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HOMOTOPY METHOD FOR POSITIVE SOLUTIONS OF p-LAPLACE INCLUSIONS

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Abstract. In this paper the compression-expansion fixed point theorems are extended to operators which are compositions of two multi-valued nonlinear maps and satisfy compactness conditions of Mönch type with respect to the weak or the strong topology. As an application, the existence of positive solutions for p-Laplace inclusions is studied.

1. Introduction

Various mathematical models for nonlinear problems are expressed as boundary value problems for differential, integro-differential, or more generally, functional-differential equations or inclusions. These can be equivalently reformulated as an operator inclusion with decomposable maps,

$$x \in \Psi\Phi x \tag{1.1}$$

where $\Psi\Phi$ stands for the composition $\Psi \circ \Phi$ of two single or multi-valued operators Ψ and Φ. In [14], a fixed point approach has been used to develop an existence theory for inclusion (1.1). The key result in [14] is the following fixed point principle for decomposable non-convex-valued maps. Before we state it we

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introduce the notation:

$$\begin{aligned} P_{fc}(X) &= \{A \subset X : A \text{ is nonempty, closed, convex}\}, \\ P_{k^w c}(X) &= \{A \subset X : A \text{ is nonempty, weakly compact, convex}\}. \end{aligned}$$

Here X is a closed convex subset of a normed linear space.

Theorem 1.1. *Let X, Y be normed linear spaces or, more generally, metrizable locally convex linear topological spaces, let A and B be non-empty weakly compact convex subsets of X and Y, respectively, and let*

$$\Phi: A \to P_{fc}(B), \quad \Psi: B \to P_{fc}(A)$$

be two multi-valued maps. Assume Φ and Ψ are sequentially weakly upper semi-continuous (w-u.s.c. *for short*). *Then there exists at least one $x \in A$ with $x \in \Psi\Phi x$ and, equivalently, there exists at least one $y \in B$ with $y \in \Phi\Psi y$.*

Remark 1.2. The result remains true if A, B are compact with respect to the strong topology and Φ, Ψ are upper semi-continuous (u.s.c. for short). Indeed, any u.s.c. map on a compact set is sequentially w-u.s.c.

The above principle yields the following fixed point theorems for self-maps of a closed convex set.

Theorem 1.3. *Let X, Y be normed linear spaces, let C be a closed convex subset of X, and let*

$$\Phi: C \to P_{k^w c}(Y), \quad \Psi: \overline{\text{co}}\, \Phi(C) \to P_{fc}(C)$$

be two multi-valued maps. Assume that there exists $x_0 \in C$ such that the following condition holds:

(1.2) *if $A \subset C$, $A = \overline{\text{co}}\,(\{x_0\} \cup \Psi(\overline{\text{co}}\, \Phi(A)))$ then A is weakly compact and Φ, Ψ are sequentially w-u.s.c. on A and $\overline{\text{co}}\, \Phi(A)$, respectively.*

Then there exists at least one $x \in C$ with $x \in \Psi\Phi x$.

Remark 1.4. Theorem 1.3 is also true if instead of (1.2) we take the condition

if $A \subset C$, $A = \overline{\text{co}}\,(\{x_0\} \cup \Psi(\overline{\text{co}}\, \Phi(A)))$ then A is compact and Φ, Ψ are u.s.c. on A and $\overline{\text{co}}\, \Phi(A)$, respectively.

As a result, a Leray–Schauder type continuation theorem for non-self-maps was also obtained in [14].

THEOREM 1.5. *Let X and Y be normed linear spaces, K a closed convex subset of X, U a convex relatively open subset of K, $x_0 \in U$ and let*

$$\Phi: \overline{U} \to P_{k^w c}(Y), \quad \Psi: \overline{\mathrm{co}}\, \Phi(\overline{U}) \to P_{fc}(K)$$

be two multi-valued maps. Assume that for every compact convex subset A of $\overline{U}$, Φ and Ψ are sequentially w-u.s.c. on A and $\overline{\mathrm{co}}\, \Phi(A)$, respectively. Also assume that the two conditions:

(a) *if $A \subset \overline{U}$, A closed convex, $A \subset \overline{\mathrm{co}}\,(\{x_0\} \cup \Psi(\overline{\mathrm{co}}\, \Phi(A)))$ then A is compact,*
(b) *$x \notin (1-\lambda)x_0 + \lambda \Psi \Phi x$ for all $x \in \overline{U} \setminus U$, $\lambda \in (0,1)$*

are satisfied. Then there exists at least one $x \in \overline{U}$ with $x \in \Psi \Phi x$.

The main idea of the proof of Theorem 1.1 is to consider the Cartesian product map $\Pi: A \times B \to P_{fc}(A \times B)$, given by

$$\Pi(x, y) = \Psi y \times \Phi x$$

whose values are convex in $X \times Y$. This is done to overcome the difficulty that $\Psi \Phi x$ is generally non-convex. Then observe that $A \times B$ is a weakly compact convex subset of $X \times Y$ and Π is sequentially w-u.s.c. Now the Arino–Gautier–Penot fixed point theorem guarantees the existence of a fixed point (x, y) of Π. Hence $x \in \Psi y$ and $y \in \Phi x$, whence $x \in \Psi \Phi x$.

The same idea will be used in this paper to give extensions for inclusions of type (1.1), of the Krasnoselskii's compression-expansion fixed point theorem in a cone. Such type of results give information about the existence of nontrivial (non-zero) solutions, and also, in some cases, lead to multiplicity theorems. Our extensions will use compactness conditions of Mönch type with respect to the strong topology, or to the weak topology, in the same way that Theorem 1.3 does. This abstract part of the paper also relates to the continuation theorems established in [6] and [7]. As an application and mainly motivated by the recent paper [15], we present an existence principle for positive solutions of inclusions with p-Laplacian. We shall use the fixed point approach and basic results on the p-Laplacian (see [8], [4], [19]) in order to complement and extend to p-Laplacian some methods and results given for the classical Laplacian ([9], [12], [17]), and to inclusions with p-Laplacian, several results obtained for p-Laplace equations [2], [3], [13], [15] and [18].

2. Compression-expansion fixed point theorems

Let $(X, |\cdot|)$ and Y be normed linear spaces and K a wedge of X. We start with a compression type theorem.

Theorem 2.1. *Let*

$$\Phi: K \to P_{k^w c}(Y), \quad \Psi: K \times C \to P_{fc}(K)$$

be two bounded multi-valued maps, where $C = \overline{\mathrm{co}}(\{0\} \cup \Phi(K))$. *Assume that*

(2.1) *if* $A \subset K$, $A = \overline{\mathrm{co}}\,(\{0\} \cup \Psi(A \times \overline{\mathrm{co}}(\{0\} \cup \Phi(A)))$
then A *is weakly compact and* Φ, Ψ *are sequentially w-u.s.c. on* A
and $\overline{\mathrm{co}}(\{0\} \cup \Phi(A))$, *respectively.*

In addition assume that there are $0 < r < R$, $x_0 \in K$, $r \le |x_0| \le R$ *and* $h \in \Psi(x_0, \Phi(x_0))$, $h \ne 0$, *such that the following compression condition is satisfied:*

(2.2) $x \notin \lambda\Psi(x, \Phi x)$ *for* $\lambda \in (0,1)$ *and* $x \in K$ *with* $|x| = R$,

(2.3) $x \notin \Psi(x, \Phi x) + \mu h$ *for* $\mu > 0$ *and* $x \in K$ *with* $|x| = r$.

Then there exists at least one $x \in K$ *with* $x \in \Psi(x, \Phi x)$ *and* $r \le |x| \le R$.

Proof. Let $\mathcal{M}$ be the collection of all closed, convex subsets M of K with $x_0 \in M$ and

$$\overline{\mathrm{co}}(\{0\} \cup \Psi(M \times \overline{\mathrm{co}}(\{0\} \cup \Phi(M)))) \subset M.$$

Clearly, $K \in \mathcal{M}$ and 0, $h \in M$ for every $M \in \mathcal{M}$. Moreover, it is easy to see that

$$M \in \mathcal{M} \ \Rightarrow\ \overline{\mathrm{co}}(\{0\} \cup \Psi(M \times \overline{\mathrm{co}}(\{0\} \cup \Phi(M)))) \in \mathcal{M}.$$

Define the set $A = \cap\{M : M \in \mathcal{M}\}$. We have $A \in \mathcal{M}$. Also,

$$A = \overline{\mathrm{co}}(\{0\} \cup \Psi(A \times \overline{\mathrm{co}}(\{0\} \cup \Phi(A)))).$$

It follows from (2.1) that A is weakly compact and Φ, Ψ are sequentially w-u.s.c. on A and $B := \overline{\mathrm{co}}(\{0\} \cup \Phi(A))$, respectively.

Since Φ is sequentially w-u.s.c. on A, we have that B is weakly compact too.

Define the map $\Pi: A \times B \to P_{fc}(A \times B)$ by $\Pi(x, y) =$

$$\begin{cases} \left\{\dfrac{\delta}{r}h\right\} \times \{0\} & \text{if } x = 0, \\ \left[\dfrac{|x|}{r}\Psi\left(\dfrac{r}{|x|}x, \lambda(x,y)\dfrac{r}{|x|}y\right) + \dfrac{\delta}{r}h\right] \times \left[\dfrac{|x|}{r}\Phi\left(\dfrac{r}{|x|}x\right)\right] & \text{if } 0 < |x| \le r - \delta, \\ \left[\dfrac{|x|}{r}\Psi\left(\dfrac{r}{|x|}x, \lambda(x,y)\dfrac{r}{|x|}y\right) + \left(1 - \dfrac{|x|}{r}\right)h\right] & \\ \qquad \times \left[\dfrac{|x|}{r}\Phi\left(\dfrac{r}{|x|}x\right)\right] & \text{if } r - \delta < |x| < r, \\ \Psi(x,y) \times \Phi x & \text{if } r \le |x| \le R, \\ \Psi\left(\dfrac{R}{|x|}x, y\right) \times \Phi\left(\dfrac{R}{|x|}x\right) & \text{if } R < |x|. \end{cases}$$

Here for $0 < |x| < r$ and $y \in B$,

$$\lambda(x,y) = \sup\left\{\lambda : \lambda \in (0,1],\ \lambda\frac{r}{|x|}y \in B\right\}.$$

Notice

$$\lambda(x,y) = 1 \quad \text{if } y \in \frac{|x|}{r}\Phi\left(\frac{r}{|x|}x\right).$$

Also note that

$$\frac{|x|}{r}\Psi\left(\frac{r}{|x|}x, \lambda(x,y)\frac{r}{|x|}y\right) + \frac{\delta}{r}h \in A.$$

This follows from the following remark: if A is convex and $0 \in A$, then $\alpha x + \beta y \in A$ for every $x, y \in A$ and $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$. Indeed,

$$\alpha x + \beta y = (\alpha+\beta)\left[\frac{\alpha}{\alpha+\beta}x + \frac{\beta}{\alpha+\beta}y\right] \in (\alpha+\beta)A \subset A.$$

One can prove that Π is sequentially w-u.s.c. (see [11, Theorem 1.2.12]).

Now the Arino–Gautier–Penot fixed point theorem, guarantees the existence of a fixed point $(x,y) \in A \times B$ of Π. It remains to show that $r \leq |x| \leq R$. Clearly $x \neq 0$ since $h \neq 0$. If $0 < |x| \leq r - \delta$, then

$$x \in \frac{|x|}{r}\Psi\left(\frac{r}{|x|}x, \Phi\left(\frac{r}{|x|}x\right)\right) + \frac{\delta}{r}h.$$

If we denote $z = (r/|x|)x$, then $|z| = r$ and

$$z \in \Psi(z, \Phi z) + \mu h$$

where $\mu = \delta/|x| > 0$. This contradicts (2.3). We derive the same contradiction if we assume $r - \delta < |x| < r$. Finally, assume $|x| > R$. Then

$$x \in \Psi\left(\frac{R}{|x|}x, \Phi\left(\frac{R}{|x|}x\right)\right)$$

and if we denote $z = (R/|x|)x$, we see that $|z| = R$ and $z \in (R/|x|)\Psi(z, \Phi z)$ with $0 < R/|x| < 1$, which contradicts (2.2). Therefore $r < |x| < R$. $\square$

Our next abstract result is an expansion type theorem.

Theorem 2.2. *Assume the conditions of Theorem* 2.1 *except* (2.2)–(2.3) *are satisfied. In addition assume that the following expansion condition holds:*

$$x \notin \lambda\Psi(x, \Phi x) \quad \textit{for } \lambda \in (0,1) \textit{ and } x \in K \textit{ with } |x| = r$$

and

$$x \notin \Psi(x, \Phi x) + \mu h \quad \textit{for } \mu > 0 \textit{ and } x \in K \textit{ with } |x| = R.$$

Then there exists at least one $x \in K$ with $x \in \Psi(x,\Phi x)$ and $r \le |x| \le R$.

Proof. Define $\Phi' : K \to P_{k^w c}(Y)$, by

$$\begin{aligned}
\Phi'(x) &= \Phi\bigg(\bigg(\frac{R}{|x|}+\frac{r}{|x|}-1\bigg)x\bigg) && \text{if } r \le |x| \le R,\\
\Phi'(x) &= \Phi\bigg(\bigg(\frac{R}{r}\bigg)x\bigg) && \text{if } |x| < r,\\
\Phi'(x) &= \Phi\bigg(\bigg(\frac{r}{R}\bigg)x\bigg) && \text{if } |x| > R.
\end{aligned}$$

Also define $\Psi' : K \times C \to P_{f\,c}(K)$ by

$$\begin{aligned}
\Psi'(x,y) &= \bigg(\frac{R}{|x|}+\frac{r}{|x|}-1\bigg)^{-1}\Psi\bigg(\bigg(\frac{R}{|x|}+\frac{r}{|x|}-1\bigg)x,y\bigg) && \text{if } r \le |x| \le R,\\
\Psi'(x) &= \bigg(\frac{r}{R}\bigg)\Psi\bigg(\bigg(\frac{R}{r}\bigg)x,y\bigg) && \text{if } |x| < r,\\
\Psi'(x) &= \bigg(\frac{R}{r}\bigg)\Psi\bigg(\bigg(\frac{r}{R}\bigg)x,y\bigg) && \text{if } |x| > R.
\end{aligned}$$

Notice, if $\overline{x} = (R/|x| + r/|x| - 1)x$, then $r \le |\overline{x}| \le R$ whenever $x \in K$ and $r \le |x| \le R$. Also $|\overline{x}| = r$ for $|x| = R$ and $|\overline{x}| = R$ if $|x| = r$. Now observe that conditions (2.2)–(2.3) in Theorem 2.1 holds for Φ' and Ψ' instead of Φ and Ψ, respectively, with $\overline{x}_0$ and $\overline{h} := (R/|x_0| + r/|x_0| - 1)h$. Consequently, there exists $x' \in K$ with $x' \in \Psi'(x',\Phi' x')$ and $r \le |x'|` \le R$. Then $x := (R/|x'| + r/|x'| - 1)x'$ satisfies $x \in \Psi(x,\Phi x)$ and $r \le |x| \le R$. □

In particular, if $\Psi(x,y) = \Psi(y)$, we immediately derive from Theorems 2.1 and 2.2, an existence principle for (1.1).

Theorem 2.3. *Let $\Phi : K \to P_{k^w c}(Y)$, $\Psi : C \to P_{fc}(K)$ be two bounded multi-valued maps. Assume that*

(2.4) $A \subset K,\ A = \overline{\mathrm{co}}\,(\{0\} \cup \Psi(\overline{\mathrm{co}}(\{0\} \cup \Phi(A)))\ \Rightarrow\ A$ *is weakly compact and Φ, Ψ are sequentially w-u.s.c. on A and $\overline{\mathrm{co}}(\{0\} \cup \Phi(A))$, respectively.*

In addition assume either

$$(2.5)\qquad \begin{cases} x \notin \lambda\Psi\Phi x & \textit{for } \lambda \in (0,1) \textit{ and } x \in K \textit{ with } |x| = R; \textit{ and}\\ x \notin \Psi\Phi x + \mu h & \textit{for } \mu > 0 \textit{ and } x \in K \textit{ with } |x| = r,\end{cases}$$

or

$$(2.6)\qquad \begin{cases} x \notin \lambda\Psi\Phi x & \textit{for } \lambda \in (0,1) \textit{ and } x \in K \textit{ with } |x| = r; \textit{ and}\\ x \notin \Psi\Phi x + \mu h & \textit{for } \mu > 0 \textit{ and } x \in K \textit{ with } |x| = R.\end{cases}$$

Then there exists at least one $x \in K$ with $x \in \Psi\Phi x$ and $r \leq |x| \leq R$.

Theorem 2.3 yields in particular the following compression-expansion principle for convex-valued maps.

Corollary 2.4. *Let $\Phi: K \to P_{k^w c}(K)$ be a bounded multi-valued map. Assume that Φ is sequentially w-u.s.c. and*

$$A \subset K,\ A = \overline{\mathrm{co}}\,(\{0\} \cup \Phi(A)) \ \Rightarrow\ A \text{ is weakly compact and } \Phi \text{ is sequentially w-u.s.c. on } A.$$

In addition assume that one of the following conditions is satisfied

$$\begin{cases} x \notin \lambda\Phi x & \text{for } \lambda \in (0,1) \text{ and } x \in K \text{ with } |x| = R, \text{ and} \\ x \notin \Phi x + \mu h & \text{for } \mu > 0 \text{ and } x \in K \text{ with } |x| = r; \end{cases}$$

$$\begin{cases} x \notin \lambda\Phi x & \text{for } \lambda \in (0,1) \text{ and } x \in K \text{ with } |x| = r, \text{ and} \\ x \notin \Phi x + \mu h & \text{for } \mu > 0 \text{ and } x \in K \text{ with } |x| = R. \end{cases}$$

Then there exists at least one $x \in K$ with $x \in \Phi x$ and $r \leq |x| \leq R$.

Our next result gives an useful sufficient condition to guarantee the second part of (2.5) and of (2.6).

Theorem 2.5. *Assume that for some $r_0 > 0$, $\mu_0 \geq 0$, and $h \in K$, $h \neq 0$, the following conditions hold:*

$$x \notin \Psi\Phi(x + \mu h) \quad \text{for all } x \in K,\ \mu \geq \mu_0; \tag{2.7}$$

$$x \notin \Psi\Phi(x + \mu h) \quad \text{for all } x \in K,\ |x| > r_0,\ \mu \in (0, \mu_0). \tag{2.8}$$

Then

$$x \notin \Psi\Phi x + \mu h \quad \text{for } \mu > 0 \text{ and } x \in K \text{ with } |x| \geq \rho,$$

where $\rho = r_0 + \mu_0|h|$.

Proof. Assume the contrary. Then there exists an $x \in K$ with $|x| \geq \rho$ and $x \in \Psi\Phi x + \mu h$ for some $\mu > 0$. Denote $y := x - \mu h$. Then

$$y \in \Psi\Phi(y + \mu h).$$

Since Ψ has values in K, the last relation says that $y \in K$. The case $\mu \geq \mu_0$ is impossible by (2.7). Now if $\mu < \mu_0$, then from (2.8), $|y| \leq r_0$. But

$$|y| \geq |x| - \mu|h| > |x| - \mu_0|h| \geq \rho - \mu_0|h| = r_0,$$

which is a contradiction. □

Remark 2.6. If (2.7) holds and there exists constants $r_0, r_1 > 0$ such that $|x| \neq r_0$ for all solutions $x \in K$ to

$$x \in \lambda\Psi\Phi x$$

and every $\lambda \in (0,1)$, and $|x| \le r_1$ for every solution $x \in K$ to

$$x \in \Psi\Phi(x + \mu h)$$

and $\mu \in (0,\mu_0)$, then (2.5) or (2.6) is satisfied with $R := r_0$ and $r := r_1 + \mu_0|h|$ if $r_0 > r_1 + \mu_0|h|$ and respectively with $r = r_0$ and $R > r_1 + \mu_0|h|$ if $r_0 \le r_1 + \mu_0|h|$.

3. Positive solutions of p-Laplace inclusions

Consider the boundary value problem

$$\begin{cases} -\Delta_p u \in f(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$, and $|\cdot|$ is the Euclidean norm in $\mathbb{R}^N$. We seek weak solutions to (3.1), i.e. a function $u \in W_0^{1,p}(\Omega)$, for which there is $w \in L^\infty(\Omega)$ with $w(x) \in f(x,u(x),\nabla u(x))$ for a.e. $x \in \Omega$ and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v\,dx = \int_\Omega wv\,dx \quad \text{for all } v \in W_0^{1,p}(\Omega).$$

Notice that by a well-known regularity result (see [3] and [4]), any weak solution of (3.1) belongs to $C^1(\overline{\Omega})$. By a positive solution to (3.1) we mean a weak solution u with $u > 0$ in Ω.

We shall assume that $f : \Omega \times \mathbb{R}^{N+1} \to 2^{\mathbb{R}}$ is such that

$$f(x,v) \subset \mathbb{R}_+ \quad \text{for all } v \in \mathbb{R}_+ \times \mathbb{R}^N \text{ and a.e. } x \in \Omega. \tag{3.2}$$

Also assume that

(3.3) the map $F\colon L^m(\Omega;\mathbb{R}^{N+1}) \to 2^{L^n(\Omega)}$ given by $Fv := \{w \in L^n(\Omega) : w(x) \in f(x,v(x)) \text{ for a.e. } x \in \Omega\}$ has non-empty, closed, convex values, is u.s.c. and $F(C(\overline{\Omega};\mathbb{R}^{N+1})) \subset L^\infty(\Omega)$

for some $m,n \in [1,\infty)$. Sufficient conditions for (3.3) can be found in [5].

In what follows by λ_1 and ϕ_1 we shall denote the first eigenvalue of the p-Laplacian and its associated positive eigenfunction.

From Theorem 2.3, we obtain the following existence principle for (3.1), under an "a priori bounds" assumption.

Theorem 3.1. *Assume* (3.2), (3.3), *there exists* $r_0 > 0$ *such that*

$$|u|_{W^{1,p}(\Omega)} \ne r_0 \tag{3.4}$$

for every solution $u \ge 0$ *to* $-\Delta_p u \in \lambda f(x,u,\nabla u)$ *and all* $\lambda \in (0,1)$, *and that for each* $\mu_0 > 0$ *there exists* $r_1 > 0$ *with*

$$|u|_{W^{1,p}(\Omega)} \le r_1 \tag{3.5}$$

for every solution $u \geq 0$ to $-\Delta_p u \in f(x, u+\mu, \nabla u)$ and all $\mu \in (0, \mu_0)$. In addition assume that one of the following conditions holds:

(a) *there exists constants $a > \lambda_1$ and $b \geq 0$ with*

$$au^{p-1} - b \leq f(x, u, v) \tag{3.5}$$

for all $u \in \mathbb{R}_+$, $v \in \mathbb{R}^N$ and a.e. $x \in \Omega$;

(b) *there exists constants $a > 0$, $b, c \geq 0$ and $q > p-1$, $\alpha \in (p-1, pq/(q+1))$ with*

$$au^q - b|v|^\alpha - c \leq f(x, u, v)$$

for all $u \in \mathbb{R}_+$, $v \in \mathbb{R}^N$ and a.e. $x \in \Omega$.

Then problem (3.1) *has a positive solution.*

Proof. It is well known that for every $v \in W^{-1,p'}(\Omega)$, there exists a unique $u_v \in W_0^{1,p}(\Omega)$ denoted by $(-\Delta_p)^{-1}v$ such that $-\Delta_p u_v = v$ in the weak sense, i.e.

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla w \, dx = (v, w) \quad \text{for all } w \in W_0^{1,p}(\Omega),$$

and the (nonlinear) map $(-\Delta_p)^{-1}$ is bounded and continuous from $W^{-1,p'}(\Omega)$ to $W_0^{1,p}(\Omega)$. In addition, by the strong maximum principle for the p-Laplacian (see [19]), if $v \in L^\infty(\Omega)$, $v \geq 0$ and $v \neq 0$, then $u_v > 0$ in Ω. Hence any nontrivial, nonnegative solution to (3.1) is positive.

Let $X := C^1(\overline{\Omega})$ endowed with the norm of $W^{1,p}(\Omega)$, $Y := L^\infty(\Omega)$ endowed with the norm of $L^n(\Omega)$ and $K := \{u \in C^1(\overline{\Omega}) : u \geq 0\}$.

Let $\Phi : X \to 2^Y$ be defined by $\Phi = FJ$, where $J : C^1(\overline{\Omega}) \to C(\overline{\Omega}; \mathbb{R}^{N+1})$, $Ju = (u, \nabla u)$. Clearly J is a bounded linear operator. Consequently, Φ has non-empty, closed, convex values and is u.s.c. Also, define $\Psi : L^\infty(\Omega) \to C^1(\overline{\Omega})$, by $\Psi v = (-\Delta_p)^{-1} v$. Clearly Ψ is completely continuous since the imbedding of $L^\infty(\Omega)$ into $W^{-1,p'}(\Omega)$ is compact. Consequently, $\Psi\Phi$ sends bounded sets into relatively compact sets, so condition (2.4) trivially holds.

Now a positive solution of (3.1) is a fixed point $u \in C^1(\overline{\Omega})$ of $\Psi\Phi$ with $u > 0$ on Ω. Let us prove that

$$u \notin \Psi\Phi(u + \mu) \quad \text{for all } u \in C^1(\overline{\Omega}),\ u \geq 0,\ u \neq 0 \text{ and } \mu \geq \mu_0$$

for some sufficiently large $\mu_0 > 0$.

Assume first (a). We shall use the arguments used in [9], [12] and [17] for the classical Laplacian. Let $u \in C^1(\overline{\Omega})$, $u \geq 0$, $u \neq 0$ and $u \in \Psi\Phi(u + \mu)$ for some $\mu \geq (b/a)^{1/(p-1)}$. Then according to (3.5), we have

$$-\Delta_p u \geq a(u + \mu)^{p-1} - b \geq a(u^{p-1} + \mu^{p-1}) - b \geq au^{p-1} \tag{3.6}$$

Now we recall Picone's identity for the p-Laplacian (see [1]):

$$(3.7)\quad |\nabla v|^p + (p-1)\frac{v^p}{u^p}|\nabla u|^p - p\frac{v^{p-1}}{u^{p-1}}\nabla v|\nabla u|^{p-2}\nabla u = |\nabla v|^p - \nabla\bigg(\frac{v^p}{u^{p-1}}\bigg)|\nabla u|^{p-2}\nabla u \geq 0$$

for $u, v \in C^1(\Omega)$, $u > 0$ and $v \geq 0$. This for $v = \phi_1$ gives

$$(3.8)\quad \int_\Omega \nabla\bigg(\frac{\phi_1^p}{u^{p-1}}\bigg)|\nabla u|^{p-2}\nabla u \leq \int_\Omega |\nabla\phi_1|^p = \lambda_1\int_\Omega \phi_1^p.$$

On the other hand, (3.6) gives

$$\int_\Omega \nabla\bigg(\frac{\phi_1^p}{u^{p-1}}\bigg)|\nabla u|^{p-2}\nabla u = -\int_\Omega \frac{\phi_1^p}{u^{p-1}}\Delta_p u \geq a\int_\Omega \phi_1^p.$$

Hence

$$a\int_\Omega \phi_1^p \leq \lambda_1\int_\Omega \phi_1^p$$

which contradicts $a > \lambda_1$.

Assume now (b). We shall basically use the same arguments as in [15] (see also [16]). Let $u \in C^1(\overline{\Omega})$, $u \geq 0$, $u \neq 0$ and $u \in \Psi\Phi(u+\mu)$ for some $\mu \geq (c/a)^{1/q}$. We have

$$\int_\Omega \nabla\bigg(\frac{\phi_1^p}{u^{p-1}}\bigg)|\nabla u|^{p-2}\nabla u = -\int_\Omega \frac{\phi_1^p}{u^{p-1}}\Delta_p u$$
$$\geq \int_\Omega [a(u+\mu)^q - b|\nabla u|^\alpha - c]\frac{\phi_1^p}{u^{p-1}} \geq \int_\Omega [au^q - b|\nabla u|^\alpha + a\mu^q - c]\frac{\phi_1^p}{u^{p-1}}.$$

This together with (3.8) gives

$$(3.9)\quad b\int_\Omega |\nabla u|^\alpha\frac{\phi_1^p}{u^{p-1}} \geq (l(\mu) - \lambda_1)\int_\Omega \phi_1^p$$

where

$$l(\mu) = \min\bigg\{\frac{at^q + a\mu^q - c}{t^{p-1}} : t > 0\bigg\}.$$

It is easy to see that $l(\mu) \to \infty$ as $\mu \to \infty$. Then (3.9) will guarantee that μ is bounded if we show that the left side of (3.9) is bounded from above. Indeed, from

$$-\Delta_p u \geq a(u+\mu)^q - b|\nabla u|^\alpha - c \geq au^q - b|\nabla u|^\alpha + a\mu^q - c \geq au^q - b|\nabla u|^\alpha,$$

taking the test function ϕ_1^p/u^{p-1} we obtain

$$\int_\Omega \nabla\bigg(\frac{\phi_1^p}{u^{p-1}}\bigg)|\nabla u|^{p-2}\nabla u \geq a\int_\Omega \phi_1^p u^{q-p+1} - b\int_\Omega \phi_1^p u^{-p+1}|\nabla u|^\alpha.$$

On the other hand, according to (3.7), the left integral is equal to

$$-(p-1)\int_\Omega \phi_1^p u^{-p}|\nabla u|^p + p\int_\Omega \phi_1^{p-1}u^{-p+1}\nabla\phi_1|\nabla u|^{p-2}\nabla u.$$

Hence

$$(3.10)\quad (p-1)\int_\Omega \phi_1^p u^{-p}|\nabla u|^p + a\int_\Omega \phi_1^p u^{q-p+1} \le p\int_\Omega \phi_1^{p-1}u^{-p+1}|\nabla\phi_1||\nabla u|^{p-1} + b\int_\Omega \phi_1^p u^{-p+1}|\nabla u|^\alpha.$$

Now using the Young inequality $x \le \varepsilon x^\gamma + C$ with $\gamma > 1$ (here we take $\gamma := p/(p-1)$), and the fact that $\nabla\phi_1$ is bounded, we obtain that

$$(3.11)\quad p\int_\Omega \phi_1^{p-1}u^{-p+1}|\nabla\phi_1||\nabla u|^{p-1} \le \frac{p-1}{2}\int_\Omega \phi_1^p u^{-p}|\nabla u|^p + C.$$

Then (3.10) and (3.11) imply

$$(3.12)\quad \frac{p-1}{2}\int_\Omega \phi_1^p u^{-p}|\nabla u|^p + a\int_\Omega \phi_1^p u^{q-p+1} \le b\int_\Omega \phi_1^p u^{-p+1}|\nabla u|^\alpha + C.$$

Now we use the Young inequality in the form

$$xy \le \varepsilon x^\gamma + \varepsilon^{1/(1-\gamma)}y^{\gamma/(\gamma-1)}, \quad \gamma > 1,\ \varepsilon > 0,$$

first with $x := \phi_1^{p-\alpha}u^{1-p+\alpha}$, $y := \phi_1^\alpha u^{-\alpha}|\nabla u|^\alpha$, $\gamma = p/(p-\alpha)$ and $\varepsilon := \varepsilon_1 > 0$ such that $b\varepsilon_1^{1/(1-\gamma)} = (p-1)/4$, to obtain

$$(3.13)\quad b\int_\Omega \phi_1^p u^{-p+1}|\nabla u|^\alpha \le \frac{p-1}{4}\int_\Omega \phi_1^p u^{-p}|\nabla u|^p + b\varepsilon_1\int_\Omega \phi_1^p u^{(1-p+\alpha)p/(p-\alpha)}.$$

Next we apply Young's inequality once again with

$$\gamma := \frac{q-p+1}{1-p+\alpha}\cdot\frac{p-\alpha}{p}, \quad x := \phi_1^{p/\gamma}u^{(1-p+\alpha)p/(p-\alpha)} \quad \text{and} \quad \varepsilon := \frac{a}{2b\varepsilon_1}.$$

Notice $\gamma > 1$ since $\alpha < pq/(q+1)$ and $p/\gamma < p$. As a result

$$(3.14)\quad b\varepsilon_1\int_\Omega \phi_1^p u^{(1-p+\alpha)p/(p-\alpha)} \le \frac{a}{2}\int_\Omega \phi_1^p u^{q-p+1} + C.$$

Now (3.13) and (3.14) give

$$(3.15)\quad b\int_\Omega \phi_1^p u^{-p+1}|\nabla u|^\alpha \le \frac{p-1}{4}\int_\Omega \phi_1^p u^{-p}|\nabla u|^p + \frac{a}{2}\int_\Omega \phi_1^p u^{q-p+1} + C.$$

Then (3.12) and (3.15) imply

$$(3.16)\quad \frac{p-1}{4}\int_\Omega \phi_1^p u^{-p}|\nabla u|^p + \frac{a}{2}\int_\Omega \phi_1^p u^{q-p+1} \le C$$

and finally (3.16) and (3.15) guarantee that

$$b\int_\Omega \phi_1^p u^{-p+1}|\nabla u|^\alpha \le C$$

as we wished.

Finally, assumptions (3.4) and (3.5) guarantee (2.5) or (2.6) as shows Remark 2.6 (here $h = 1$ and $|h| = |h|_{W^{1,p}(\Omega)} = |h|_{L^p(\Omega)}$). $\square$

Concerning the "a priori bounds" hypothesis (3.4) and (3.5), we refer to the papers [9], [10] and [17] for the classical Laplacian, to [13] for the one-dimensional p-Laplacian, and to [3] and [15], for the N-dimensional p-Laplacian.

For example, the a priori estimates in [15], together with our existence principle, Theorem 3.1, yield the following version for inclusions of the main result, Theorem 4.2 in [15].

Theorem 3.2. *Assume Ω is a bounded C^2 domain in $\mathbb{R}^N$, $1 < p < N$ and for $f\colon \Omega \times \mathbb{R}^{N+1} \to 2^{\mathbb{R}}$ the following conditions are satisfied:*

(a) *$f(x,w) \subset \mathbb{R}_+$ for all $w \in \mathbb{R}_+ \times \mathbb{R}^N$ and a.e. $x \in \Omega$;*
(b) *the graph of f belongs to the σ-field $\mathcal{L} \otimes \mathcal{B}$ (here $\mathcal{B} = \mathcal{B}(\mathbb{R}^{N+1}) \otimes \mathcal{B}(\mathbb{R})$ is the Borel σ-field in $\mathbb{R}^{N+1} \times \mathbb{R}$);*
(c) *the map $w \mapsto f(x,w)$ is u.s.c. for a.e. $x \in \Omega$;*
(d) *$u^q - M|v|^\alpha \leq f(x,u,v) \leq c_0 u^q + M|v|^\alpha$ for $u \in \mathbb{R}_+$, $v \in \mathbb{R}^N$ and a.e. $x \in \Omega$, where $c_0 \geq 1$, $M > 0$, $q \in (p-1,(p-1)N/(N-p))$ and $\alpha \in (p-1, pq/(q+1))$.*

Then problem (3.1) *has a positive solution.*

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