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EXISTENCE THEORY FOR SINGLE AND MULTIPLE SOLUTIONS TO SINGULAR BOUNDARY VALUE PROBLEMS FOR SECOND ORDER IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract. In this paper we present some new existence results for singular boundary value problems for second order impulsive differential equations. Our nonlinearity may be singular in its dependent variable.

1. Introduction

This paper is devoted to study the existence of multiple positive solutions for the singular Dirichlet boundary value problem with impulse effects

$$\tag{1.1}\begin{cases} y'' + q(t)f(t,y) = 0, & \text{for } t \neq t_k,\ t \in (0,1), \\ -\Delta y'|_{t=t_k} = I_k(y(t_k)), & \text{for } k = 1, \ldots, m, \\ y(0) = 0, \quad y(1) = 0. \end{cases}$$

Here, let $0 < t_1 < \ldots < t_m < 1$ be given, where $f(t,y) \in C((0,1)\times(0,\infty),(0,\infty))$, and nonlinearity f may be singular at $y = 0$; q may be singular at $t = 0$ and/or $t = 1$; $I_k\colon [0,\infty) \to [0,\infty)$ is continuous and nondecreasing; $\Delta y'|_{t=t_k} =$

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$y'(t_k+0)-y'(t_k-0)$, where $y'(t_k+0)$ (respectively, $y'(t_k-0)$) denote the right limit (respectively, left limit) of $y'(t)$ at $t=t_k$.

In recent years, boundary problems of second-order differential equations with impulses have been studied extensively in the literature (see for instance [1], [3], [5]–[8], [11]–[13] and their references), there are two most common techniques to approach this problem: (1) the method of lower and upper solutions with monotone iterative technique has been used (see [5]–[7]); (2) Krasnoselskiĭ's fixed point theorem in a cone has been used (see [1], [7]). The existence of positive solutions of problem (1.1) for the case of nonsingular has been studied by [13], by employing a cone index theory.

For the case of $I_k=0$, $k=1,\ldots,m$, problem (1.1) is related to two points boundary value problem of ODE. Agarwal and O'Regan [2] have applied a fixed point index theorem in cones to establish the existence of multiple positive solutions to singular problem (1.1).

Motivated by the work above, in this paper we shall extend the results of [2] to second order impulsive differential equations.

First, we present an existence principle for the nonsingular boundary value problem which will be needed in Section 2. We use Schauder's fixed point theorem and a nonlinear alternative of Leray–Schauder type to obtain a general existence principle for the Dirichlet boundary value problem for second order impulsive differential equations

$$\begin{cases} y''+f(t,y)=0 & \text{for } t\neq t_k,\ t\in(0,1),\\ -\Delta y'|_{t=t_k}=I_k(y(t_k)) & \text{for } k=1,\ldots,m,\\ y(0)=a,\quad y(1)=b. \end{cases} \tag{1.2}$$

Let $J=[0,1]$, $PC[J,R]=\{y\colon J\to R \mid y(t)$ is continuous, $y'(t_k+0)$ and $y'(t_k-0)$ are existence, and $y'(t_k)=y'(t_k-0)\}$, then $PC[J,R]$ is a Banach space with $|y|_0=\sup_{t\in[0,1]}|y(t)|$. Let $J'=(0,1)$, $J^0=(0,1)/\{t_1,\ldots,t_m\}$, $J_0=(0,t_1]$, $J_1=(t_1,t_2]\ldots, J_{m-1}=(t_{m-1},t_m]$, $J_m=(t_m,1)$.

If $y\in PC[J,R]\bigcap C^2[J^0,R]$ satisfies all of the equations of (1.2), we call y is a solution of (1.2).

Theorem 1.1. *Suppose the following two conditions are satisfied:*

$$f\colon J\times R\to R \quad \text{is continuous,} \tag{1.3}$$

$$I_k\colon R\to R \quad \text{is continuous.} \tag{1.4}$$

(a) *Assume*

$$\begin{cases} \textit{for each } r>0 \ \textit{ there exists } h_r\in L^1_{loc}(J') \\ \textit{with } \int_0^1 t(1-t)h_r(t)\,dt<\infty \\ \textit{such that } |y|\le r \textit{ implies } |f(t,y)|\le h_r(t) \textit{ for } t\in J' \end{cases} \tag{1.5}$$

holds. In addition suppose there is a constant $M > |a|+|b|$, independent of λ, with

$$|y|_0 = \sup_{t\in[0,1]} |y(t)| \neq M \tag{1.6}$$

for any solution $y \in PC[J,R]\bigcap C^2[J^0,R]$ to

$$(1.7)_\lambda \qquad \begin{cases} y'' + \lambda f(t,y) = 0 & \text{for } t \in J^0, \\ -\Delta y'|_{t=t_k} = \lambda I_k(y(t_k)) & \text{for } k = 1,\dots,m, \\ y(0) = a, \quad y(1) = b, \end{cases}$$

for each $\lambda \in (0,1)$. Then (1.2) *has a solution y with $|y|_0 \le M$.*

(b) *Assume*

$$\begin{cases} \text{there exists } h \in L^1_{\text{loc}}(J') \text{ with } \int_0^1 t(1-t)h(t)\,dt < \infty \\ \text{such that } |f(t,y)| \le h(t) \text{ for } t \in J' \text{ and } y \in R \end{cases} \tag{1.8}$$

holds. Then (1.2) *has a solution.*

PROOF. (a) We begin by showing that solving $(1.7)_\lambda$ is equivalent to finding a solution $y \in PC[J,R]\bigcap C^2[J^0,R]$ to

$$(1.9)_\lambda \quad y(t) = a(1-t)+bt+\lambda \int_0^1 G(t,s)\,f(s,y(s))\,ds + \lambda \sum_{k=1}^m G(t,t_k)\,I_k(y(t_k)),$$

where $G(t,s)$ is Green's function to the Dirichlet boundary value problem $-x'' = 0$, $x(0) = x(1) = 0$, and

$$G(t,s) := \begin{cases} (1-t)s & \text{for } 0 \le s \le t \le 1, \\ (1-s)t & \text{for } 0 \le t \le s \le 1. \end{cases}$$

To see this notice if $y \in PC[J,R]\bigcap C^2[J^0,R]$ satisfies $(1.9)_\lambda$ then it is easy to see (since (1.6) holds; see [9], [10]) that $y' \in L^1[J_k]$, and note for $t \in J^0$ we have

$$\begin{aligned} y'(t) = {}& -a+b-\lambda \int_0^t s\,f(s,y(s))\,ds \\ & +\lambda \int_t^1 (1-s)\,f(s,y(s))\,ds + \lambda \sum_{0<t_k<1} G_t'(t,t_k)\,I_k(y(t_k)). \end{aligned}$$

Now we prove that if $y \in PC[J,R]\bigcap C^2[J^0,R]$ satisfy $(1.9)_\lambda$, then y is a solution of $(1.7)_\lambda$. Since $G''(t,t_k) = 0$, $t \neq t_k$, so

$$y''(t) = -\lambda t f(t,y(t)) - \lambda(1-t)f(t,y(t)) = -\lambda f(t,y(t)), \quad t \in J^0$$

and

$$y'(t_k+0)-y'(t_k-0) = \lambda[G_t'(t_k+0,t_k)-G_t'(t_k-0,t_k))]I_k(y(t_k)) = -\lambda I_k(y(t_k)).$$

Integrate $y'(t)$ from 0 to $x (x \in (0, t_1))$ and interchange the order of the integration to get

$$\begin{aligned}
\int_0^x y'(t)\,dt = \int_0^x dy(t) &= y(x) - y(0) \\
&= a\,(1-x) + b\,x + \lambda \int_0^1 G(x,s) f(s,y(s))\,ds \\
&\quad + \lambda \sum_{k=1}^m G(x,t_k)\, I_k(y(t_k)) - y(0).
\end{aligned}$$

$$\begin{aligned}
\int_0^x y'(t)\,dt &= \int_0^x (-a + b + \lambda \int_0^1 G'_t(t,s)\, f(s,y(s))\,ds \\
&\quad + \lambda \sum_{k=1}^m G'_t(t,t_k)\, I_k(y(t_k)) \\
&= -\,ax + bx + \lambda \int_0^1 [G(x,s) - G(0,s)] f(s,y(s))\,ds \\
&\quad + \lambda \sum_{k=1}^m [G(x,t_k) - G(0,t_k)]\, I_k(y(t_k)) \\
&= -\,ax + bx + \lambda \int_0^1 G(x,s) f(s,y(s))\,ds + \lambda \sum_{k=1}^m G(x,t_k)\, I_k(y(t_k)),
\end{aligned}$$

so $y(0) = a$. Similarly integrate $y'(t)$ from x $(x \in (t_m, 1))$ to 1 and interchange the order of integration to get $y(1) = b$. Thus if $y \in PC[J,R] \cap C^2[J^0,R]$ satisfies $(1.9)_\lambda$ then y is a solution of $(1.7)_\lambda$.

Define the operator $N\colon C[0,1] \to C[0,1]$ by

$$(1.10)\quad N\,y(t) = a(1-t) + bt + \int_0^1 G(t,s)\, f(s,y(s))\,ds + \sum_{k=1}^m G(t,t_k)\, I_k(y(t_k)).$$

Then $(1.9)_\lambda$ is equivalent to the fixed point problem

$$(1.11)_\lambda \qquad y = (1-\lambda)p + \lambda\, N\, y, \quad \text{where } p = a(1-t) + bt.$$

Set $U = \{u \in C[0,1] : |u|_0 < M\}$. We will show $N\colon \overline{U} \to C[0,1]$ is uniformly bounded, equicontinuous and continuous on $[0,1]$. Without loss of generality, we assume that $a = 0$, $b = 0$.

$$\begin{aligned}
(1.12)\quad |(Ny)(t)| &= \bigg| \int_0^1 G(t,s) f(s,y(s))\,ds + \sum_{k=1}^m G(t,t_k)\, I_k(y(t_k)) \bigg| \\
&\le \int_0^1 G(t,s) |f(s,y(s))|\,ds + \sum_{k=1}^m G(t,t_k) |I_k(y(t_k))| \\
&\le \int_0^1 G(t,s) h_M(s)\,ds + \sum_{k=1}^m G(t,t_k) \cdot \sup_{|x| \le M} |I_k(x)| := Y(t),
\end{aligned}$$

$t \in [0,1]$, where $Y(t)$ is the solution of following equations

$$
(1.13) \qquad \begin{cases} Y'' + h_M(t) = 0 & \text{for } t \in J^0, \\ -\Delta Y'|_{t=t_k} = \sup\limits_{|x| \le M} |I_k(x)|, & \text{for } k = 1, \dots, m, \\ Y(0) = 0, \quad Y(1) = 0, \end{cases}
$$

we have $Y \in PC[J,R] \bigcap C^2[J^0,R]$. So N is uniformly bounded on [0,1]. Noting the facts that $Y(0) = Y(1) = 0$ and the continuity of $Y(t)$ on [0,1], we have from (1.12) that for any $\varepsilon > 0$, one can find a $\delta_1 > 0$ such that $0 < \delta_1 < 1/8$ and $t_1, \dots, t_m \in (\delta_1, 1-\delta_1)$, we have

$$
(1.14) \qquad |(Ny)(t)| < \frac{\varepsilon}{2}, \quad t \in [0, 2\delta_1] \cup [1-2\delta_1, 1]
$$

We also have

$$
\begin{aligned}
(1.15) \qquad |(Ny)'(t)| &= \left| \int_0^1 G'_t(t,s) f(s,y(s))\, ds + \sum_{k=1}^m G'_t(t,t_k)\, I_k(y(t_k)) \right| \\
&\le \int_t^1 (1-s)|f(s,y(s))|\, ds + \int_0^t s|f(s,y(s))|\, ds \\
&\quad + \sum_{t<t_k} (1-t_k)|\, I_k(y(t_k))| + \sum_{t_k<t} t_k|\, I_k(y(t_k))| \\
&\le \int_{\delta_1}^1 (1-s)\, h_M(s)\, ds + \int_0^{1-\delta_1} s\, h_M(s)\, ds \\
&\quad + \sum_{k=1}^m \sup_{|x|\le M} |I_k(x)| := L,
\end{aligned}
$$

$t \in [\delta_1, 1-\delta_1]$, $t \ne t_k$, $k = 1, \dots, m$ and if $t = t_k$ $(k = 1, \dots, m)$, thus

$$
|(Ny)'(t_k+0)| \le |(Ny)'(t_k-0)| + \sup_{|x|\le M} |I_k(x)| \le L + \sup_{|x|\le M} |I_k(x)|,
$$

which is also bounded. So $(Ny)'(t)$ is bounded on $[\delta_1, 1-\delta_1]$ i.e.

$$
(1.16) \qquad |(Ny)'(t)| \le L, \quad t \in [\delta_1, 1-\delta_1].
$$

Let $\delta_2 = \varepsilon/2L$, then for $t, s \in [\delta_1, 1-\delta_1]$, $|t-s| < \delta_2$, we have

$$
(1.17) \qquad |(Ny)(t) - (Ny)(s)| \le L|t-s| < \frac{\varepsilon}{2}.
$$

Define $\delta = \min\{\delta_1, \delta_2\}$, then by using (1.14), (1.17), we obtain that

$$
(1.18) \qquad |(Ny)(t) - (Ny)(s)| < \varepsilon, \quad \text{for } s \in [0,1], \ |t-s| < \delta.
$$

This shows that $\{(Ny)(t) : y \in \overline{U}\}$ is equicontinuous on $[0,1]$. We can obtain the continuity of N in a similar way above. In fact, if $y_n, y \in \overline{U}$ and $|y_n - y|_0 \to 0$

as $n \to \infty$, then we have

$$(1.19)\quad |(Ny_n)(t) - (Ny)(t)| \le 2\bigg[\int_0^1 G(t,s)h_M(s)\,ds + \sum_{k=1}^{m} G(t,t_k)\cdot \sup_{|x|\le M}|I_k(x)|\bigg] = 2Y(t)$$

for $t \in [0,1]$. Noting the facts that $Y(0) = Y(1) = 0$ and the continuity of $Y(t)$ on $[0,1]$, then for $0 < \delta_1 < 1/8$ we have

$$(1.20)\qquad |(Ny_n)(t) - (Ny)(t)| < \varepsilon, \quad t \in [0,\delta_1] \cup [1-\delta_1, 1].$$

On the other hand, from the continuity of f, one has

$$(1.21)\qquad |(Ny_n)(t) - (Ny)(t)| \to 0, \quad t \in [\delta_1, 1-\delta_1] \text{ as } n \to \infty.$$

This together with (1.20) implies that $|(Ny_n)(t) - (Ny)(t)|_0 \to 0$ as $n \to \infty$. Therefore, $N\colon \overline{U} \to C[0,1]$ is completely continuous.

Now the nonlinear alternative of Leray–Schauder type (see [10]) guarantees that N has a fixed point i.e. $(1.9)_\lambda$ has a solution.

(b) Solving (1.2) is equivalent to the fixed point problem $y = Ny$ where N is as in (1.10). It is easy to see that $N\colon C[0,1] \to C[0,1]$ is continuous and compact (since (1.8) holds). The result follows from Schauder's fixed point theorem (see [10]). □

2. Singular boundary value problems

LEMMA 2.1 ([4]). *$S \subset PC[J,R]$ is a relative compact set if and only if for each function of S is uniform bounded on J and equicontinuous on every $J_k (k = 0,\dots,m)$.*

Consider the Dirichlet boundary value problem

$$(2.1)\qquad \begin{cases} y'' + q(t)f(t,y) = 0 & \text{for } t \in J^0, \\ -\Delta y'|_{t=t_k} = I_k(y(t_k)) & \text{for } k = 1,\dots,m, \\ y(0) = 0, \quad y(1) = 0. \end{cases}$$

Here the nonlinearity f may be singular at $y = 0$ and q may be singular at $t = 0$ and/or $t = 1$; $I_k\colon [0,\infty) \to [0,\infty)$ is continuous and nondecreasing. We begin by showing that (2.1) has a $PC[J,R] \cap C^2[J^0,R]$ solution. To do so, we first establish, via Theorem 1.1, the existence of a $PC[J,R] \cap C^2[J^0,R]$ solution, for each sufficiently large n, to the "modified" problem

$$(2.2)^n\qquad \begin{cases} y'' + q(t)f(t,y) = 0 & \text{for } t \in J^0, \\ -\Delta y'|_{t=t_k} = I_k(y(t_k)) & \text{for } k = 1,\dots,m, \\ y(0) = 1/n, \quad y(1) = 1/n. \end{cases}$$

To show that (2.1) has a solution we let $n \to \infty$; the key idea in this step is the Arzela–Ascoli theorem.

THEOREM 2.2. *Suppose the following conditions are satisfied:*

$$q \in C(0,1), \quad q > 0 \quad \textit{on } J' \quad \textit{and} \quad \int_0^1 t(1-t)q(t)\,dt < \infty \tag{2.3}$$

$$f\colon J \times (0,\infty) \to (0,\infty) \quad \textit{is continuous.} \tag{2.4}$$

$$\begin{cases} 0 \le f(t,y) \le g(y) + h(y) \ \textit{on } J \times (0,\infty) \\ \textit{with } g > 0 \textit{ continuous and nonincreasing on } (0,\infty), \\ h \ge 0 \textit{ continuous on } [0,\infty), \\ \ \textit{and } h/g \textit{ nondecreasing on } (0,\infty) \end{cases} \tag{2.5}$$

$$\begin{cases} \textit{for each constant } H > 0 \textit{ there exists a function } \psi_H \\ \textit{continuous on } J \textit{ and positive on } J' \\ \ \textit{such that } f(t,u) \ge \psi_H(t) \ \textit{on } J' \times (0,H] \end{cases} \tag{2.6}$$

and

$$\textit{there exists } r > 0 \textit{ with } \int_0^r \frac{du}{g(u)} > \frac{\sum_{k=1}^m I_k(r)}{2g(r)} + b_0\left\{1 + \frac{h(r)}{g(r)}\right\} \tag{2.7}$$

hold; here

$$b_0 = \max\left\{2\int_0^{1/2} t(1-t)q(t)\,dt,\ 2\int_{1/2}^1 t(1-t)q(t)\,dt\right\}. \tag{2.8}$$

Then (2.1) *has a solution* $y \in PC[J,R] \cap C^2[J^0,R]$ *with* $y > 0$ *on* J' *and* $|y|_0 < r$.

PROOF. Choose $\varepsilon > 0$, $\varepsilon < r$, with

$$\int_\varepsilon^r \frac{du}{g(u)} > \frac{\sum_{k=1}^m I_k(r)}{2g(r)} + b_0\left\{1 + \frac{h(r)}{g(r)}\right\}. \tag{2.9}$$

Let $n_0 \in \{1,2,\dots\}$ be chosen so that $1/n_0 < \varepsilon/2$ and let $N_0 = \{n_0, n_0+1, \dots\}$. To show $(2.2)^n$, $n \in N_0$, has a solution we examine

$$\begin{cases} y'' + q(t)F_n(t,y) = 0 & \text{for } t \in J^0, \\ -\Delta y'|_{t=t_k} = \widetilde{I}_k(y(t_k)) & \text{for } k = 1,\dots,m, \\ y(0) = 1/n, \quad y(1) = 1/n, \end{cases} \tag{2.10^n}$$

where

$$F_n(t,u) = \begin{cases} f(t,u) & \text{for } u \ge 1/n, \\ f(t,1/n) & \text{for } u < 1/n, \end{cases} \qquad \widetilde{I}_k(u) = \begin{cases} I_k(u) & \text{for } u \ge 0, \\ I_k(0) & \text{for } u < 0. \end{cases}$$

To show $(2.10)^n$ has a solution for each $n \in N_0$ we will apply Theorem 1.1. Consider the family of problems

$$(2.11)^n_\lambda \qquad \begin{cases} y'' + \lambda q(t) F_n(t,y) = 0 & \text{for } t \in J^0, \\ -\Delta y'|_{t=t_k} = \lambda \widetilde{I}_k(y(t_k)) & \text{for } k = 1, \dots, m, \\ y(0) = 1/n, \quad y(1) = 1/n & \text{for } n \in N_0, \end{cases}$$

where $0 < \lambda < 1$. Let y be a solution of $(2.11)^n_\lambda$, thus by $(1.9)_\lambda$, we obtain

$$y(t) = \frac{1}{n} + \lambda \int_0^1 G(t,s)\, F_n(s, y(s))\, ds + \lambda \sum_{k=1}^m G(t,t_k)\, \widetilde{I}_k(y(t_k)),$$

then

$$y(t) \geq 1/n, \quad t \in [0,1], \quad \widetilde{I}_k(y(t_k)) = I_k(y(t_k)), \quad F_n(t, y(t)) = f(t, y(t)).$$

Since $y'' \leq 0$ on J^0, there exists τ_n with $y(\tau_n) = \max_{t \in [0,1]}\{y(t)\}$, and $y'(\tau_n - 0) \geq 0$, $y'(\tau_n + 0) \leq 0$. Since $y'' \leq 0$ $(t \in J^0)$, then y' is nonincreasing on J_k. Suppose $t_1, \dots, t_p$ are the impulsive points in $(0, \tau_n)$. Then we have

$$\begin{gathered} y'(t) \geq y'(\tau_n - 0) \geq 0, \quad t \in (t_p, \tau_n), \\ \Delta y'|_{t=t_p} = -\lambda I_p(y(t_p)) \leq 0. \end{gathered}$$

So

$$y'(t_p) = y'(t_p - 0) \geq y'(t_p + 0) \geq y'(\tau_n - 0) \geq 0.$$

Similarly $y'(t) \geq 0$ on $J_0, \dots, J_{p-1}$, so $y'(t) \geq 0$ on $(0, \tau_n)$. Similarly $y'(t) \leq 0$ on $(\tau_n, 1)$.

For $x \in J^0$, we have

$$(2.12) \qquad -y''(x) \leq g(y(x)) \left\{ 1 + \frac{h(y(x))}{g(y(x))} \right\} q(x).$$

Integrate from t $(t \leq \tau_n)$ to τ_n to obtain

$$-\bigg(y'(\tau_n - 0) - y'(t+0) - \sum_{t < t_k < \tau_n} \Delta y'|_{t=t_k} \bigg) \leq g(y(t)) \left\{ 1 + \frac{h(y(\tau_n))}{g(y(\tau_n))} \right\} \int_t^{\tau_n} q(x)\, dx,$$

so we have

$$y'(t+0) \leq y'(\tau_n - 0) + \sum_{t < t_k < \tau_n} I_k(y(t_k)) + g(y(t)) \left\{ 1 + \frac{h(y(\tau_n))}{g(y(\tau_n))} \right\} \int_t^{\tau_n} q(x)\, dx.$$

Since $y'(\tau_n + 0) - y'(\tau_n - 0) = -I_k(y(\tau_n))$, so

$$\begin{aligned} y'(t+0) \leq{} & y'(\tau_n + 0) + I_k(y(\tau_n)) \\ & + \sum_{t < t_k < \tau_n} I_k(y(t_k)) + g(y(t)) \left\{ 1 + \frac{h(y(\tau_n))}{g(y(\tau_n))} \right\} \int_t^{\tau_n} q(x)\, dx \\ \leq{} & \sum_{k=1}^m I_k(y(\tau_n)) + g(y(t)) \left\{ 1 + \frac{h(y(\tau_n))}{g(y(\tau_n))} \right\} \int_t^{\tau_n} q(x)\, dx, \end{aligned}$$

and then integrate from 0 to τ_n to obtain

$$\int_{1/n}^{y(\tau_n)} \frac{du}{g(u)} \leq \tau_n \frac{\sum\limits_{k=1}^{m} I_k(y(\tau_n))}{g(y(\tau_n))} + \left\{1 + \frac{h(y(\tau_n))}{g(y(\tau_n))}\right\} \int_0^{\tau_n} xq(x)\,dx,$$

and so

$$\begin{aligned}(2.13)\qquad \int_{\varepsilon}^{y(\tau_n)} \frac{du}{g(u)} &\leq \tau_n \frac{\sum\limits_{k=1}^{m} I_k(y(\tau_n))}{g(y(\tau_n))} \\ &\quad + \left\{1 + \frac{h(y(\tau_n))}{g(y(\tau_n))}\right\} \frac{1}{1-\tau_n} \int_0^{\tau_n} x(1-x)\,q(x)\,dx.\end{aligned}$$

Similarly if we integrate (2.12) from τ_n to t $(t \geq \tau_n)$ and then from τ_n to 1 we obtain

$$\begin{aligned}(2.14)\qquad \int_{\varepsilon}^{y(\tau_n)} \frac{du}{g(u)} &\leq (1-\tau_n) \frac{\sum\limits_{k=1}^{m} I_k(y(\tau_n))}{g(y(\tau_n))} \\ &\quad + \left\{1 + \frac{h(y(\tau_n))}{g(y(\tau_n))}\right\} \frac{1}{\tau_n} \int_{\tau_n}^{1} x(1-x)\,q(x)\,dx.\end{aligned}$$

Now (2.13) and (2.14) imply

$$\int_{\varepsilon}^{y(\tau_n)} \frac{du}{g(u)} \leq \frac{\sum\limits_{k=1}^{m} I_k(y(\tau_n))}{2g(y(\tau_n))} + b_0 \left\{1 + \frac{h(y(\tau_n))}{g(y(\tau_n))}\right\}$$

This together with (2.9) implies $|y|_0 \neq r$. Then Theorem 1.1 implies that $(2.10)^n$ has a solution y_n with $|y_n|_0 \leq r$. In fact (as above),

$$\frac{1}{n} \leq y_n(t) < r \quad \text{for } t \in J.$$

Next we obtain a sharper lower bound on y_n, namely we will show that there exists a constant $k > 0$, independent of n, with

$$(2.15)\qquad y_n(t) \geq kt\,(1-t) \quad \text{for } t \in J.$$

To see this notice (2.6) guarantees the existence of a function $\psi_r(t)$ continuous on J and positive on J' with $f(t,u) \geq \psi_r(t)$ for $(t,u) \in J' \times (0,r]$. Now, using the Green's function representation for the solution of $(2.10)^n$, we have

$$y_n(t) = \frac{1}{n} + \int_0^1 G(t,x)\,q(x)\,f(x,y_n(x))\,dx + \sum_{k=1}^{m} G(t,t_k) I_k(y_n(t_k))$$

and so

$$(2.16)\qquad y_n(t) \geq \int_0^1 G(t,x)\,q(x)\,\psi_r(x)\,dx \equiv \Phi_r(t).$$

Now it is easy to check (as in Theorem 1.1) that

$$\begin{aligned}\Phi_r(t) &= \int_0^1 G(t,x)\,q(x)\psi_r(x)\,dx \\ &\geq t(1-t)\int_0^1 G(x,x)\,q(x)\psi_r(x)\,dx \geq t(1-t)|\Phi_r|_0\end{aligned}$$

on J, let $k = |\Phi_r|_0 > 0$, then (2.15) is true.

Next we will show

$$\{y_n\}_{n\in N_0} \quad \text{is a bounded, equicontinuous family on } J. \tag{2.17}$$

Returning to (2.12) (with y replaced by y_n) we have

$$-y_n''(x) \leq g(y_n(x))\left\{1 + \frac{h(r)}{g(r)}\right\} q(x) \quad \text{for } x \in J^0. \tag{2.18}$$

Now since $y_n'' \leq 0$ on J^0 and $y_n \geq 1/n$ on J, as discussing as from $(2.11)_\lambda^n$–(2.12), we know there exists $\tau_n \in J'$ with $y_n' \geq 0$ on $(0,\tau_n)$ and $y_n' \leq 0$ on $(\tau_n, 1)$. Integrate (2.18) from t $(t < \tau_n)$ to τ_n to obtain

$$\frac{y_n'(t+0)}{g(y_n(t))} \leq \frac{\sum\limits_{k=1}^{m} I_k(r)}{g(r)} + \left\{1 + \frac{h(r)}{g(r)}\right\}\int_t^{\tau_n} q(x)\,dx. \tag{2.19}$$

On the other hand integrate (2.18) from τ_n to t $(t > \tau_n)$ to obtain

$$\frac{-\,y_n'(t-0)}{g(y_n(t))} \leq \frac{\sum\limits_{k=1}^{m} I_k(r)}{g(r)} + \left\{1 + \frac{h(r)}{g(r)}\right\}\int_{\tau_n}^{t} q(x)\,dx. \tag{2.20}$$

We now claim that there exists a_0 and a_1 with $a_0 > 0$, $a_1 < 1$, $a_0 < a_1$ with

$$a_0 < \inf\{\tau_n : n \in N_0\} \leq \sup\{\tau_n : n \in N_0\} < a_1. \tag{2.21}$$

REMARK 2.3. Here τ_n (as before) is the unique point in $(0,1)$ with $y_n(\tau_n) = \max_{t\in[0,1]}\{y_n(t)\}$.

We now show $\inf\{\tau_n : n \in N_0\} > 0$. If this is not true then there is a subsequence S of N_0 with $\tau_n \to 0$ as $n \to \infty$ in S. Now integrate (2.19) from 0 to τ_n to obtain

$$\int_0^{y_n(\tau_n)} \frac{du}{g(u)} \leq \tau_n \frac{\sum\limits_{k=1}^{m} I_k(r)}{g(r)} + \left\{1 + \frac{h(r)}{g(r)}\right\}\int_0^{\tau_n} xq(x)\,dx + \int_0^{1/n} \frac{du}{g(u)} \tag{2.22}$$

for $n \in S$. Since $\tau_n \to 0$ as $n \to \infty$ in S, we have from (2.22) that $y_n(\tau_n) \to 0$ as $n \to \infty$ in S. However since the maximum of y_n on J occurs at τ_n we have $y_n \to 0$ as $n \to \infty$ in S. This contradicts (2.15). Consequently $\inf\{\tau_n : n \in N_0\} > 0$.

A similar argument shows $\sup\{\tau_n : n \in N_0\} < 1$. Let a_0 and a_1 be chosen as in (2.21). Now (2.19)–(2.21) imply

$$\frac{|y_n'(t)|}{g(y_n(t))} \le \frac{\sum_{k=1}^{m} I_k(r)}{g(r)} + \left\{1 + \frac{h(r)}{g(r)}\right\} v(t) \quad \text{for } t \in J' \tag{2.23}$$

where

$$v(t) = \int_{\min\{t,a_0\}}^{\max\{t,a_1\}} q(x)\,dx.$$

It is easy to see that $v \in L^1[J]$. Let $B\colon [0,\infty) \to [0,\infty)$ be defined by

$$B(z) = \int_0^z \frac{du}{g(u)}.$$

Note B is an increasing map from $[0,\infty)$ onto $[0,\infty)$ (notice $B(\infty) = \infty$ since $g > 0$ is nonincreasing on $(0,\infty)$) with B continuous on $[0,a]$ for any $a > 0$. Notice

$$\{B(y_n)\}_{n \in N_0} \quad \text{is a bounded, equicontinuous family on } J. \tag{2.24}$$

The equicontinuity follows from (here $t, s \in J$)

$$\begin{aligned}|B(y_n(t)) - B(y_n(s))| &= \left| \int_s^t \frac{d(y_n(x))}{g(y_n(x))} \right| \\ &\le \left\{1 + \frac{h(r)}{g(r)}\right\} \left| \int_s^t v(x)\,dx \right| + \frac{1}{2}|t-s| \frac{\sum_{k=1}^{m} I_k(r)}{g(r)}.\end{aligned}$$

This inequality, the uniform continuity of B^{-1} on $[0, B(r)]$, and

$$|y_n(t) - y_n(s)| = |B^{-1}(B(y_n(t))) - B^{-1}(B(y_n(s)))|$$

now establishes (2.17).

The Arzela–Ascoli Theorem guarantees the existence of a subsequence N of N_0 and a function $y \in PC[J,R] \cap C^2[J^0,R]$ with y_n converging uniformly on J to y as $n \to \infty$ through N. Also $y(0) = y(1) = 0$, $|y|_0 \le r$ and $y(t) \ge kt(1-t)$ for $t \in J$. In particular $y > 0$ on J'. Fix $t \in (0,t_1)$, then y_n $(n \in N)$ satisfies the integral equation

$$y_n(x) = y_n\left(\frac{t_1}{2}\right) + y_n'\left(\frac{t_1}{2}\right)\left(x - \frac{t_1}{2}\right) + \int_{t_1/2}^{x} (s-x)\,q(s)\,f(s,y_n(s))\,ds,$$

for $x \in (0,t_1)$. Notice that $\{y_n'(t_1/2)\}$, $n \in N$, is a bounded sequence since $ks(1-s) \le y_n(s) \le r$ for $s \in J'$. Thus $\{y_n'(t_1/2)\}_{n \in N}$ has a convergent subsequence; so $\{y_n(t)\}_{n \in N}$ is relative compact on $(0,t_1)$. For convenience, let $\{y_n'(t_1/2)\}_{n \in N}$ denote this subsequence also let $r_0 \in R$ be its limit and let $n \to \infty$ through N

(we note here that f is uniformly continuous on compact subsets of $[\min(t_1/2, t), \max(t_1/2, t)] \times (0, r]$) to obtain

$$y(t) = y\left(\frac{t_1}{2}\right) + r_0\left(t - \frac{t_1}{2}\right) + \int_{t_1/2}^{t} (s - t)\, q(s)\, f(s, y(s))\, ds.$$

We can do this argument for each $t \in (0, t_1)$ and so $y''(t) + q(t)\, f(t, y(t)) = 0$ for $t \in (0, t_1)$.

Similarly, we can obtain the same results in $(t_1, t_2), (t_2, t_3), \ldots, (t_m, 1)$.

Finally it is easy to see that $|y|_0 < r$ (note if $|y|_0 = r$ then following essentially the argument from (2.12)–(2.14) will yield a contradiction). □

Next we establish the existence of two nonnegative solutions to the singular second order Dirichlet problem

$$\begin{cases} y''(t) + q(t)[g(y(t)) + h(y(t))] = 0 & \text{for } t \in J^0, \\ -\Delta y'|_{t=t_k} = I_k(y(t_k)) & \text{for } k = 1, \ldots, m, \\ y(0) = y(1) = 0; \end{cases} \tag{2.25}$$

here our nonlinear term $g + h$ may be singular at $y = 0$. First we state the fixed point result.

THEOREM 2.4 ([1]). *Let $E = (E, \|\cdot\|)$ be a Banach space, $K \subset E$ a cone and let $\|\cdot\|$ be increasing with respect to K. Also r, R are constants with $0 < r < R$. Suppose $A\colon \overline{\Omega_R} \cap K \to K$ (here $\Omega_R = \{x \in E : \|x\| < R\}$) is a continuous, compact map and assume the following conditions hold:*

$$x \neq \lambda A(x) \quad \text{for } \lambda \in [0, 1) \text{ and } x \in \partial_E \Omega_r \cap K \tag{2.26}$$

and

$$\|A\,x\| > \|x\| \quad \text{for } x \in \partial_E \Omega_R \cap K. \tag{2.27}$$

Then A has a fixed point in $K \cap \{x \in E : r \leq \|x\| \leq R\}$.

REMARK 2.5. In Theorem 2.4 if (2.26) and (2.27) are replaced by

$$x \neq \lambda\, A(x) \quad \text{for } \lambda \in [0, 1) \text{ and } x \in \partial_E \Omega_R \cap K \tag{2.28}$$

and

$$\|A\,x\| > \|x\| \quad \text{for } x \in \partial_E \Omega_r \cap K. \tag{2.29}$$

then A has a fixed point in $K \cap \{x \in E : r \leq \|x\| \leq R\}$.

medskip Now $E = (PC[J, R], |\cdot|_0)$ (here $|u|_0 = \sup_{t \in [0,1]} |u(t)|$, $u \in PC[J, R]$) will be our Banach space and

$$K = \{y \in PC[J, R] : y(t) \geq 0,\ t \in J \text{ and } y(t) \geq t(1 - t)|y|_0 \text{ on } J\}. \tag{2.30}$$

From Theorem 2.2 we have immediately the following existence result for (2.25).

Theorem 2.6. *Suppose the following conditions are satisfied:*

$$q \in C(0,1), \quad q > 0 \quad \text{on } J' \quad \text{and} \quad \int_0^1 t(1-t)q(t)\,dt < \infty, \tag{2.31}$$

$$g > 0 \quad \text{is continuous and nonincreasing on } (0,\infty), \tag{2.32}$$

$$h \geq 0 \text{ continuous on } [0,\infty) \text{ with } h/g \text{ nondecreasing on } (0,\infty) \tag{2.33}$$

and

$$\text{there exists } r > 0 \text{ with } \int_0^r \frac{du}{g(u)} > \frac{\sum_{k=1}^m I_k(r)}{2g(r)} + b_0\left\{1 + \frac{h(r)}{g(r)}\right\}; \tag{2.34}$$

here

$$b_0 = \max\left\{2\int_0^{1/2} t(1-t)q(t)\,dt,\ 2\int_{1/2}^1 t(1-t)q(t)\,dt\right\}. \tag{2.35}$$

Then (2.25) *has a solution* $y \in PC[J,R] \cap C^2[J^0,R]$ *with* $y > 0$ *on* J' *and* $|y|_0 < r$.

Proof. The result follows from Theorem 2.2 with $f(t,u) = g(u) + h(u)$. Notice (2.6) is clearly satisfied with $\psi_H(t) = g(H)$. $\square$

Theorem 2.7. *Assume* (2.31)–(2.34) *hold. Choose* $a \in (0,1/2)$ *and fix it and suppose there exists* $R > r$ *with*

$$R < g(R)\left\{1 + \frac{h(a(1-a)R)}{g(a(1-a)R)}\right\}\int_a^{1-a} G(\sigma,s)\,q(s)\,ds + \sum_{k=1}^m G(\sigma,t_k)I_k(t_k(1-t_k)R); \tag{2.36}$$

here $0 \leq \sigma \leq 1$ *is such that*

$$\int_a^{1-a} G(\sigma,s)\,q(s)\,ds = \sup_{t\in[0,1]} \int_a^{1-a} G(t,s)\,q(s)\,ds \tag{2.37}$$

and

$$G(t,s) = \begin{cases} (1-t)s & \text{for } 0 \leq s \leq t, \\ (1-s)t & \text{for } t \leq s \leq 1. \end{cases}$$

Then (2.25) has a solution $y \in PC[0,1] \cap C^2[J^0,R]$ *with* $y > 0$ *on* J' *and* $r < |y|_0 \leq R$.

Proof. To show the existence of the solution described in the statement of Theorem 2.10 we will apply Theorem 2.4. First however choose $\varepsilon > 0$ and $\varepsilon < r$

with

$$\int_\varepsilon^r \frac{du}{g(u)} > \frac{\sum\limits_{k=1}^m I_k(r)}{2g(r)} + b_0\bigg\{1+\frac{h(r)}{g(r)}\bigg\}. \tag{2.38}$$

Let $n_0 \in \{1,2,\ldots\}$ be chosen so that $(1/n_0) < (\varepsilon/2)$ and $(1/n_0) < a(1-a)R$ and let $N_0 = \{n_0, n_0+1, \ldots\}$. We first show that

$$(2.39)^n \qquad \begin{cases} y''(t) + q(t)\,[g(y(t)) + h(y(t))] = 0 & \text{for } t \in J^0, \\ -\Delta y'|_{t=t_k} = I_k(y(t_k)) & \text{for } k = 1,\ldots,m, \\ y(0) = y(1) = 1/n, \end{cases}$$

has a solution y_n for each $n \in N_0$ with $y_n(t) \geq 1/n$ on J and $r \leq |y_n|_0 \leq R$. To show $(2.39)^n$ has such a solution for each $n \in N_0$, we will look at

$$(2.40)^n \qquad \begin{cases} y''(t) + q(t)\,[g^*(y(t)) + h(y(t))] = 0 & \text{for } t \in J^0, \\ -\Delta y'|_{t=t_k} = \widetilde{I}_k(y(t_k)) & \text{for } k = 1,\ldots,m, \\ y(0) = y(1) = 1/n, \end{cases}$$

with

$$g^\star(u) = \begin{cases} g(u) & \text{for } u \geq 1/n, \\ g(1/n), & \text{for } 0 \leq u < 1/n. \end{cases} \qquad \widetilde{I}_k(u) = \begin{cases} I_k(u) & \text{for } u \geq 0, \\ I_k(0) & \text{for } u < 0. \end{cases}$$

REMARK 2.8. Notice $g^\star(u) \leq g(u)$ for $u \geq 0$.

Fix $n \in N_0$. Let $E = (PC[J,R], |\cdot|_0)$ and

$$K = \{u \in PC[J,R] : u(t) \geq 0,\ t \in J \text{ and } u(t) \geq t(1-t)|u|_0 \text{ on } J\}. \tag{2.41}$$

Clearly K is a cone of E. Let $A\colon K \to PC[J,R]$ be defined by

$$Ay(t) = \frac{1}{n} + \int_0^1 G(t,s)\,q(s)\,[g^\star(y(s)) + h(y(s))]\,ds + \sum_{k=1}^m G(t,t_k)\widetilde{I}_k(y(t_k)). \tag{2.42}$$

A standard argument implies $A\colon K \to PC[J,R]$ is continuous and completely continuous. Next we show $A\colon K \to K$. If $u \in K$ then clearly $A\,u(t) \geq 0$ for $t \in J$. Also notice that

$$\begin{aligned} Ay(t) \geq \frac{1}{n} + t(1-t)\int_0^1 s(1-s)q(s)[g^\star(y(s) + hy(s))]\,ds \\ + t(1-t)\sum_{k=1}^m t_k(1-t_k)\widetilde{I}_k(y(t_k)) \end{aligned}$$

and

$$t(1-t)|Ay|_0 \le t(1-t)\left[\frac{1}{n} + \int_0^1 s(1-s)q(s)[g^\star(y(s)+hy(s))\right] ds$$
$$+ \sum_{k=1}^{m} t_k(1-t_k)\widetilde{I}_k(y(t_k))],$$

so $Ay(t) \ge t(1-t)|Ay|_0$ and

$$\begin{cases} (Au)''(t) \le 0 & \text{on } J^0, \\ Au(0) = A\,u(1) = 1/n. \end{cases}$$

Consequently $Au \in K$ so $A\colon K \to K$. Let $\Omega_1 = \{u \in PC[J,R] : |u|_0 < r\}$ and $\Omega_2 = \{u \in PC[J,R] : |u|_0 < R\}$. We first show

$$y \ne \lambda\,Ay \quad \text{for } \lambda \in [0,1) \text{ and } y \in K \cap \partial\Omega_1. \tag{2.43}$$

Suppose this is false i.e. suppose there exists $y \in K \cap \partial\Omega_1$ and $\lambda \in [0,1)$ with $y = \lambda\,Ay$. We can assume $\lambda \ne 0$. Now since $y = \lambda\,Ay$ we have

$$\begin{cases} y''(t) + \lambda\,q(t)[g^\star(y(t)) + h(y(t))] = 0 & \text{for } t \in J^0, \\ -\Delta y'|_{t=t_k} = \lambda I_k(y(t_k)) & \text{for } k = 1,\dots,m, \\ y(0) = y(1) = 1/n. \end{cases} \tag{2.44}$$

Since $y'' \le 0$ on J^0 and there exists $t_0 \in J'$ with $y(t_0) = \max_{t\in[0,1]}\{y(t)\}$ and $y'(t_0-0) \ge 0$, $y'(t_0+0) \le 0$. As the same way as in Theorem 2.4, $y'(t) \ge 0$ on $(0,t_0)$ and $y'(t) \le 0$ on $(t_0,1)$ and $y \ge (1/n)$ on J and $y(t_0) = |y|_0 = r$ (note $y \in K \cap \partial\Omega_1$). Also notice

$$g^\star(y(t)) + h(y(t)) \le g(y(t)) + h(y(t)) \quad \text{for } t \in J'$$

since g is nonincreasing on $(0,\infty)$. For $x \in J^0$ we have

$$-y''(x) \le g(y(x))\left\{1 + \frac{h(y(x))}{g(y(x))}\right\}q(x). \tag{2.45}$$

Integrate from t ($t \le t_0$) to t_0 to obtain

$$y'(t+0) \le \sum_{k=1}^{m} I_k(r) + g(y(t))\left\{1 + \frac{h(r)}{g(r)}\right\}\int_t^{t_0} q(x)\,dx$$

and then integrate from 0 to t_0 to obtain

$$\int_{1/n}^{r} \frac{du}{g(u)} \le t_0\frac{\sum_{k=1}^{m} I_k(r)}{g(r)} + \left\{1 + \frac{h(r)}{g(r)}\right\}\int_0^{t_0} xq(x)\,dx.$$

Consequently

$$\int_\varepsilon^r \frac{du}{g(u)} \le t_0 \frac{\sum\limits_{k=1}^m I_k(r)}{g(r)} + \left\{1 + \frac{h(r)}{g(r)}\right\} \int_0^{t_0} x\, q(x)\, dx$$

and so

$$\int_\varepsilon^r \frac{du}{g(u)} \le t_0 \frac{\sum\limits_{k=1}^m I_k(r)}{g(r)} + \left\{1 + \frac{h(r)}{g(r)}\right\} \frac{1}{1-t_0} \int_0^{t_0} x(1-x)\, q(x)\, dx. \tag{2.46}$$

Similarly if we integrate (2.45) from t_0 to t ($t \ge t_0$) and then from t_0 to 1 we obtain

$$\int_\varepsilon^r \frac{du}{g(u)} \le (1-t_0) \frac{\sum\limits_{k=1}^m I_k(r)}{g(r)} + \left\{1 + \frac{h(r)}{g(r)}\right\} \frac{1}{t_0} \int_{t_0}^{1} x(1-x)\, q(x)\, dx. \tag{2.47}$$

Now (2.46) and (2.47) imply

$$\int_\varepsilon^r \frac{du}{g(u)} \le \frac{\sum\limits_{k=1}^m I_k(r)}{2g(r)} + b_0 \left\{1 + \frac{h(r)}{g(r)}\right\} \tag{2.48}$$

where b_0 is as defined in (2.35). This contradicts (2.38) and consequently (2.43) is true.

Next we show

$$|A\,y|_0 > |y|_0 \quad \text{for } y \in K \cap \partial\Omega_2. \tag{2.49}$$

To see this let $y \in K \cap \partial\Omega_2$ so $|y|_0 = R$. Also since $y(t)$ is satisfied

$$y(t) \ge t(1-t)|y|_0 \ge t(1-t)R, \quad \text{for } \in J.$$

Also for $s \in [a, 1-a]$ we have

$$g^\star(y(s)) + h(y(s)) = g(y(s)) + h(y(s))$$

since $y(s) \ge a(1-a)\, R > (1/n_0)$ for $s \in [a, 1-a]$. Note in particular that

$$y(s) \in [a(1-a)\, R,\, R] \quad \text{for } s \in [a, 1-a]\,. \tag{2.50}$$

With σ as defined in (2.37) we have using (2.50) and (2.36),

$$\begin{aligned}
A\,y(\sigma) &= \frac{1}{n} + \int_0^1 G(\sigma, s)\, q(s)\, [g^\star(y(s)) + h(y(s))]\, ds + \sum_{k=1}^m G(\sigma, t_k) I_k(y(t_k)) \\
&\ge \int_a^{1-a} G(\sigma, s) q(s)\, [g^\star(y(s)) + h(y(s))]\, ds + \sum_{k=1}^m G(\sigma, t_k) I_k(t_k(1-t_k)R) \\
&= \int_a^{1-a} G(\sigma, s)\, q(s)\, g(y(s)) \left\{1 + \frac{h(y(s))}{g(y(s))}\right\} ds
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{m} G(\sigma,t_k)I_k(t_k(1-t_k)R)\\
&\geq g(R)\left\{1+\frac{h(a\,(1-a)\,R)}{g(a\,(1-a)\,R)}\right\}\int_a^{1-a} G(\sigma,s)\,q(s)\,ds\\
&+\sum_{k=1}^{m} G(\sigma,t_k)I_k(t_k(1-t_k)R) > R = |y|_0,
\end{aligned}
$$

and so $|A\,y|_0 > |y|_0$. Hence (2.49) is true.

Now Theorem 2.4 implies A has a fixed point $y_n \in K \cap (\overline{\Omega_2} \setminus \Omega_1)$ i.e. $r \leq |y_n|_0 \leq R$. In fact $|y_n|_0 > r$ (note if $|y_n|_0 = r$ then following essentially the same argument from (2.45)–(2.48) will yield a contradiction). Consequently $(2.40)^n$ (and also $(2.39)^n$) has a solution $y_n \in PC[J,R] \cap C^2[J^0,R]$, $y_n \in K$, with

$$\frac{1}{n} \leq y_n(t) \quad \text{for } t \in J,\ r < |y_n|_0 \leq R \tag{2.51}$$

and (note $y_n \in K$)

$$y_n(t) \geq t(1-t)\,r \quad \text{for } t \in J. \tag{2.52}$$

Next we will show

$$\{y_n\}_{n\in N_0} \quad \text{is a bounded, equicontinuous family on } J. \tag{2.53}$$

Returning to (2.45) (with y replaced by y_n) we have

$$-y_n''(x) \leq g(y_n(x))\left\{1+\frac{h(R)}{g(R)}\right\}q(x) \quad \text{for } x \in J^0. \tag{2.54}$$

Now since $y_n'' \leq 0$ on J^0 and $y_n \geq \frac{1}{n}$ on J. As discussing as in Theorem 2.2, there exists $\tau_n \in J'$ with $y_n' \geq 0$ on $(0,\tau_n)$ and $y_n' \leq 0$ on $(\tau_n, 1)$. Integrate (2.54) from t $(t < \tau_n)$ to τ_n to obtain

$$\frac{y_n'(t+0)}{g(y_n(t))} \leq \frac{\sum\limits_{k=1}^{m} I_k(R)}{g(R)} + \left\{1+\frac{h(R)}{g(R)}\right\}\int_t^{\tau_n} q(x)\,dx. \tag{2.55}$$

On the other hand integrate (2.54) from τ_n to t $(t > \tau_n)$ to obtain

$$\frac{-y_n'(t-0)}{g(y_n(t))} \leq \frac{\sum\limits_{k=1}^{m} I_k(R)}{g(R)} + \left\{1+\frac{h(R)}{g(R)}\right\}\int_{\tau_n}^{t} q(x)\,dx. \tag{2.56}$$

We now claim that there exists a_0 and a_1 with $a_0 > 0$, $a_1 < 1$, $a_0 < a_1$ with

$$a_0 < \inf\{\tau_n : m \in N_0\} \leq \sup\{\tau_n : n \in N_0\} < a_1. \tag{2.57}$$

REMARK 2.9. Here τ_n (as before) is the unique point in $(0,1)$ with $y_n(\tau_n) = \max_{t\in[0,1]}\{y_n(t)\}$.

We now show $\inf\{\tau_n : n \in N_0\} > 0$. If this is not true then there is a subsequence S of N_0 with $\tau_n \to 0$ as $n \to \infty$ in S. Now integrate (2.55) from 0 to τ_n to obtain

$$\int_0^{y_n(\tau_n)} \frac{du}{g(u)} \le \tau_n \frac{\sum_{k=1}^{m} I_k(R)}{g(R)} + \left\{1 + \frac{h(R)}{g(R)}\right\} \int_0^{\tau_n} x\, q(x)\, dx + \int_0^{1/n} \frac{du}{g(u)} \tag{2.58}$$

for $n \in S$. Since $\tau_n \to 0$ as $n \to \infty$ in S, we have from (2.58) that $y_n(\tau_n) \to 0$ as $n \to \infty$ in S. However since the maximum of y_n on J occurs at τ_n we have $y_n \to 0$ in $PC[J, R]$ as $n \to \infty$ in S. This contradicts (2.52). Consequently $\inf\{\tau_n : n \in N_0\} > 0$. A similar argument shows $\sup\{\tau_n : n \in N_0\} < 1$. Let a_0 and a_1 be chosen as in (2.57). Now (2.55)–(2.57) imply

$$\frac{|y_n'(t)|}{g(y_n(t))} \le \frac{\sum_{k=1}^{m} I_k(R)}{g(R)} + \left\{1 + \frac{h(R)}{g(R)}\right\} v(t) \quad \text{for } t \in J' \tag{2.59}$$

where

$$v(t) = \int_{\min\{t,a_0\}}^{\max\{t,a_1\}} q(x)\, dx.$$

It is easy to see that $v \in L^1[J]$. Let $B\colon [0,\infty) \to [0,\infty)$ be defined by

$$B(z) = \int_0^z \frac{du}{g(u)}.$$

Note B is an increasing map from $[0,\infty)$ onto $[0,\infty)$ (notice $B(\infty) = \infty$ since $g > 0$ is nonincreasing on $(0,\infty)$) with B continuous on $[0, a]$ for any $a > 0$. Notice

$$\{B(y_n)\}_{n \in N_0} \quad \text{is a bounded, equicontinuous family on } J. \tag{2.60}$$

The equicontinuity follows from (here $t, s \in J$)

$$\begin{aligned} |B(y_n(t)) - B(y_n(s))| &= \left| \int_s^t \frac{d(y_n(x))}{g(y_n(x))} \right| \\ &\le \frac{1}{2}|t - s| \frac{\sum_{k=1}^{m} I_k(R)}{g(R)} + \left\{1 + \frac{h(R)}{g(R)}\right\} \left| \int_s^t v(x)\, dx \right|. \end{aligned}$$

This inequality, the uniform continuity of B^{-1} on $[0, B(R)]$, and

$$|y_n(t) - y_n(s)| = |B^{-1}(B(y_n(t))) - B^{-1}(B(y_n(s)))|$$

now establishes (2.53).

The Arzela–Ascoli Theorem guarantees the existence of a subsequence N of N_0 and a function $y \in PC[J, R] \cap C^2[J^0, R]$ with y_n converging uniformly on J to y as $n \to \infty$ through N. Also $y(0) = y(1) = 0$, $r \le |y|_0 \le R$ and $y(t) \ge t(1-t)\, r$ for $t \in J$. In particular $y > 0$ on J'.

In the same way as in Theorem 2.2, we can prove $y''(t) + q(t)\,[g(y(t)) + h(y(t))] = 0$ for $t \in J^0$. Finally, it is easy to see that $|y|_0 > r$ (note if $|y|_0 = r$ then following essentially the argument from (2.45)–(2.48) will yield a contradiction). $\square$

THEOREM 2.10. *Assume* (2.31)–(2.34) *and* (2.36) *hold. Then* (2.25) *has two solutions* $y_1, y_2 \in PC[J,R] \cap C^2[J^0,R]$ *with* $y_1 > 0$, $y_2 > 0$ *on* J' *and* $|y_1|_0 < r < |y_2|_0 \le R$.

PROOF. The existence of y_1 follows from Theorem 2.6 and the existence of y_2 follows from Theorem 2.2. $\square$

EXAMPLE 2.11. Consider the singular boundary value problem for second order impulsive differential equation:

$$\begin{cases} y'' + \dfrac{1}{\alpha+1}(y^{-\alpha} + y^{\beta} + 1) = 0 & \text{for } t \in J^0, \\ -\Delta y'|_{t=t_k} = c_k y(t_k), & \text{for } k = 1, \dots, m,\ c_k \ge 0, \\ y(0) = y(1) = 0, \quad \alpha > 0, \quad \beta > 1. \end{cases} \tag{2.61}$$

Suppose $0 < \sum_{k=1}^m c_k < (1/\alpha + 1)$, then (2.61) has two solutions $y_1, y_2 \in PC[J,R] \cap C^2[J^0,R]$ with $y_1 > 0$, $y_2 > 0$ on J' and $|y_1|_0 < 1 < |y_2|_0$.

To see this we will apply Theorem 2.10 with $q = (1/\alpha + 1)$, $g(u) = u^{-\alpha}$ and $h(u) = u^{\beta} + 1$. Clearly (2.31)–(2.33) hold. Also note

$$b_0 = \max\left\{ \frac{2}{\alpha+1} \int_0^{1/2} t(1-t)\,dt,\ \frac{2}{\alpha+1} \int_{1/2}^1 t(1-t)\,dt \right\} = \frac{1}{6(\alpha+1)}.$$

Consequently (2.34) holds (with $r = 1$) since

$$\begin{aligned} \frac{\displaystyle\int_0^r \frac{du}{g(u)} - \frac{1}{2g(r)} \sum_{k=1}^m I_k(r)}{\left\{1 + \dfrac{h(r)}{g(r)}\right\}} &= \frac{\dfrac{r^{\alpha+1}}{\alpha+1} - \dfrac{1}{2} r^{\alpha} \displaystyle\sum_{k=1}^m c_k r}{(1 + r^{\alpha+\beta} + r^{\alpha})} \\ &= \frac{\dfrac{1}{(\alpha+1)} - \dfrac{1}{2} \displaystyle\sum_{k=1}^m c_k}{3} \\ &> \frac{\dfrac{1}{(\alpha+1)} - \dfrac{1}{2(\alpha+1)}}{3} = \frac{1}{6(\alpha+1)} > b_0. \end{aligned}$$

Finally note (since $\beta > 1$), take $a = 1/4$, that

$$\lim_{R\to\infty} \frac{\Big[R - \sum_{k=1}^{m} G(\sigma, t_k) I_k(t_k(1-t_k)R)\Big]\, g\Big(\dfrac{3\,R}{16}\Big)}{g(R)\, g\Big(\dfrac{3\,R}{16}\Big) + g(R)\, h\Big(\dfrac{3\,R}{16}\Big)}$$

$$= \lim_{R\to\infty} \frac{\Big[R - \sum_{k=1}^{m} G(\sigma, t_k) c_k t_k (1-t_k) R\Big] \Big(\dfrac{3}{16}R\Big)^{-\alpha}}{R^{-\alpha}\Big[\Big(\dfrac{3}{16}R\Big)^{\beta} + 1 + \Big(\dfrac{3}{16}\Big)^{-\alpha}\Big]}$$

$$= \lim_{R\to\infty} \frac{R\Big[1 - \sum_{k=1}^{m} G(\sigma, t_k) c_k t_k (1-t_k)\Big] \Big(\dfrac{16}{3}\Big)^{\alpha}}{\Big[\Big(\dfrac{3}{16}R\Big)^{\beta} + 1 + \Big(\dfrac{16}{3R}\Big)^{\alpha}\Big]} = 0,$$

so there exists $R > 1$ with (2.36) holding. The result now follows from Theorem 2.10.

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