Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 30, 2007, 335–343

LOCALLY EXPANDING MAPPINGS AND HYPERBOLICITY

Jacek Tabor

Abstract. The aim of the following paper is to propose and investigate the partial generalization of hyperbolicity to metric spaces. The locally expanding mappings, as we call them, possess many similar behaviour to that characteristic to hyperbolic mappings, in particular, they have lipschitz shadowing property. As a direct corollary we obtain for example shadowing on the Julia set.

1. Introduction

Hyperbolicity is one of the crucial dynamical notions, as it guarantees many basic properties, like stability, expansivity, shadowing etc. [4], [5]. Due to its importance it has also found many generalizations [2], [1].

The direction of the paper is an approach to deal with noninvertible mappings in metric spaces using the ideas based on hyperbolicity.

Motivation comes in particular from Julia sets. One can show that the mapping $f_c : z \to z^2 + c$ generating Julia set has shadowing (for some parameter values c), but clearly is not invertible. We show that the proof of shadowing on Julia set can be modified to fit a fairly general setting in metric spaces.

It occurs that the right assumption is that the given mapping is locally invertible and that its local inverse is a strong contraction (we call such mappings locally expanding).

2000 *Mathematics Subject Classification.* 37C50, 34D30, 65L20.
Key words and phrases. Stability, hyperbolicity, shadowing, Julia set.
The research was partially supported by the KBN grant 1/P03A/002/29.

At the end of the introduction let us mention that we use local versions of some ideas from [3].

2. Locally expanding mappings

In the whole paper we assume that X, Y are complete metric spaces and M, N are complete manifolds.

DEFINITION 2.1. Let $x_0 \in X$ be fixed and let $r, R > 0$ be given. We say that $f: X \to Y$ is *locally* (r, R)*-invertible at* x_0 if for every $y \in B(f(x_0), R)$ there exists a unique $x \in B(x_0, r)$ such that $y = f(x)$.

The following example shows that the class of locally invertible mappings is quite large.

EXAMPLE 2.2. Let M, N be C^1-manifolds and let $f: M \to N$ be C^1. By the Banach inverse mapping theorem we obtain that f is locally invertible at every point x_0 such that $d_{x_0} f$ is an isomorphism.

If f is locally (r, R)-invertible at x_0 then by

$$f_{x_0}^{-1}: B(f(x_0), R) \to B(x_0, r)$$

we denote the function which maps an arbitrary $y \in B(f(x_0), R)$ into the unique x which satisfies $y = f(x)$.

By $\mathrm{lip}_R(f_{x_0}^{-1})$ we denote the lipschitz constant of $f_{x_0}^{-1}$.

DEFINITION 2.3. Let $A \subset X$. We say that f is *locally* (r, R)*-invertible on* A if f is locally (r, R)-invertible for every $x_0 \in A$.

If f is locally (r, R)-invertible on A we put

$$\mathrm{lip}_R(f^{-1}, A) := \sup_{a \in A} \mathrm{lip}_R(f_a^{-1}),$$

and $\mathrm{lip}_R(f^{-1}) := \mathrm{lip}_R(f^{-1}, X)$. We say that f is *locally expanding* on the set A if $\mathrm{lip}_R(f^{-1}, A) < 1$ for a certain $R > 0$.

The following example gives a fairly large class of expanding mappings.

EXAMPLE 2.4. Let M, N be Riemann manifolds and let $f: M \to N$ be C^1. Let $A \subset M$ be compact. If $d_x f$ is invertible for every $x \in A$ and

$$\|(d_x f)^{-1}\| < 1 \quad \text{for } x \in A$$

then f is locally expanding on A.

REMARK 2.5. Suppose that f is locally (r, R)-invertible on A and

$$\mathrm{lip}_R(f^{-1}, A) < 1.$$

Let $S := \min\{r/l, R\}$. Then f is locally (lS, S)-invertible and

$$\mathrm{lip}_S(f^{-1}, A) < 1.$$

Before proceeding further for the convenience of the reader let us establish notation and formulate some basic definitions concerning shadowing.

By $\mathbb{N}$ we denote the set of all nonnegative integers. Let X be a metric space and let $f\colon U \to X$, where $U \subset X$. Given $\delta \geq 0$, we say that a sequence $\mathrm{x} = (x_n)_{n\in\mathbb{N}} \subset U$ is a *positive δ-pseudoorbit* (for f) if

$$d(x_{n+1}, f(x_n)) \leq \delta \quad \text{for } n \in \mathbb{N}.$$

A positive orbit for f is a sequence $\mathrm{x} = (x_n)_{n\in\mathbb{N}} \subset U$ such that $x_{n+1} = f(x_n)$. Thus a positive 0-pseudoorbit is simply a positive orbit.

We need the (extended) metric on the space of sequences:

$$d_{\sup}(\mathrm{x}, \mathrm{y}) := \sup_{n\in\mathbb{N}} d(x_n, y_n)$$

for sequences $\mathrm{x} = (x_n), \mathrm{y} = (y_n) \subset X$.

Substituting in the above $\mathbb{N}$ by $\mathbb{Z}$ in an obvious way we obtain the notion of a δ-pseudoorbit and orbit.

Definition 2.6. We say that a (positive) δ-pseudoorbit x is ε-shadowed by a (positive) orbit y if

$$d_{\sup}(\mathrm{x}, \mathrm{y}) \leq \varepsilon.$$

Definition 2.7. We say that f has *the shadowing* (*shadowing*$^+$) *property* on $A \subset U$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every (positive) δ-pseudoorbit in A is ε-shadowed. Shadowing is sometimes called pseudo-orbit tracing property (POTP shortly).

As shows the following simple proposition, usually shadowing$^+$ implies shadowing. Since the assumptions of the proposition will be satisfied in the examples we discuss, from now on we restrict our attention to shadowing$^+$.

Proposition 2.8. *Let X be a metric space and let $\varepsilon > 0$, $\delta > 0$ be given. Assume that every positive δ-pseudorobit is ε-shadowed by some positive orbit. If additionally one of the following conditions holds*

(a) *for every positive orbits* x, y*, if* $d_{\sup}(\mathrm{x}, \mathrm{y}) \leq 2\varepsilon$ *then* $\mathrm{x} = y$*;*

(b) *every closed ball in X with radius ε is compact;*

then every δ-pseudoorbit is ε-shadowed by an orbit.

Since the proof of the above proposition is classical, we skip it.

One of the interesting problems in the theory of stability of dynamical systems is to check which mappings have shadowing or shadowing$^+$. We prove that this is the case for locally expanding ones.

Theorem 2.9. *Let X be complete metric space, let $A, U \subset X$ be such that $A \subset U$. Let $l \in (0,1)$, $R > 0$. We assume that $f\colon U \to X$ is locally (lR, R)-invertible on A and that $\operatorname{lip}_R(f^{-1}, A) \le l$. Let* x *be an arbitrary positive δ-pseudorbit in A with $\delta \le (1-l)R$. Then there exists a unique positive orbit* y *in U such that*

$$d_{\sup}(\mathrm{x}, \mathrm{y}) \le lR. \tag{2.1}$$

Moreover, $d_{\sup}(\mathrm{x}, \mathrm{y}) \le l\delta/(1-l)$.

Proof. We are going to construct a complete metric space on which our orbit is a fixed point. We define

$$\mathcal{X} := \{\mathrm{y} \subset X : d_{\sup}(\mathrm{x}, \mathrm{y}) \le lR\}.$$

In $\mathcal{X}$ we use the metric $d_{\sup}$. One can easily check that $\mathcal{X}$ is a complete metric space.

Since x is a δ-pseudoorbit $d(x_{n+1}, f(x_n)) \le \delta$ for every $n \in \mathbb{N}$. We define the mapping $\mathcal{P}\colon \mathcal{X} \to \mathcal{X}$ by the formula

$$(\mathcal{P}\mathrm{y})_n := f_{x_n}^{-1}(y_{n+1}) \quad \text{for } n \in \mathbb{N}.$$

We are first going to show that $\mathcal{P}$ is a well-defined contraction. By the definition, the domain of $f_{x_n}^{-1}$ is $B(f(x_n), R)$ and its image is contained in $B(x_n, lR)$. Thus to show that $\mathcal{P}$ is well-defined we need to prove that

$$B(x_{n+1}, lR) \subset B(f(x_n), R) \quad \text{and} \quad f_{x_n}^{-1}(B(x_{n+1}, lR)) \subset B(x_n, lR).$$

We prove the first inclusion. Let $y \in B(x_{n+1}, lR)$. Since $\delta \le (1-l)R$ we get

$$\begin{aligned} d(y, f(x_n)) &\le d(y, x_{n+1}) + d(x_{n+1}, f(x_n)) \\ &\le lR + \delta \le lR + (1-l)R = R. \end{aligned}$$

Now we deal with the second one. Let us first observe that by the definition of $f_{x_n}^{-1}$ we have $f_{x_n}^{-1}(f(x_n)) = x_n$. For an arbitrary $y \in B(x_{n+1}, lR)$ by applying the above we get

$$d(f_{x_n}^{-1}(y), x_n) = d(f_{x_n}^{-1}(y), f_{x_n}^{-1}(f(x_n))) \le l d(y, f(x_n)) \le lR.$$

Thus we see that $\mathcal{P}$ is well-defined. Since $\mathcal{P}$ is lipschitz with constant $l < 1$ by the Banach contraction principle we obtain that it has a unique fixed point y and

$$\begin{aligned} d_{\sup}(\mathrm{x}, \mathrm{y}) &\le \frac{d(\mathrm{x}, \mathcal{P}\mathrm{x})}{1-l} = \frac{1}{1-l} \sup_{n \in \mathbb{N}} d(x_n, f_{x_n}^{-1}(x_{n+1})) \\ &= \frac{1}{1-l} \sup_{n \in \mathbb{N}} d(f_{x_n}^{-1}(f(x_n)), f_{x_n}^{-1}(x_{n+1})) \le \frac{l\delta}{1-l}. \end{aligned}$$

Since y is a fixed point for $\mathcal{P}$, it satisfies $f_{x_n}^{-1}(y_{n+1}) = y_n$ for $n \in \mathbb{N}$, i.e. $y_{n+1} = f(y_n)$ which clearly means that y is a positive orbit.

Now we prove the uniqueness part. By the construction we obtain that $\mathrm{y} \in B_{\sup}(\mathrm{x}, lR)$. Let an arbitrary positive orbit $\mathrm{z} \in B(\mathrm{x}, lR)$ be given. We are going to show that z = y. Since we know that y is a unique fixed point for $\mathcal{P}$ it is enough to show that z is also a fixed point for $\mathcal{P}$. This is equivalent to proving that

$$f_{x_n}^{-1}(z_{n+1}) = z_n \quad \text{for } n \in \mathbb{N}. \tag{2.2}$$

Since $d(x_{n+1}, z_{n+1}) \leq lR$ we get

$$d(z_{n+1}, f(x_n)) \leq d(z_{n+1}, x_{n+1}) + d(x_{n+1}, f(x_n)) \leq lR + \delta \leq R.$$

Thus $z_{n+1} \in B(f(x_n), R)$. As z is a positive orbit we have $z_{n+1} = f(z_n)$. By the definition of $f_{x_n}^{-1}$ we get (2.2). □

Remark 2.10. We would not like to discuss it here in more details, but the above theorem shows in fact that f has *lipschitz shadowing*$^+$, that is the approximation constant to depends linearly on δ.

As a consequence we obtain that locally expanding mappings are expansive.

Corollary 2.11. *Let $U \subset X$, $f: U \to X$, $A \subset X$. Let $l \in (0,1)$, $R > 0$ be given. We assume that f is locally (lR, R)-invertible on A and that* $\mathrm{lip}_R(f^{-1}, A) \leq l$. *Let* x, y *be arbitrary positive orbits in X such that* x *lies in A and* $d_{\sup}(\mathrm{x}, \mathrm{y}) \leq lR$. *Then* x = y.

Proof. We apply Theorem 2.9 with $\delta = 0$. □

Let us now present a global version of Theorem 2.9:

Corollary 2.12. *Let $f: X \to X$. We assume that f is locally (lR, R)-invertible on X and that* $\mathrm{lip}_R(f^{-1}) \leq l < 1$. *Let* x *be an arbitrary positive δ-pseudoorbit with $\delta \leq (1-l)R$. Then there exists a unique positive orbit* y *in X such that* $d_{\sup}(\mathrm{x}, \mathrm{y}) \leq lR$. *Moreover,* $d_{\sup}(\mathrm{x}, \mathrm{y}) \leq l\delta/(1-l)$.

Example 2.13. Let M be a compact Riemann manifold, let $U \subset M$ be open and let $f: U \to M$ be a C^1-mapping such that $\|(d_x f)^{-1}\| < 1$ for every $x \in U$. Then f is locally expanding, and in particular has the lipschitz shadowing$^+$ property.

A simplest example is given by a function $f: S^1 \to S^1$ defined by

$$f(z) := z^n, \quad \text{where } n \geq 2.$$

At the end of this section we would like to discuss the case how the shadowing is correlated between semiconjugate systems. Given a positive δ-pseudoorbit x for f we denote by $\mathrm{orb}_f^+(\mathrm{x})$ the orbit which shadows x constructed in Theorem 2.9.

Proposition 2.14. *Let $f\colon X \to X$, $g\colon Y \to Y$ be locally invertible. We assume that* $\mathrm{lip}_R(f^{-1}) \le l, \mathrm{lip}_S(f^{-1}) \le l$ *for certain R, S. Let $\delta \le (1-l)R$ and let $a\colon X \to Y$ be such that*

$$a \circ f = g \circ a. \tag{2.3}$$

We assume additionaly that

$$\mathrm{lip}(a)R \le S. \tag{2.4}$$

Let x *be an arbitrary δ-pseudorobit for f. Then $a(\mathrm{x})$ is a* $\mathrm{lip}(a)\delta$*-pseudoorbit for g and*

$$a(\mathrm{orb}_f^+(\mathrm{x})) = \mathrm{orb}_g^+(a(\mathrm{x})).$$

Proof. In the proof we use the following convention: given a sequence $\mathrm{z} = (z_n) \subset X$ and $h\colon X \to Y$ by $h(\mathrm{x})$ we understand the sequence $(h(z_n))_n \subset Y$.

Since x is a positive δ-pseudorobit for f, by (2.3) and (2.4) we trivially obtain that $a(\mathrm{x})$ is a positive $\mathrm{lip}(a)\delta$-pseudoorbit of g.

By Theorem 2.9 we obtain unique positive orbits $\mathrm{orb}_f^+(\mathrm{x}) \subset X$, $\mathrm{orb}_g^+(a(\mathrm{x})) \subset Y$ such that

$$d_{\sup}(\mathrm{orb}_f^+(\mathrm{x}), \mathrm{x}) \le lR, \quad d_{\sup}(\mathrm{orb}_g^+(a(\mathrm{x})), a(\mathrm{x})) \le lS.$$

On the other hand, since x is a positive orbit such that $d(\mathrm{orb}_f^+(\mathrm{x}), \mathrm{x}) \le lR$, by (2.3) and (2.4) we get that $a(\mathrm{x})$ is a positive orbit for g and

$$d_{\sup}(a(\mathrm{orb}_f^+(\mathrm{x})), a(\mathrm{x})) \le \mathrm{lip}(a)lR.$$

Since $\mathrm{lip}(a)lR \le lS$, by the uniqueness we obtain that

$$a(\mathrm{orb}_f^+(\mathrm{x})) = \mathrm{orb}_g^+(a(\mathrm{x})). \qquad \square$$

3. Shadowing on subsets

As we have seen in Theorem 2.9, even if the pseudoorbit belongs to the set A, the orbit which shadows may not belong to A, but to its small neighbourhood. In many cases we would like to obtain that orbit from the set A. This happens if the set A is invariant.

Definition 3.1. Let $U \subset X$ and let $f\colon U \to X$. We say that $A \subset U$ is *invariant* if $f(A) = f^{-1}(A) = A$.

Now we are ready to state the shadowing result on invariant sets.

PROPOSITION 3.2. *Let $f\colon U \to X$, let $A \subset U$ be closed and invariant subset of X. Let $l \in (0,1)$, $R > 0$ be given. We assume that f is locally (lR, R)-invertible on A and that $\mathrm{lip}_R(f^{-1}, A) \le l$. Let δ be such that $\delta \le (1-l)R$. Let* x *be an arbitrary positive δ-pseudorbit in A. Then there exists a unique positive orbit* y *in X such that*

$$d_{\sup}(\mathrm{x}, \mathrm{y}) \le lR. \tag{3.1}$$

Moreover, y *lies in A and $d_{\sup}(\mathrm{x}, \mathrm{y}) \le l\delta/(1-l)$.*

PROOF. By Theorem 2.9 we know that there exists a unique orbit y satisfying (3.1). Moreover, $d_{\sup}(\mathrm{x}, \mathrm{y}) \le l\delta/(1-l)$.

To check that the positive orbit x lies in A let us define f_A as a restriction of f to A. We study the semisystem in A with generator f_A.

Since A is invariant, f_A is locally (lR, R)-invertible (on A) and $\mathrm{lip}_R(f_A^{-1}) \le l < 1$. Let $x \in A$ be arbitrary and let $y \in B(f(x), R) \cap A$. Clearly $f_x^{-1}(y) \in B(x, lR)$ and $\mathrm{lip}((f_x^{-1})|_A) \le l$. The question remains if $f_x^{-1}(y) \in A$? But this is the case due to the invariance of A.

By applying once more Theorem 2.9 for f_A and the positive pseudoorbit x we obtain the existence of a unique positive orbit y_A in A such that $d_{\sup}(\mathrm{x}, \mathrm{y}_A) \le \min\{r, lR\}$.

By the uniqueness $\mathrm{y} = \mathrm{y}_A$. Consequently y is a positive orbit in A. $\square$

However, even the above proposition assumes slightly two much, as we cannot usually be certain that the pseudoorbit is taken from A, but only from a neighbourhood of A. To obtain that the tracing orbit lies in A, we need to additionally assume that f is lipschitz on a neighbourhood of A.

By $B(A, w)$ we denote the set $\{x \in X : d(x, A) < w\}$

THEOREM 3.3. *Let $U \subset X$, $f\colon U \to X$ and let $A \subset U$ be closed and invariant subset of X. Let $l \in (0,1)$ and $R > 0$ be given. We assume that f is locally (lR, R)-invertible on A and that*

$$\mathrm{lip}_R(f^{-1}, A) \le l, \quad L := \mathrm{lip}(f) < \infty.$$

Let $\delta > 0$ and $w > 0$ be such that

$$\frac{l}{1-l}\delta + \frac{Ll+1}{1-l}w \le \frac{lR}{2}. \tag{3.2}$$

Let x *be an arbitrary positive δ-pseudorbit in $B(A, w)$. Then there exists a unique positive orbit* y *in X such that*

$$d_{\sup}(\mathrm{x}, \mathrm{y}) \le lR/2. \tag{3.3}$$

Moreover, y *is an orbit in A and $d_{\sup}(\mathrm{x}, \mathrm{y}) \le (l\delta + w + Llw)/(1-l)$.*

PROOF. Let $\delta_w := \delta + w + Lw$. By (3.2) we trivially obtain that $\delta_w \le (1-l)R$.

Since $x_k \in \{x \in X: d(x,A) < w\}$ for every $k \in \mathbb{Z}_+$, there exists $v_k \in A$ such that $d(v_k, w_k) \le w$ for $k \in \mathbb{Z}_+$. We check that $\mathrm{v} = (v_k)$ is a positive δ_w-pseudoorbit in A:

$$d(v_{k+1}, f(v_k)) \le d(v_{k+1}, x_{k+1}) + d(x_{k+1}, f(x_k)) + d(f(x_k), f(v_k)) \le w + \delta + Lw.$$

By applying Proposition 3.2 we obtain a positive orbit y in A such that $d(v_k, y_k) \le l\delta_w/(1-l)$ for $k \in \mathbb{Z}_+$, and consequently

$$d(x_k, y_k) \le d(x_k, v_k) + d(y_k, v_k) \le w + \frac{l\delta_w}{1-l} = \frac{l\delta + w + Llw}{1-l}.$$

We prove the uniqueness part. Let $\widetilde{\mathrm{y}}$ be another positive orbit in X such that $d_{\sup}(\mathrm{x}, \widetilde{y}) \le lR/2$. Then $d_{\sup}(\mathrm{y}, \widetilde{y}) \le lR$, and since y is an orbit in A by Corollary 2.11 we obtain that $\widetilde{\mathrm{y}} = \mathrm{y}$. □

As a direct corollary we get:

COROLLARY 3.4. *Let X be a complete metric space, let $U \subset X$ and let $f: U \to X$. Let $A \subset U$ be a closed subset of X. We assume that f is locally expanding on A. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every positive δ-pseudoorbit* x *in $B(A,\delta)$ there exists a unique positive orbit* y *in X such that $d_{\sup}(\mathrm{x},\mathrm{y}) \le \varepsilon$. Moreover,* y *lies in A.*

EXAMPLE 3.5. Let M be a Riemann manifold, let $U \subset M$ be open and let $f: U \to M$ be C^1. Assume that $A \subset U$ is compact and invariant, $d_x f$ is invertible for $x \in A$ and

$$\|(d_x f)^{-1}\| < 1 \quad \text{for } x \in A.$$

Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every positive δ-pseudoorbit x in $B(A,\delta)$ there exists a unique positive orbit y in A such that $d_{\sup}(\mathrm{x},\mathrm{y}) \le \varepsilon$.

Now applying the above example we are ready to present a simple proof of shadowing for the Julia map with a specified parameter value. In fact this method works for all sufficiently large parameters.

EXAMPLE 3.6. Let $c \in \mathbb{C}$ be fixed and let $f_c: \mathbb{C} \to \mathbb{C}$ be defined by

$$f_c(z) := z^2 + c \quad \text{for } z \in \mathbb{C}.$$

The Julia set J_c (corresponding to the parameter c) is the boundary of R_c — the set of all points which do not escape to infinity in the semisystem generated by f_c.

Let $c = 4$. Let us first notice that if z is such that $|z| > 3$ then $z \notin R_c$:

$$|f_c(z)| \ge |z|^2 - |c| \ge 5,$$

and consequently $|f_c^n(z)| \to \infty$. Analogously, if $|z| < 1$ then $z \notin R_c$:

$$|f_c(z)| \ge |c| - |z| \ge 4 - 1 = 3,$$

and the previous works. Thus we get

$$R_c \subset U_c := \{z : 1 \leq |z| \leq 3\}.$$

Consequently we obtain that $f := f_c|_{U_c}$ is such that J_c is an invariant set for f_c. Moreover, one can easily check that $|(d_x f)^{-1}| < 1$ for $x \in J_c$, which by Example 3.5 proves that f has the shadowing$^+$ property on J_c.

Acknowledgements. I would like to thank my father for the helpful comments and suggestions.

References

[1] P. Diamond, P. Kloeden, V. Kozyakin and A. Pokrovskiĭ, *Semihyperbolic mappings*, J. Nonlinear Sci. **5** (1995), 419–431.

[2] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge Univ. Press, 1997.

[3] J. Ombach, *The simplest shadowing*, Ann. Polon. Math. **LVIII.3** (1993), 253–258.

[4] K. Palmer, *Shadowing in Dynamical Systems*, Kluwer Academic Press, 2000.

[5] S. Pilyugin, *Shadowing in Dynamical Systems*, Springer–Verlag, Berlin, Heidelberg, 1999.

Manuscript received December 12, 2005

Jacek Tabor
Institute of Mathematics
Jagiellonian University
Reymonta 4
30-059 Kraków, POLAND

E-mail address: tabor@im.uj.edu.pl