \Phi^{-1} \in C(\gamma A)\}.$$

(b) $\{u^\nu\}: \Omega \to \mathbb{R}^m$ is the given sequence of μ-measurable functions which satisfies the condition

$$\sup_\nu \int_{x: u^\nu(x) \in A} g(u^\nu)\mu(dx) < \infty.$$

We will use the following lemmas.

Lemma 3.5 ([14, Lemma 3.1]). *Suppose that Assumption B is satisfied and define the sequence of measures $\{L^\nu\}_{\nu \in \mathbb{N}}$ on $\Omega \times \gamma A$ by expressions*

$$(F, L^\nu) := \int_{\{x: u^\nu(x) \in A\}} F(x, \Phi(u^\nu(x)))g(u^\nu(x))\mu(dx), \quad \text{where } F \in C(\Omega \times \gamma A).$$

Then we have:

(a) *There exists a subsequence of $\{L^\nu\}$ still denoted by the same expression, measures $L \in \mathcal{M}(\Omega \times \gamma A)$, $\widetilde{m} \in \mathcal{M}(\Omega)$ and a family of probability measures $\{\widetilde{\nu}_x\}_{x \in \Omega} \in \mathcal{P}(\Omega, \gamma A, \widetilde{m})$ such that*

$$L^\nu \overset{*}{\rightharpoonup} L \quad \text{in } \mathcal{M}(\Omega \times \gamma A),$$
$$(F, L) = \int_\Omega \int_{\gamma A} F(x, \lambda)\widetilde{\nu}_x(d\lambda))\widetilde{m}(dx) \quad \text{where } F \in C(\Omega \times \gamma A),$$
$$\operatorname{supp} \widetilde{m} \subseteq \operatorname{supp} \mu.$$

(b) *Let $\widetilde{m} = p(x)\mu + \widetilde{m}_s$ be the Lebesgue–Nikodym decomposition of $\widetilde{m}$ with respect to μ. Then if $\gamma A \setminus A \neq \emptyset$ we have $\widetilde{\nu}_x(\gamma A \setminus A) = 1$ for $\widetilde{m}_s$ almost all $x \in \Omega$, while if $\gamma A \setminus A = \emptyset$ we have $\widetilde{m}_s = 0$.*

Lemma 3.6 ([15, Lemma 4.2]). *Let the Assumption* B *be satisfied and $\widetilde{m}$ and $\{\widetilde{\nu}_x\}_{x \in \Omega}$ are the same as in Lemma* 3.5. *Suppose further that $\{U^r\}_{r \in \mathbb{N}}$ is a decreasing family of open subsets in γA such that $\overline{U}^{r+1} \subseteq U^r$ and $K = \bigcap_r U^r$. Then for every $f \in \mathcal{F}_A$ and every $l \in C(\Omega)$ there exists the subsequence of $\{u^\nu\}_{\nu \in \mathbb{N}}$ denoted by the same expression such that*

$$\lim_{r \to \infty} \left(\lim_{\nu \to \infty} \int_{\{x: u^\nu(x) \in A, \Phi(u^\nu(x)) \in U^r\}} l(x) f(u^\nu(x))\mu(dx) \right)$$
$$= \int_\Omega l(x) \int_K \widetilde{f}(\lambda)\widetilde{\nu}_x(d\lambda)\widetilde{m}(dx).$$

Now we prove the theorem. Substituting $|f|$ by f we can assume that f is nonnegative.

(a) Let $K \subseteq P \cap \operatorname{int} A_i$ be an arbitrary compact subset. As μ_x is a regular measure for μ-almost every x, it suffices to prove that $\int_K f(\lambda)\mu_x(d\lambda) = 0$ for μ-almost all x. This follows from the same arguments as in the proof of part i) in Theorem 3.1 given in [15] but instead of g we can substitute f.

(b) We apply Lemma 3.5 with $A = A_i$ and $\Phi = \Phi_i$. This implies the existence of measures $\widetilde{\nu}_x^i, \widetilde{m}^i$ such that

$$\int_{u^\nu(x)\in A_i} F(x, \Phi_i(u^\nu(x)))g(u^\nu)\mu(dx) \to \int_\Omega \int_{\gamma A_i} F(x,\lambda)\widetilde{\nu}_x^i(d\lambda)\widetilde{m}^i(dx),$$

whenever $F \in C(\Omega \times \gamma A_i)$.

Let $V_R^r := V^r \cap \{x : |x| < R + 1/r\}$. Then (3.1) holds true with $\{V_R^r\}_{r\in\mathbb{N}}$ used instead of $\{V^r\}_{r\in\mathbb{N}}$ and we also have $\overline{V}_R^{r+1} \subseteq V_R^r$.

Consider the following subsets of γA_i (see (2.1)):

$$U_R^r = \overline{\Phi_i(V_R^r \cap A_i)} \setminus \partial_{\gamma A_i}\overline{\Phi_i(V_R^r \cap A_i)}.$$

We observe that every set U_R^r is open in γA_i and $\overline{U}_R^{r+1} \subseteq U_R^r$ for every $r \in \mathbb{N}$. This is justified in Steps 1 and 2 in the proof of part ii) of Theorem 3.1 in [15]. Next, we show that

$$\lim_{r\to\infty}\left(\lim_{\nu\to\infty}\int_{\{x:u^\nu(x)\in A_i,\Phi_i(u^\nu(x))\in U_R^r\}} f(u^\nu(x))\mu(dx)\right) = 0. \tag{3.3}$$

To prove this at first we observe that $U_R^r \cap \Phi_i(A_i) \subseteq \Phi_i(V_R^{r-1} \cap A_i)$. This was shown in Step 3 in the proof of part ii) of Theorem 3.1 in [15]. Then we use condition (3.1).

Let $K_R = \bigcap_r U_R^r$. Lemma 3.6 and (3.3) give

$$\int_\Omega \int_{K_R} \widetilde{f}_i(\lambda)\widetilde{\nu}_x^i(d\lambda)\widetilde{m}^i(dx) = 0.$$

After letting $R \to \infty$ and using Lebesgue's Dominated Convergence Theorem we get

$$\int_{K_i^0} \widetilde{f}_i(\lambda)\widetilde{\nu}_x^i(d\lambda) = 0 \quad \widetilde{m}^i\text{-almost everywhere.} \tag{3.4}$$

Now we use change of variable formulae linking the measures $\widetilde{m}^i$, $\widetilde{\nu}_x^i$ with $\overline{m}^i$ and $\overline{\nu}_x^i$, see the formulae (18), (21) and (22) in the proof of Theorem 3.1 in [14]. We have

$$\overline{m}_i(dx) = w_i(x)p_i(x)\mu(dx),$$

where

$$w_i(x) = \int_{\Phi_i(\partial A_i \cap A_i)} \frac{1}{g(\Phi_i^{-1}(\lambda))}\widetilde{\nu}_x^i(d\lambda),$$

and p_i is the same as in Lebesgue–Nikodym decomposition of $\widetilde{m}^i$ with respect to μ, i.e. $\widetilde{m}^i = p_i(x)\mu + \widetilde{m}^i_s$,

$$(3.5)\qquad (G,\overline{\nu}^i_x) = \begin{cases} \dfrac{1}{w_i(x)} \displaystyle\int_{\Phi_i(\partial A_i\cap A_i)} (G/g)(\Phi_i^{-1}(\lambda))\widetilde{\nu}^i_x(d\lambda) & \text{if } w_i(x) > 0,\\ G(a) & \text{if } w_i(x) = 0, \end{cases}$$

for μ-almost all x, whenever $G \in C(\partial A_i \cap A_i)$, where $a \in \partial A_i \cap A_i$ can be chosen arbitrary.

We use the functions $G_\varepsilon \in C(\mathbb{R}^m)$ such that $0 \le G_\varepsilon \le 1$ and $G_\varepsilon \to \chi_P$, $G_\varepsilon \circ \Phi_i^{-1} \to \chi_{\Phi_i(P)}$ as $\varepsilon \to 0$ in the pointwise sense and plug $G_\varepsilon f$ to the formulae (3.5). This gives

$$\int_{\partial A_i\cap A_i} G_\varepsilon(\lambda) f(\lambda)\overline{\nu}^i_x(d\lambda) = \frac{1}{w_i(x)} \int_{\Phi_i(\partial A_i\cap A_i)} G_\varepsilon(\Phi_i^{-1}(\lambda))\widetilde{f}_i(\lambda)\widetilde{\nu}^i_x(d\lambda),$$

for μ-almost all x such that $w_i(x) \neq 0$. After letting $\varepsilon \to 0$ we obtain

$$(3.6)\qquad \int_{\partial A_i\cap A_i\cap P} f(\lambda)\overline{\nu}^i_x(d\lambda) = \frac{1}{w^i(x)} \int_{\Phi_i(\partial A_i\cap A_i)\cap\Phi_i(P)} \widetilde{f}(\lambda)\widetilde{\nu}^i_x(d\lambda),$$

for $\overline{m}^i$-almost all x.

Next we observe that $\Phi_i(P) \cap \Phi_i(A_i) \subseteq K_i^0$. This follows from the following sequence of inclusions

$$\begin{aligned} \Phi_i(P) \cap \Phi_i(A_i) &= \Phi_i(P \cap A_i)\\ &= \Phi_i\Big(\Big(\bigcup_R \bigcap_r \overline{V}^r_R\Big) \cap A_i\Big) = \bigcup_R \Phi_i\Big(\bigcap_r \overline{V}^r_R \cap A_i\Big)\\ &\subseteq \bigcup_R \bigcap_r \Phi_i(\overline{V}^r_R \cap A_i) \subseteq \bigcup_R \bigcap_r \overline{\Phi_i(V^r_R \cap A_i)} = K_i^0. \end{aligned}$$

Therefore according to (3.4) the right hand in (3.6) is zero. This justifies (b).

(c) We use the following formulae linking measures m^i, ν^i_x and $\widetilde{m}^i$, $\widetilde{\nu}^i_x$ (see (18)–(20) in the proof of Theorem 3.1 in [14])

$$\begin{aligned} &m^i(dx) = h_i(x)\widetilde{m}^i(dx), \quad \text{where } h_i(x) = \widetilde{\nu}^i_x(\gamma A_i \setminus A_i),\\ (3.7)\qquad &(G,\nu^i_x) = \begin{cases} \dfrac{1}{h_i(x)} \displaystyle\int_{\gamma A_i\setminus A_i} G(\lambda)\widetilde{\nu}^i_x(d\lambda) & \text{if } h_i(x) \neq 0,\\ G(y) & \text{if } h_i(x) = 0, \end{cases} \end{aligned}$$

for $\widetilde{m}^i$ almost all x, where $y \in \gamma A_i \setminus A_i$ can be chosen arbitrary and $G \in C(\gamma A_i \setminus A_i)$.

Let us take such x that $h_i(x) > 0$. Formulae (3.7) gives

$$(3.8)\qquad \int_{\gamma A_i\setminus A_i} G(\lambda)\widetilde{f}_i(\lambda)\nu^i_x(d\lambda) = \frac{1}{h_i(x)} \int_{\gamma A_i\setminus A_i} G(\lambda)\widetilde{f}_i(\lambda)\widetilde{\nu}^i_x(d\lambda),$$

whenever $G \in C(\gamma A_i)$. As the reminder $\gamma A_i \setminus A_i$ is closed, Tietze's extension theorem (see e.g. [8]) implies that identity (3.8) folds for every $G \in C(\gamma A_i \setminus A_i)$.

Let x be such that $h_i(x) \neq 0$ and define two measures on $\gamma A_i \setminus A_i$:

$$\alpha = \widetilde{f}_i \nu_x^i \quad \text{and} \quad \beta = \frac{1}{h_i(x)} \widetilde{f}_i \widetilde{\nu}_x^i \angle(\gamma A_i \setminus A_i),$$

where the symbol $\mu\angle R$ stands for the restriction of the measure $\mu \in \mathcal{M}(S)$ to the Borel subset $R \subseteq S$, i.e. $\mu\angle R(A) = \mu(R \cap A)$. Identity (3.8) applied to every $G \in C(\gamma A_i \setminus A_i)$ gives

$$\int_{\gamma A_i \setminus A_i} G(\lambda)\alpha(d\lambda) = \int_{\gamma A_i \setminus A_i} G(\lambda)\beta(d\lambda).$$

Hence $\alpha = \beta$ as functionals on $C(\gamma A_i \setminus A_i)$. Therefore by Riesz Theorem we see that $\alpha = \beta$ as regular measures on $\gamma A_i \setminus A_i$. In particular $\alpha((\gamma A_i \setminus A_i) \cap K_i^0) = \beta((\gamma A_i \setminus A_i) \cap K_i^0)$, which reads as

$$\int_{(\gamma A_i \setminus A_i) \cap K_i^0} \widetilde{f}_i(\lambda)\nu_x^i(d\lambda) = \frac{1}{h_i(x)} \int_{(\gamma A_i \setminus A_i) \cap K_i^0} \widetilde{f}_i(\lambda)\widetilde{\nu}_x^i(d\lambda).$$

According to (3.4) the right hand side above is 0 for $\widetilde{m}_i$-almost all x. As the set $\{x \in \Omega: h_i(x) = 0\}$ is of m^i measure 0, therefore the statement (c) follows. □

Proof of Theorem 3.2.. It follows the same line as the proof of Theorem 2.7 presented in [15]) (see the proof of Theorem 3.2 on page 1321). The only difference is that we have to substitute g by f. Therefore we omit it. □

Proof of Theorem 3.3.. We use the fact that $K_i \cap R_i^b = K_i^0 \cap R_i^b$ (see Remark 4.2 in [15]) and the statement (c) in Theorem 3.1 which asserts that for m^i-almost all x

$$\int_{K_i^0 \cap R_i^b} \widetilde{|f|}_i(\lambda)\nu_x^i(d\lambda) = 0. \qquad \square$$

4. Compactness criterion

Our goal now is to obtain the following theorem.

Theorem 4.1. *Suppose that $\Omega \subseteq \mathbb{R}^n$ is the compact set equipped with the Radon measure μ and that Assumption* A2 *with bricks $\{A_i\}_{i=1,\dots,k}$ and density function $g: \mathbb{R}^m \to (0,\infty)$ is satisfied. Assume further that there is given the sequence $\{u^\nu\}_{\nu\in\mathbb{N}}$ of μ-measurable functions, $u^\nu: \Omega \to \mathbb{R}^m$ such that*

$$\sup_{\nu\in\mathbb{N}} \int_{\mathbb{R}^m} g(u^\nu)\mu(dx) < \infty \tag{4.1}$$

and $\{u^\nu\}_{\nu\in\mathbb{N}}$ generates measures

$$\{\mu_x\}_{x\in\Omega}, \quad (\overline{m}^i, \{\overline{\nu}_x^i\}_{x\in\Omega})_{i=1,\dots,k}, \quad (m^i, \{\nu_x^i\}_{x\in\Omega})_{i=1,\dots,k}$$

(*see Definition* 2.5). *Then for an arbitrary $f \in \mathcal{F}$ the following conditions are equivalent to each other:*

(a) *The sequence $\{f(u^\nu)\}_{\nu\in\mathbb{N}}$ is weakly compact in $L^1(\Omega,\mu)$.*

(b) *For every* $i \in \{1, \ldots, k\}$*, we have*

$$A_\varepsilon^i = \lim_{\nu \to \infty} \int_{\{x \in \Omega : u^\nu(x) \in A_i : \mathrm{dist}(\Phi_i(u^\nu(x)), R_i^\infty) < \varepsilon\}} |f(u^\nu(x))| \mu(dx) \xrightarrow{\varepsilon \to 0} 0. \tag{4.2}$$

(c) *For every* $i \in \{1, \ldots, k\}$ *we have*

$$\int_\Omega \int_{R_i^\infty} \widetilde{|f|}_i(\lambda) \nu_x^i(d\lambda) m^i(dx) = 0. \tag{4.3}$$

Proof. Substituting f by $|f|$ we may assume that f is nonnegative.

(b) $\Rightarrow$ (a) It suffices to show that for every $i \in \{1, \ldots, k\}$ two conditions in Dunford–Pettis criterion

$$\sup_\nu \int_{\{u^\nu \in A_i\}} f(u^\nu(x)) \mu(dx) < \infty,$$

and

$$\lim_{l \to \infty} \left(\limsup_{\nu \to \infty} \int_{\{u^\nu \in A_i, f(u^\nu(x)) > l\}} f(u^\nu(x)) \mu(dx) \right) = 0$$

are satisfied.

The first condition follows from inequality $f(\lambda) \le g(\lambda) \|f/g\|_{L^\infty(A_i)}$ and assumption (4.1). To check the second condition we take an arbitrary $\varepsilon > 0$, and let

$$D_\nu^l = \{x : u^\nu(x) \in A_i, f(u^\nu(x)) > l\}.$$

Then $D_\nu^l = B_{\nu,\varepsilon}^l \cup C_{\nu,\varepsilon}^l$ where

$$\begin{aligned} B_{\nu,\varepsilon}^l &= \{x \in D_\nu^l : \mathrm{dist}(\Phi_i(u^\nu(x)), R_i^\infty) < \varepsilon\}, \\ C_{\nu,\varepsilon}^l &= \{x \in D_\nu^l : \mathrm{dist}(\Phi_i(u^\nu(x)), R_i^\infty) \ge \varepsilon\}. \end{aligned}$$

It suffices to prove that both expressions:

$$I_\varepsilon^l = \limsup_{\nu \to \infty} \int_{B_{\nu,\varepsilon}^l} f(u^\nu(x)) \mu(dx) \quad \text{and} \quad II_\varepsilon^l = \limsup_{\nu \to \infty} \int_{C_{\nu,\varepsilon}^l} f(u^\nu(x)) \mu(dx)$$

can be arbitrary small if we take l is large enough. Let us take an arbitrary $\delta > 0$. Using assumption (b) we find $\varepsilon = \varepsilon_0$ small enough, so that $I_{\varepsilon_0}^l < \delta$. The second expression vanishes for k large enough. Indeed, let us define

$$\mathcal{A}_\varepsilon^i = \{\lambda \in A_i \colon \mathrm{dist}(\Phi_i(\lambda), R_i^\infty) \ge \varepsilon\}.$$

We show that there exists a constant C_ε such that

$$g(\lambda) \le C_\varepsilon \quad \text{if } \lambda \in \mathcal{A}_\varepsilon^i. \tag{4.4}$$

To see that we use the continuity of the function $\widetilde{\chi}_{A_i}$ extending the mapping $1/g(\Phi_i^{-1}(\lambda)) \colon \Phi_i(A_i) \to \mathbb{R}$ to γA_i and prove by contradiction that there exists the number $L_\varepsilon > 0$ such that $G(m) \geq L_\varepsilon$ for every $m \in \{\overline{m} \in \gamma A_i : \mathrm{dist}(\overline{m}, R_i^\infty) \geq \varepsilon\}$. This gives (4.4). Let us define

$$E_\varepsilon^l = \{\lambda \in A_i : f(\lambda) > l,\ \mathrm{dist}(\Phi_i(\lambda), R_i^\infty) \geq \varepsilon\}.$$

The condition (4.4) implies that for every $\lambda \in E_\varepsilon^l$ we have

$$l < f(\lambda) \leq \|f/g\|_{L^\infty(A_i)} g(\lambda) \leq C_\varepsilon \|f/g\|_{L^\infty(A_i)} := b_\varepsilon.$$

Therefore $E^l = \emptyset$ and also $C_{\nu,\varepsilon}^l = \emptyset$ for every ν if we take $l > b_\varepsilon$. Therefore $II_{\varepsilon_0}^l = 0$ for every $l > b_{\varepsilon_0}$ and property (a) follows.

(a) $\Rightarrow$ (b) Continuity of the function $\widetilde{\chi}_{A_i}$ (defined on γA_i) implies the following condition:

for all $l \in \mathbb{N}$ there exists $\varepsilon_l > 0$ such that
for all $\lambda \in A_i$ if $\mathrm{dist}(\Phi_i(\lambda), R_i^\infty) < \varepsilon_l$ then $g(\lambda) > l$.

We may assume that the sequence $\{\varepsilon_l\}_{l \in \mathbb{N}}$ defined above converges to 0 as $l \to \infty$. Let us denote

$$B_l = \{\lambda \in A_i : \mathrm{dist}(\Phi_i(\lambda), R_i^\infty) < \varepsilon_l\}.$$

Then $B_l \subseteq \{\lambda \in A_i : g(\lambda) > l\} := C_l$. Therefore

$$A_{\varepsilon_l}^i = \int_{\{x : u^\nu(x) \in B_l\}} f(u^\nu(x))\mu(dx) \leq \int_{\{x : u^\nu(x) \in C_l\}} f(u^\nu(x))\mu(dx) \tag{4.5}$$

and Chebyshev inequality yields

$$\begin{aligned} \sup_\nu \mu(\{x : u^\nu(x) \in C_l\}) &\leq \sup_\nu \mu(\{x \in \Omega : g(u^\nu(x)) > l\}) \\ &\leq \frac{1}{l} \sup_\nu \int_\Omega g(u^\nu)\mu(dx) \xrightarrow{l \to \infty} 0. \end{aligned}$$

Uniform integrability condition for the sequence $\{f(u^\nu)\}_{\nu \in \mathbb{N}}$ implies that the supremum over ν on the right hand side of (4.5) converges to 0 as $l \to \infty$ and property (4.2) follows.

(c) $\Leftrightarrow$ (b) Denote by B the left hand side in (4.3). Let $h_\varepsilon : A_i \to [0,1]$ be defined by

$$h_\varepsilon(\lambda) = \begin{cases} 0 & \text{if } \mathrm{dist}(\Phi_i(\lambda), R_i^\infty) > \varepsilon, \\ -2\varepsilon^{-1}\mathrm{dist}(\Phi_i(\lambda), R_i^\infty) + 2 & \text{if } \mathrm{dist}(\Phi_i(\lambda), R_i^\infty) \in [\varepsilon/2, \varepsilon], \\ 1 & \text{if } \mathrm{dist}(\Phi_i(\lambda), R_i^\infty) < \varepsilon/2, \end{cases}$$

and consider

$$f_\varepsilon(\lambda) = f(\lambda) h_\varepsilon(\lambda) \chi_{A_i}(\lambda).$$

Then $f_\varepsilon \in \mathcal{F}$ and according to Theorem 2.4 the sequence $\{f_\varepsilon(u^\nu(x))\mu(dx)\}_{\nu\in\mathbb{N}}$ converges weakly $*$ in measures to the measure

$$\begin{aligned} B_\varepsilon(dx) = \int_{\operatorname{int} A_i} & f_\varepsilon(\lambda)\mu_x(d\lambda)\mu(dx) \\ & + \int_{\partial A_i \cap A_i} f_\varepsilon(\lambda)\overline{\nu}^i_x(d\lambda)p_i(x)\mu(dx) + \int_{\gamma A_i \setminus A_i} \widetilde{f}_\varepsilon(\lambda)\nu^i_x(d\lambda)m^i(dx), \end{aligned}$$

where $\widetilde{f}_\varepsilon \in C(\gamma A_i)$ is the unique continuous extension of $(f_\varepsilon/g)\circ\Phi_i^{-1}$ defined on $\Phi_i(A_i)$ and $\overline{m}^i(dx) = p_i(x)\mu(dx)$. Note that

$$h_\varepsilon(\lambda) \le \chi_{\{\lambda\in A_i:\operatorname{dist}(\Phi_i(\lambda),R_i^\infty)<\varepsilon\}} \le h_{2\varepsilon}(\lambda).$$

Therefore for μ-almost every x

$$f_\varepsilon(u^\nu(x)) \le (f\chi_{\{\lambda\in A_i:\operatorname{dist}(\Phi_i(\lambda),R_i^\infty)<\varepsilon\}})(u^\nu(x)) \le f_{2\varepsilon}(u^\nu(x)).$$

After integrating this over Ω and letting $\nu\to\infty$ we get $B_\varepsilon(\Omega) \le A^i_\varepsilon \le B_{2\varepsilon}(\Omega)$.

Now we let $\varepsilon \to 0$ and observe that

$$f_\varepsilon \to 0, \quad f_\varepsilon \le f \quad \text{and} \quad \widetilde{f}_\varepsilon \to \widetilde{f}_i\chi_{R_i^\infty}, \quad \widetilde{f}_\varepsilon \le \|\widetilde{f}_i\|_{L^\infty(\gamma A_i)}.$$

Lebesgue Dominated Convergence Theorem yields

$$\lim_{\varepsilon\to0} A^i_\varepsilon = \lim_{\varepsilon\to0} B_\varepsilon(\Omega) = \int_\Omega\int_{R_i^\infty} \widetilde{f}_i\nu_x(d\lambda)m(dx) = B.$$

The above identity shows that (a) is equivalent to (c). □

As an immediate corollary we obtain the following result.

Corollary 4.2. *Let the assumptions of Theorem* 4.1 *be satisfied. Then the following conditions are equivalent to each other:*

(a) *The sequence $\{g(u^\nu)\}_{\nu\in\mathbb{N}}$ is weakly compact in $L^1(\Omega,\mu)$.*
(b) *For every $i\in\{1,\dots,k\}$ and for m^i-almost all x we have $\nu^i_x(R_i^\infty) = 0$ and m^i is absolutely continuous with respect to μ.*

Proof. We plug $f = g$ in Theorem 4.1 and use part (a) of Theorem 2.8. □

Remark 4.3. In the case $k = 1$ (i.e. $A_1 = \mathbb{R}^m$) when $\mathbb{R}^m$ is compactified by adding the sphere and $g(\lambda) = 1 + |\lambda|$ the result was was obtained in part (ii) in Theorem 2.9 in [1]. For $k = 1$ and an arbitrary compactification of $\mathbb{R}^m$ with $g(\lambda) = 1 + |\lambda|^p$, $1 \le p < \infty$ this is Lemma 3.2.14 in [27]. The other cases were not discussed.

5. The oscillation effects

Our goal now is to obtain the following generalization of Young's theorem.

THEOREM 5.1. *Suppose that $\Omega \subseteq \mathbb{R}^n$ is the compact set equipped with the Radon measure μ and Assumption* A2 *with bricks $\{A_i\}$ and density function g on $\mathbb{R}^m$ is satisfied. Assume further that there is given the sequence $\{u^\nu\}_{\nu\in\mathbb{N}}$ of μ-measurable functions, $u^\nu\colon \Omega \to \mathbb{R}^m$ such that*

$$\sup_\nu \int_{\mathbb{R}^m} g(u^\nu)\mu(dx) < \infty \quad \text{and} \quad u^\nu(x) \to L \text{ in a measure } \mu,$$

where $L \subseteq \mathbb{R}^m$ is given closed subset. Then there exist

(a) *a subsequence of $\{u^\nu\}_{\nu\in\mathbb{N}}$ denoted by the same expression,*
(b) *the functions $p_i(x), q_i(x) \in L^1(\mu)$ where $i \in \{1,\dots,k\}$,*
(c) *families of probability measures $\{\mu_x\}_{x\in\Omega} \in \mathcal{P}(\Omega,\mathbb{R}^m,\mu)$, $\{\overline{\nu}^i_x\}_{x\in\Omega} \in \mathcal{P}(\Omega,\partial A_i\cap A_i,\mu)$ and $\{\nu^i_x\}_{x\in\Omega} \in \mathcal{P}(\Omega,\gamma A_i\setminus A_i,\mu)$ where $i \in \{1,\dots,k\}$,*

such that if $f\in\mathcal{F}$ and the sequence $\{f(u^\nu(x))\}_{\nu\in\mathbb{N}}$ is weakly compact in $L^1(\Omega,\mu)$ then it weakly converges in $L^1(\Omega,\mu)$ to

$$\overline{f}(x) = \sum_{i=1}^k \bigg(\int_{L\cap \operatorname{int} A_i} f(\lambda)\mu_x(d\lambda) + p_i(x)\int_{L\cap\partial A_i\cap A_i} f(\lambda)\overline{\nu}^i_x(d\lambda) + q_i(x)\int_{X^i}\widetilde{f}_i(\lambda)\nu^i_x(d\lambda)\bigg), \tag{5.1}$$

where $\widetilde{f}_i$ is defined by (2.3), $X^i = R^b_i \setminus \bigcup_{\varepsilon>0}\overline{\Phi_i(L^c_\varepsilon\cap A_i)}$, $L_\varepsilon = \{x\in\mathbb{R}^m : \operatorname{dist}(x,L)\le\varepsilon\}$. Moreover, the measures $\{\mu_x\}_{x\in\Omega}$, $\{\overline{\nu}^i_x\}_{x\in\Omega}$ and $\{\nu^i_x\}_{x\in\Omega}$ are the same as that in Theorem 2.4 *and the measures $\overline{m}^i$ and m^i in Theorem* 2.4 *are linked with p_i, q_i by:*

$$\overline{m}^i = p_i(x)\mu, \qquad m^i = q_i(x)\mu + m^i_s,$$

where m^i_s is the singular part in the Lebesgue's–Nikodym decomposition of m^i.

PROOF. Let $f\in\mathcal{F}$ and assume at first that the sequence $\{f(u^\nu)\}_{\nu\in\mathbb{N}}$ is weakly compact in $L^1(\Omega,\mu)$. Using Theorem 2.4 we may extract the subsequence (also denoted by $\{u^\nu\}_{\nu\in\mathbb{N}}$) such that $\{f(u^\nu)\mu(dx)\}_{\nu\in\mathbb{N}}$ converges weakly $*$ in the space of measures to the measure

$$M = \sum_{i=1}^k \bigg(\int_{\operatorname{int} A_i} f(\lambda)\mu_x(d\lambda)\mu(dx) + \int_{\partial A_i\cap A_i} f(\lambda)\overline{\nu}^i_x(d\lambda)\overline{m}^i(dx) + \int_{\gamma A_i\setminus A_i}\widetilde{f}_i(\lambda)\nu^i_x(d\lambda)m^i(dx)\bigg) = \sum_{i=1}^k(\mathcal{A}_i+\mathcal{B}_i+\mathcal{C}_i).$$

It suffices to show that $M = \overline{f}\mu$.

As $\overline{m}^i$ is absolutely continuous with respect to μ, it represents as

$$\overline{m}^i = p_i(x)\mu, \tag{5.2}$$

with some $p_i \in L^1(\Omega, \mu)$. Let us assume that $L \neq \mathbb{R}^m$ and define

$$V^i_{r,s} = \mathbb{R}^m \setminus L_{1/s-1/r} = L^c_{1/s-1/r} \quad \text{for } r > s.$$

We easily check that $\overline{V}^i_{r+1,s} \subseteq V^i_{r,s}$. Moreover, the condition $u^\nu \to L$ in a measure implies that for every $i \in \{1, \ldots, k\}$ and for every $r, s \in \mathbb{N}$ such that $r > s$

$$\lim_{\nu\to\infty} \mu(\{x : u^\nu(x) \in V_{r,s} \cap A_i\}) = 0.$$

This and uniform integrability of the sequence $\{f(u^\nu)\}_{\nu\in\mathbb{N}}$ implies

$$B^i_r = \lim_{\nu\to\infty} \int_{\{x\in\Omega: u^\nu(x)\in V_{r,s}\cap A_i\}} |f(u^\nu(x))|\mu(dx) = 0 \quad \text{as} \quad r \to \infty.$$

According to parts (a) and (b) of Theorem 3.1, for

$$P_s := \bigcap_{r:r>s} V^i_{r,s} = \{x \in \mathbb{R}^m : \operatorname{dist}(x, L) \geq 1/s\}$$

and every $i \in \{1, \ldots, k\}$

$$\begin{aligned} \int_{P_s\cap(\bigcup_i \operatorname{int} A_i)} |f(\lambda)|\mu_x(d\lambda) &= 0, \quad \text{for } \mu\text{-almost all } x,\\ \int_{P_s\cap\partial A_i\cap A_i} |f(\lambda)|\overline{\nu}^i_x(d\lambda) &= 0 \quad \text{for } \overline{m}^i\text{-almost all } x. \end{aligned} \tag{5.3}$$

As $\bigcup_s P_s = \mathbb{R}^m \setminus L$, we can substitute

$$\mathcal{A}_i = \int_{L\cap\operatorname{int} A_i} f(\lambda)\mu_x(d\lambda)\mu(dx) \quad \text{and} \quad \mathcal{B}_i = p_i(x)\int_{L\cap\partial A_i\cap A_i} f(\lambda)\overline{\nu}^i_x(d\lambda)\mu(dx).$$

The above identities remain valid also if $L = \mathbb{R}^m$. Now we deal with the expression $\mathcal{C}_i$. Using Theorem 4.1 we recognize that, for every $i \in \{1, \ldots, k\}$,

$$\biggl(\int_{R^\infty_i} |\widetilde{f}_i|(\lambda)\nu^i_x(d\lambda)\biggr)m^i(dx) = 0.$$

Therefore

$$\mathcal{C}_i = \biggl(\int_{R^b_i} \widetilde{f}_i(\lambda)\nu^i_x(d\lambda)\biggr)m^i(dx).$$

The statement (a) in Theorem 2.8 asserts that for m^i_s almost all x the measure ν^i_x is supported in R^∞_i. Therefore

$$\mathcal{C}_i = q_i(x)\biggl(\int_{R^b_i} \widetilde{f}_i(\lambda)\nu^i_x(d\lambda)\biggr)\mu(dx),$$

where $q_i \in L^1(\Omega, \mu)$ is such that

$$m^i = q_i\mu + m^i_s \tag{5.4}$$

in the Lebesgue's–Nikodym decomposition of m^i. Let

$$K_i^s = \bigcap_{r:r>s} \overline{\Phi_i(V_{r,s}^i \cap A_i)}.$$

Theorem 3.3 implies that for every $i = 1, \dots, k$ and m^i-almost all x

$$\int_{K_i^s \cap R_b^i} \widetilde{|f|}_i(\lambda)\nu_x^i(d\lambda) = 0.$$

Consequently

$$\int_{\bigcup_s K_i^s \cap R_b^i} \widetilde{|f|}_i(\lambda)\nu_x^i(d\lambda) = 0, \tag{5.5}$$

for m^i-almost all x, whenever $i \in \{1, \dots, k\}$. Therefore

$$\mathcal{C}_i = q_i(x)\biggl(\int_{\widetilde{X}^i} \widetilde{f}_i(\lambda)\nu_x^i(d\lambda)\biggr)\mu(dx),$$

where

$$\widetilde{X}^i = R_i^b \setminus \bigcup_s \bigcap_{r:r>s} \overline{\Phi_i(L_{1/s-1/r}^c \cap A_i)}.$$

Inequalities $1/(s(s+1)) \le 1/s - 1/r \le 1/s$ satisfied for every $r > s$ imply

$$\overline{\Phi_i(L_{1/(s(s+1))}^c \cap A_i)} \supseteq \overline{\Phi_i(L_{1/s-1/r}^c \cap A_i)} \supseteq \overline{\Phi_i(L_{1/s}^c \cap A_i)}.$$

Therefore for $Y = \bigcup_{\varepsilon>0} \overline{\Phi_i(L_\varepsilon^c \cap A_i)}$ we get

$$\begin{aligned} Y &= \bigcup_s \overline{\Phi_i(L_{1/(s(s+1))}^c \cap A_i)} \\ &\supseteq \bigcup_s \bigcap_{r:r>s} \overline{\Phi_i(L_{1/s-1/r}^c \cap A_i)} \supseteq \bigcup_s \overline{\Phi_i(L_{1/s}^c \cap A_i)} = Y. \end{aligned} \tag{5.6}$$

Hence $\widetilde{X}^i = R_i^b \setminus Y = X^i$, which implies (5.1). The last statement follows from (5.2) and (5.4). $\square$

Remark 5.2. The set X^i consists of elements $x \in R_i^b$ that if $\{y_l\}_{l\in\mathbb{N}} \subseteq A_i$ is such that $\Phi_i(y_l) \to x$ as $l \to \infty$, then limit of $\{y_l\}_{l\in\mathbb{N}}$ belongs to $L \cap \overline{A_i}$. From the very definition of R_i^b every such a sequence $\{y_l\}_{l\in\mathbb{N}}$ is bounded. Therefore according to the condition (2.2) the limit of $\{y_l\}_{l\in\mathbb{N}}$ exists.

Our next result deals with supports of the resulting measures, provided that the given sequence converges to the closed set L in the measure.

Theorem 5.3. *Let the assumptions of Theorem* 5.1 *be satisfied and the quantities $\{\mu_x\}_{x\in\Omega}$, $\{\overline{\nu}_x^i\}_{x\in\Omega}$, $\{\nu_x^i\}_{x\in\Omega}$, p_i, q_i and X^i be as in its statement. Then we have*

(a) $\operatorname{supp}\mu_x \subseteq L$ *for μ-almost every x;*
(b) $\operatorname{supp}\overline{\nu}_x^i \subseteq L \cap \partial A_i \cap A_i$ *for $p_i(x)\mu$ almost every x;*
(c) $\operatorname{supp}\nu_x^i \subseteq (X^i \cap R_i^b) \cup R_i^\infty$ *for $q_i(x)\mu$-almost every x.*

Proof. The first statement is the known property of Young measures (see e.g. Theorem 1 in [12]). To prove the second statement we recognize that the function $f \equiv 1$ belongs to $\mathcal{F}$ and for that the sequence $\{f(u^\nu)\}$ is weakly compact in $L^1(\Omega)$. Taking into account (5.3) applied with $f \equiv 1$ we get $\overline{\nu}^i_x((\partial A_i \cap A_i)\setminus L) = 0$ for $\overline{m}^i$ almost every x, which implies the thesis. To prove the last statement we deduce from (5.5) and (5.6) that

$$\int_{R_i^b \setminus X^i} |\widetilde{f}_i|(\lambda)\nu^i_x(d\lambda) = 0,$$

for every $i = 1,\dots,k$ and for m^i-almost every x, if only $\{f(u^\nu)\}$ is weakly compact in $L^1(\Omega)$. We substitute $f \equiv 1$ and note that for that f the function $\widetilde{f}_i = 1/g \circ \Phi_i^{-1}$ is strictly positive on R_i^b. Therefore $\nu^i_x(R_i^b \setminus X^i) = 0$ for m^i-almost every x, in particular also for $q_i(x)\mu$-almost every x. □

6. Illustrations

In this section we illustrate Theorems: 2.4, 3.1–3.3, 4.1, 5.1 and 5.3 on examples. We start with the illustration of Theorem 2.4.

Example 6.1 (Illustration of Theorem 2.4). Let $k = 2$, $m = 2$, $A_1 = \mathbb{R}^2 \setminus \{(0,0)\}$, $A_2 = \{(0,0)\}$, $\Phi_1(x) := x/(1+|x|)$, $\Phi_2(x) = x$. In particular $\Phi_1 : \mathbb{R}^2 \setminus \{(0,0)\} \to \overline{B(1)} \subseteq \mathbb{R}^2$ is an embedding into a unit ball: $\gamma A_1 = \overline{B(1)}$ and $\gamma A_1 \setminus A_1 = S^1 \cup \{(0,0)\}$ is the reminder (where S^1 is the unit sphere in $\mathbb{R}^2$).

Consider $\Omega = [0,1]$, let $\mu = dx$ be the Lebesgue measure and $g(\lambda) = 1 + |\lambda|$. Define the sequence $u^\nu : \Omega \to \mathbb{R}^2$ by the formulae:

$$u^\nu(x) = \begin{cases} -\nu e_1 & \text{for } x \in \left[0, \dfrac{1}{1+\nu}\right), \\ \nu e_1 & \text{for } x \in \left[\dfrac{1}{1+\nu}, \dfrac{2}{1+\nu}\right), \\ \dfrac{1}{\nu} e_1 & \text{for } x \in \left[\dfrac{2}{1+\nu}, 1\right]. \end{cases} \tag{6.1}$$

Let us compute the resulting measures

$$\{\mu_x\}_{x\in\Omega}, \quad (\overline{m}^i, \{\overline{\nu}^i_x\}_{x\in\Omega}), \quad (m^i, \{\nu^i_x\}_{x\in\Omega})$$

generated by the sequence where $i = 1,2$. We start with the computation of the Young measure $\{\mu_x\}_{x\in\Omega}$. An easy computation gives that for every $g \in L^1(0,1)$ and $f \in C_0(\mathbb{R}^2)$ we have

$$\int_{[0,1]} g(x) f(u^\nu(x))\,dx = \int_{[0,1/(1+\nu))} g(x) f(-\nu e_1)\,dx \\ + \int_{[1/(1+\nu),2/(1+\nu))} g(x) f(\nu e_1)\,dx + \int_{[2/(1+\nu),1]} g(x) f\left(\frac{1}{\nu} e_1\right)dx.$$

This converges to $\int_{[0,1]} g(x)f((0,0))\,dx$, whatever g and f we take. We conclude that

$$\mu_x = \delta_{(0,0)} \quad \text{(Dirac delta concentrated at } (0,0)\text{), for every } x \in [0,1]. \tag{6.2}$$

To proceed further we need to explain the structure of set $\mathcal{F}$ in Assumption A1. As $\Phi_1^{-1}(y) = y/(1-|y|)$, we need to explain when the function

$$\widetilde{f}_1(y) = f\left(\frac{y}{1-|y|}\right)(1-|y|)\colon B(1)\setminus\{(0,0)\} \to \mathbb{R}$$

has continuous extension to $\overline{B(1)}$. At first we note that the extension of this function to zero is equivalent to the fact that $f|_{\mathbb{R}^2\setminus\{(0,0)\}}$ has limit at $(0,0)$. Moreover, the radial limits at ∞: $\lim_{t\to\infty} f(t\theta)/t =: f^\infty(\theta)$ exist and define the continuous function on the sphere S^1.

Now we compute the resulting measures for $i = 1$. For this we proceed similarly as in Example 3.1 in [15]. As $\partial A_1 \cap A_1 = \emptyset$, it follows that measures $\overline{m}^1$, $\overline{\nu}_x^1$ will not appear in the representation formulae (2.5). Let us take an arbitrary $h \in C([0,1])$ and $f \in \mathcal{F}$ such that $f((0,0)) = 0$. Then we have

$$\begin{aligned} I^\nu &= \int_{[0,1]} h(x)f(u^\nu(x))\,dx \\ &= \int_{[0,1/(1+\nu))} h(x)f(-\nu e_1)\,dx + \int_{[1/(1+\nu),2/(1+\nu))} h(x)f(\nu e_1)\,dx \\ &\quad + \int_{[2/(1+\nu),1]} h(x)f\left(\frac{1}{\nu}e_1\right)dx =: I_1^\nu + I_2^\nu + I_3^\nu. \end{aligned}$$

As $f = (f/g)\cdot g$ and $g(-\nu e_1) = 1+\nu =: 1/k_\nu$, we compute that

$$I_1^\nu = \frac{f(\Phi_1^{-1}(z_\nu))}{g(\Phi_1^{-1}(z_\nu))}\frac{1}{k_\nu}\int_{[0,k_\nu)} h(x)\,dx = \widetilde{f}_1(z_\nu) ⨍_{[0,k_\nu)} h(x)\,dx,$$

where $z_\nu = -\nu e_1/(1+\nu) = \Phi_1(-\nu e_1) \to -e_1$ as $\nu \to \infty$ (here symbol $⨍_A$ denotes the barintegral over A). By the continuity of the function $\widetilde{f}_1$ we get $I_1^\nu \to \widetilde{f}_1(-e_1)h(0)$ as $\nu \to \infty$.

By similar arguments

$$I_2^\nu = \widetilde{f}_1(w_\nu) ⨍_{[k_\nu,2k_\nu)} h(x)\,dx \xrightarrow{\nu\to\infty} \widetilde{f}_1(e_1)h(0), \quad \text{where } w_\nu = \frac{\nu e_1}{1+\nu} \xrightarrow{\nu\to\infty} e_1,$$

$$I_3^\nu = \widetilde{f}_1(r_\nu)\left(1+\frac{1}{\nu}\right)\int_{[2k_\nu,1]} h(x)\,dx \xrightarrow{\nu\to\infty} \widetilde{f}_1((0,0))\int_{[0,1]} h(x)dx,$$

$$\text{where } r_\nu = \frac{(1/\nu)e_1}{1+1/\nu} \xrightarrow{\nu\to\infty} (0,0).$$

Therefore

$$I^\nu \xrightarrow{\nu\to\infty} \widetilde{f}_1((0,0))\int_{[0,1]} h(x)\,dx + h(0)(\widetilde{f}_1(-e_1)+\widetilde{f}_1(e_1)) := I \tag{6.2}$$

According to Theorem 2.6

$$\begin{aligned} I &= \int_{[0,1]} h(x)\Big(\int_{\mathbb{R}^2\setminus\{(0,0)\}} f(\lambda)\mu_x(d\lambda)\Big)dx \\ &\quad + \int_{[0,1]} h(x)\Big(\int_{S^1\cup\{(0,0)\}} \widetilde{f}_1(\lambda)\nu_x^1(d\lambda)\Big)m^1(dx) \\ &= \int_{[0,1]} h(x)\Big(\int_{S^1\cup\{(0,0)\}} \widetilde{f}_1(\lambda)\nu_x^1(d\lambda)\Big)m^1(dx), \end{aligned} \tag{6.4}$$

where the last equation follows from (6.2). Taking into account (6.3) and (6.4) we get

$$\begin{aligned} &\widetilde{f}_1((0,0))\int_{[0,1]} h(x)\,dx + 2h(0)\frac{1}{2}\Big(\widetilde{f}_1(-e_1)+\widetilde{f}_1(e_1)\Big) \\ &\qquad = \int_{[0,1]} h(x)\Big(\int_{S^1\cup\{(0,0)\}} \widetilde{f}_1(\lambda)\nu_x^1(d\lambda)\Big)m^1(dx), \end{aligned} \tag{6.5}$$

no matter what $f\in\mathcal{F}$ (such that $f((0,0))=0$) and $h\in C([0,1])$ we take. This allows to define ν_x^1 uniquely. Namely, taking $f=g\chi_{A_1}$, we recognize that $\widetilde{f}_1\equiv 1$, so the left hand side above reads as $\int_{[0,1]} h(x)\,dx + 2h(0)$, while the right hand side is the same as $\int_{[0,1]} h(x)m^1(dx)$. This and (6.5) implies

$$\begin{aligned} m^1(dx) &= dx + 2\delta_{(0)}, \\ \nu_x^1 &= \delta_{(0,0)} \qquad\qquad \text{for } \mathcal{L}^1 \text{ almost all } x\in(0,1], \\ \nu_0^1 &= \frac{1}{2}\delta_{e_1} + \frac{1}{2}\delta_{-e_1}. \end{aligned} \tag{6.6}$$

Now let us compute the measures: $(\overline{m}^2, \{\overline{\nu}_x^2\}_{x\in\Omega})$, $(m^2,\{\nu_x^2\}_{x\in\Omega})$ generated by $\{u^\nu\}_{\nu\in\mathbb{N}}$. This case is simpler. As $int A_2=\emptyset$ and $\gamma A_2\setminus A_2=\emptyset$, only the measures: $\overline{\nu}_x^2$ and $\overline{m}^2 = p_2(x)\,dx$ can appear in the representation formulae. Moreover, $\partial A_2\cap A_2 = \{(0,0)\}$, therefore we get: $\overline{\nu}_x^2=\delta_{(0,0)}$. But $p_2\equiv 0$. Indeed, the limit of $\{\chi_{\{u^\nu\in A_2\}}\,dx\}$ equals 0 in the space of measures, as the set $\{u^\nu\in A_2\}$ is empty. At the same time by Theorem 2.4 it is the same as

$$\Big(\int_{\partial A_2\cap A_2}\chi_{A_2}\overline{\nu}_x^2(d\lambda)\Big)p_2(x)\,dx = \Big(\int_{\{(0,0)\}}\chi_{(0,0)}\delta_{(0,0)}(d\lambda)\Big)p_2(x)dx = p_2(x)\,dx.$$

Therefore we have

$$\overline{\nu}_x^2=\delta_{(0,0)}, \quad p_2\equiv 0 \quad \text{almost everywhere.} \tag{6.7}$$

All our remaining examples will be based on the model presented in Example 6.1.

EXAMPLE 6.2 (Illustration of Theorem 3.1). We consider the same sequence as discussed in Example 6.1. Let us fix $i = 1$ and consider the family of open sets:

$$V^r := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 > r\} \cup \{x \in \mathbb{R}^2 : |x| < 1/r\}.$$

According to the notation in Theorem 3.1 we have $P = \{(0,0)\}$ and $P \cap intA_1 = \emptyset = \partial A_1 \cap A_1$. In particular the conditions (a) and (b) in its formulation are trivially satisfied whatever sequence we chose. Now we discuss condition (c) for our sequence. As $K_1^0 = \{(0,0)\} \in \gamma A_1 \setminus A_1$, we deduce from (6.6) that $\nu_x^1(K_1^0 \cap (\gamma A_1 \setminus A_1)) = 1$ for $\mathcal{L}^1$ almost every $x \in [0,1]$. Therefore condition (c) in Theorem 2.6 does not hold. In particular it is not true that

$$\int_{K_1^0 \cap (\gamma A_1 \setminus A_1)} |\widetilde{f}_1(\lambda)| \nu_x^1(d\lambda) = 0 \quad m^1\text{-almost everywhere,}$$

for every $f \in \mathcal{F}$ as it is not true for $f = g$.

Moreover, direct computation shows that for the given r and sufficiently big ν we have

$$\{x : u^\nu(x) \in A_1 \cap V^r\} = \left[\frac{1}{1+\nu}, 1\right]. \tag{6.8}$$

Therefore according to the notation in Theorem 2.6 the expression

$$\begin{aligned} A_r^1 &= \limsup_{\nu\to\infty} \int_{\{x: u^\nu(x) \in A_1 \cap V^r\}} g(u^\nu(x))\, dx \\ &= \limsup_{\nu\to\infty} \left(\int_{[1/(1+\nu), 2/(1+\nu))} (1+\nu)\, dx + \int_{[2/(1+\nu),1]} \left(1 + \frac{1}{\nu}\right) dx \right) = 2 \end{aligned}$$

does not converge to 0 as $r \to \infty$. The assumption (2.6) in Theorem 2.6 is not satisfied.

We show how to chose $f \in \mathcal{F}$ admitted to (3.1). From (6.1) and (6.8) we get

$$\begin{aligned} B_r^1 &= \lim_{\nu\to\infty} \int_{\{x: u^\nu(x) \in A_1 \cap V^r\}} |f(u^\nu(x))|\, dx \\ &= \lim_{\nu\to\infty} \int_{[1/(1+\nu), 2/(1+\nu)) \cup [2/(1+\nu),1]} |f(u^\nu(x))|\, dx \\ &= \lim_{\nu\to\infty} \left(\frac{|f|(\nu e_1)}{g(\nu e_1)} + \left| f\left(\frac{1}{\nu} e_1\right)\right| \left(1 - \frac{2}{1+\nu}\right)\right) = |\widetilde{f}_1(e_1)| + |\widetilde{f}_1(0)|. \end{aligned}$$

Consequently,

$$\lim_{r\to\infty} B_r^1 = 0 \text{ if and only if } \widetilde{f}_1(e_1) = \widetilde{f}_1((0,0)) = 0. \tag{6.9}$$

The above implies

$$0 = |\widetilde{f}_1((0,0))|\nu_x^1(\{(0,0)\}) = \int_{\{(0,0)\}} |\widetilde{f}_1(\lambda)|\nu_x^1(d\lambda) = \int_{K_1^0\cap(\gamma A_1\setminus A_1)} |\widetilde{f}_1(\lambda)|\nu_x^1(d\lambda).$$

This illustrates part (c) in Proposition 3.1.

EXAMPLE 6.3 (Illustration of Theorem 3.2). We consider the same sequence and admitted function as in Example 6.1. One easily shows that the topological condition (3.2) is satisfied. Moreover, $K_1 = \{(0,0)\} \cup \{e_1\}$. Therefore if f satisfies (3.1) we get from (6.9) that

$$\int_{K_1\cap(\gamma A_1\setminus A_1)} |\widetilde{f}_1(\lambda)|\nu_x^1(d\lambda) = |\widetilde{f}_1((0,0))|\nu_x^1(\{(0,0)\}) + |\widetilde{f}_1(e_1)|\nu_x^1(\{e_1\}) = 0$$

as we have claimed. Note that the statement is not true with f substituted by g (see (6.6)).

EXAMPLE 6.4 (Illustration of Theorem 3.3). We study the same model as before and start with the verification of Assumption A2. For this, we have to verify if the function

$$G_i = \frac{1}{1+|\Phi_i^{-1}(\lambda)|} : \Phi_1(A_i) \to \mathbb{R}$$

has the unique continuous extension to γA_i. As $A_2 = \Phi_2(A_2)$ is just one point there is nothing to check (the reminder is empty). Let us deal with case $i = 1$. We have $\Phi_1^{-1}(y) = y/(1-|y|)$, so that $G_1(y) = 1-|y| : B(1)\setminus\{(0,0)\} = \Phi_1(A_1) \to \mathbb{R}$ has a continuous extension to $\gamma A_1 = \overline{B(1)}$. Moreover,

$$R_1^\infty = G_1^{-1}(0) = S^1, \quad R_1^b = (S^1 \cup \{(0,0)\}) \setminus S^1 = \{(0,0)\}. \tag{6.10}$$

As we have verified in (6.9) the function $f \in \mathcal{F}$ satisfies (3.1) provided that $\widetilde{f}_1(e_1) = \widetilde{f}_1((0,0)) = 0$. Moreover, $K_1 = \{(0,0)\} \cup \{e_1\}$ and $K_1 \cap R_1^b = \{(0,0)\}$. Therefore the statement of Theorem 3.3 holds.

We are now to illustrate conditions (4.2) and (4.3) in Theorem 4.1.

EXAMPLE 6.5 (Illustration of Theorem 4.1). We consider the sequence from Example 6.1. At first we note that the sequence $\{1+|u^\nu|\}_{\nu\in\mathbb{N}}$ is not compact in $L^1(\Omega)$. Indeed, for an arbitrary $\varepsilon > 0$ and the sufficiently big ν

$$\int_{[0,\varepsilon)} (1+|u^\nu|)\,dx \ge \varepsilon + \int_{[0,1/(1+\nu))} \nu\,dx = \varepsilon + \frac{\nu}{1+\nu} \xrightarrow{\nu\to\infty} \varepsilon + 1,$$

therefore the equintegrability condition is not satisfied. Let us verify the validity of the condition (4.2) equivalent to the fact that the sequence $\{f(u^\nu)\}$ is weakly compact in $L^1(\Omega)$. For $i = 1$ we have $R_1^\infty = S^1$ (as computed in previous example) and

$$\begin{aligned}\lim_{\varepsilon\to 0} A_\varepsilon^1 &= \lim_{K\to\infty}\lim_{\nu\to\infty}\int_{\{x\in\Omega:|u^\nu(x)|>K\}} |f(u^\nu(x))|\,dx \\ &= \lim_{\nu\to\infty}\int_{[0,2/(1+\nu))} |f(u^\nu(x))|\,dx \\ &= \lim_{\nu\to\infty} |f(-\nu e_1)|\frac{1}{1+\nu} + |f(\nu e_1)|\frac{1}{1+\nu} = |\widetilde{f}(-e_1)| + |\widetilde{f}(e_1)|.\end{aligned}$$

Hence condition (4.2) for $i = 1$ is equivalent to the fact that $\widetilde{f}(-e_1) = \widetilde{f}(e_1) = 0$. For $i = 2$ it is trivially satisfied (set $\{u^\nu \in A_2\}$ is empty). Now we verify the condition (4.3). When $i = 2$ it is trivially satisfied ($R_2^\infty = \emptyset$) while when $i = 1$ we have $R_1^\infty = S^1$ and according to (6.6) we have

$$\begin{aligned}&\int_\Omega\int_{R_1^\infty} |\widetilde{f}_1(\lambda)|\nu_x^1(d\lambda)m^1(dx) = \int_\Omega\int_{S^1} |\widetilde{f}_1(\lambda)|\nu_x^1(d\lambda)(dx + 2\delta_0) \\ &= \int_\Omega\int_{S^1} |\widetilde{f}_1(\lambda)|\delta_{(0,0)}(d\lambda)dx + 2\int_{S^1} |\widetilde{f}_1(\lambda)|\nu_0^1(d\lambda) == |\widetilde{f}_1(-e_1)| + |\widetilde{f}_1(e_1)|.\end{aligned}$$

We see that the conditions (4.2) and (4.3) are equivalent. Direct computation shows that their validity is equivalent to the weak compactness in $L^1(\Omega)$ of the sequence $\{f(u^\nu)\}$.

Example 6.6 (Illustration of Theorem 5.1). We consider the sequence from Example 6.1. It is clear that $u^\nu \to \{(0,0)\} := L$ in the measure. Measures $\{\mu_x\}_{x\in\Omega}$, $(\overline{m}^i, \{\overline{\nu}_x^i\}_{x\in\Omega})$, $(m^i, \{\nu_x^i\}_{x\in\Omega})$ (where $i = 1, 2$) have already been constructed in Example 6.1. Assume now that the sequence $\{f(u^\nu)\}$ is weakly compact in $L^1(\Omega)$. By previous example this is equivalent to the fact that $\widetilde{f}_1(e_1) = \widetilde{f}_1(-e_1) = 0$, but we will not use this fact here. Theorem 5.1 asserts that $\{f(u^\nu)\}$ weakly converges in $L^1(\Omega)$ to

$$\begin{aligned}\overline{f}(x) = \sum_{i=1}^2 \bigg(\int_{L\cap \operatorname{int} A_i} f(\lambda)\mu_x(d\lambda) &+ p_i(x)\int_{L\cap\partial A_i\cap A_i} f(\lambda)\overline{\nu}_x^i(d\lambda) \\ &+ q_i(x)\int_{X^i} \widetilde{f}_i(\lambda)\nu_x^i(d\lambda)\bigg) := \sum_{i=1}^2 (I_i).\end{aligned}$$

Let us compute X^1. It is clear that $L_\varepsilon = \{x \in \mathbb{R}^2 : |x| < \varepsilon\}$ and so $L_\varepsilon^c = \{x \in \mathbb{R}^2 : |x| \geq \varepsilon\} = L_\varepsilon^c \cap A_1$, $\Phi_1(L_\varepsilon^c \cap A_1) = \{x \in B(1) : \|x\| \geq \varepsilon/(1+\varepsilon)\} \subseteq \mathbb{R}^2$. Therefore $Y := \bigcup_{\varepsilon>0} \overline{\Phi_1(L_\varepsilon^c \cap A_1)} = \overline{B(1)} \setminus \{(0,0)\}$. By (6.10) we have $R_1^b = \{(0,0)\}$, which gives

$$X^1 = R_1^b \setminus Y = \{(0,0)\}. \tag{6.11}$$

Moreover, the formulae (6.6) implies $q_1(x) \equiv 1$, and therefore (as $L \cap intA_1 = \emptyset$ and $\partial A_1 \cap A_1 = \emptyset$)

$$I_1 = q_1(x) \int_{X^1} \widetilde{f}_1(\lambda)\nu_x^1(d\lambda) = \int_{(0,0)} \widetilde{f}_1(\lambda)\delta_{(0,0)}(d\lambda) = \widetilde{f}_1((0,0)).$$

Now we compute I_2. Obviously, only the second term:

$$p_2(x) \int_{L\cap\partial A_2\cap A_2} f(\lambda)\overline{\nu}_x^2(d\lambda)$$

appears in decomposition. It is zero, as we have shown in (6.7) that $p_2 \equiv 0$. We get: $I_2 = 0$.

We have verified using Theorem 5.1 that if the sequence $\{f(u^\nu)\}$ is weakly compact in $L^1(\Omega)$ then its weak limit equals $\widetilde{f}_1((0,0)) = \lim_{\lambda\to(0,0),\lambda\neq(0,0)} f(\lambda)$, in particular it is the constant function.

This can be verified directly, without the use of an advanced Young measure theory, as we show below. For this purpose, let us decompose: $f(u^\nu) = f(u^\nu)\chi_{\{u^\nu>\varepsilon\}} + f(u^\nu)\chi_{\{u^\nu\leq\varepsilon\}}$, take an arbitrary $g \in L^\infty(\Omega)$ and compute the limits of

$$A_\varepsilon^\nu = \int_\Omega g(x)f(u^\nu(x))\chi_{\{u^\nu(x)>\varepsilon\}}\,dx \quad \text{and} \quad B_\varepsilon^\nu = \int_\Omega g(x)f(u^\nu(x))\chi_{\{u^\nu(x)\leq\varepsilon\}}\,dx$$

separately as $\nu\to\infty$. The function inside the integral in A_ε^ν converges to 0 in a measure and is dominated by $\|g\|_{L^\infty(\Omega)}|f(u^\nu(x))|$ forming an equiintegrable sequence. Lebegue's Dominated Convergence Theorem implies that $A_\varepsilon^\nu \to 0$ as $\nu\to\infty$. For fixed ε and the sufficiently big ν the second term equals

$$B_\varepsilon^\nu = \int_{[2/(1+\nu),1]} g(x)f(\frac{1}{\nu}e_1)\,dx \xrightarrow{\nu\to\infty} \int_{[0,1]} g(x)\widetilde{f}_1((0,0))\,dx.$$

This shows by another methods that the weakly compact sequence $\{f(u^\nu)\}$ converges weakly in $L^1(\Omega)$ to the constant function: $\widetilde{f}_1((0,0))$.

Our last goal is to illustrate the last result.

EXAMPLE 6.7 (Illustration of Theorem 5.3). We deal with the sequence from Example 6.1. As computed in (6.6) we have $\mu_x = \delta_{(0,0)}$ and $L = \{(0,0)\}$ so indeed μ_x is supported in L as we have claimed in the first statement. To comment part (b) of the statement we observe that the measures $\overline{\nu}_x^1$ will not appear in Theorem 2.4. Moreover, according to (6.7) the measure $p_2(x)\,dx$ is zero, so that all the interval $[0,1]$ is of measure 0 and we have nothing to check. However, $\overline{\nu}_x^2 = \delta_{(0,0)}$ is supported in $L\cap\partial A_2\cap A_2 = \{(0,0)\}$.

To deal with the last statement we use the formulae (6.6) which gives $q_1 \equiv 1$ while q_2 is an empty object (as $\gamma A_2 \setminus A_2 = \emptyset$). Using again (6.6) we get $\nu_x^1 = \delta_{(0,0)}$ for $\mathcal{L}^1$ almost every x. According to (6.10) and (6.11) we have

$Z := (X^1 \cap R_1^b) \cup R_1^\infty = \{(0,0)\} \cup S^1$, therefore ν_x^1 is supported in Z for $\mathcal{L}^1$ almost every x as claimed in the last statement of the theorem.

Acknowledgements. The work is supported by a KBN grant no. 1/PO3A/ 008/29 and partially supported by EC FP6 Marie Curie ToK programme SPA-DE2, MTKD-CT-2004-014508 and Polish MNiSW SPB-M. I would like to thank Piotr Gwiazda for motivating me to do this work and many discussions. I also thank Tomasz Roubíček and Martin Kružík for helpful discussions during my visit of Charles University and Institute of Information Theory and Automation of the Academy of Science of the Czech Republic in December 2005. The hospitality of both institutions are gratefully acknowledged. Finally, I would like to thank the anonymous referee for helpful advices which essentially improved the quality of the paper.

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Manuscript received January 9, 2006

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