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BIFURCATION PHENOMENA IN CONTROL FLOWS

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Abstract. We study bifurcation phenomena in control flows and the bifurcation of control sets. A Mel'nikov method and the Conley index together with exponential dichotomy theory and integral manifold theory are used.

1. Introduction

The purpose of this paper is to provide information concerning the bifurcation of control sets of two-dimensional control systems of the form

$$x' = f(x,\varepsilon) + \rho u(t) g(x,\varepsilon) \quad (x \in \mathbb{R}^2). \tag{1.1}$$

Here $u: \mathbb{R} \to [-1,1]$ is a measurable control function, ε takes values in an open interval $I = (a,b) \subset \mathbb{R}$, and ρ is a real parameter which assumes small values. We will always assume that $f(0,\varepsilon) = g(0,\varepsilon) = 0$ for all $\varepsilon \in I$, so that $x = 0$ is an equilibrium point of (1.1) for all choices of ε, ρ, and $u(\,\cdot\,)$.

When $\rho = 0$ one obtains a one-parameter family of ordinary differential equations

$$x' = f(x,\varepsilon) \quad (x \in \mathbb{R}^2) \tag{1.2}$$

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each of which admits $x = 0$ as an equilibrium point. The classical Poincaré–Bendixon theory imposes strong constraints on the asymptotic behavior of the solutions of (1.2).

We pose a bifurcation problem of "Arnold type" [1], [2]. Let us explain what we mean by this term. For each $\varepsilon \in D$ let $B_\varepsilon = D_x f(0, \varepsilon)$ be the Jacobian matrix of $f(\,\cdot\,, \varepsilon)$ at $x = 0$. Suppose that there exist numbers $\varepsilon_1 < \varepsilon_2 \in I$ with the following properties. First, the eigenvalues of B_ε lie in the left half-plane for $\varepsilon < \varepsilon_1$; i.e. B_ε is a Hurwitz matrix. Second, the eigenvalues of B_ε lie in the right half-plane if $\varepsilon > \varepsilon_2$. Finally, in the intermediate regime $\varepsilon_1 < \varepsilon < \varepsilon_2$, we will assume that B_ε admits one negative (real) eigenvalue and one positive eigenvalue.

The problem we pose is the following. Suppose that the parameter ρ in (1.1) is small but non-zero. First, describe the qualitative change in the behavior of the solutions of (1.1) as ε passes through $[\varepsilon_1, \varepsilon_2]$. Then, determine the number and structure of the control sets of (1.1) when $\varepsilon < \varepsilon_1$, $\varepsilon_1 < \varepsilon < \varepsilon_2$, and $\varepsilon > \varepsilon_2$.

Classical bifurcation theory offers only limited insight into this problem because of the nonautonomous nature of equation (1.1). However L. Arnold has offered hypotheses concerning the expected behavior of solutions of 1-parameter families like (1.1) [1]. His insights have been developed by later authors [24], [32], [33]. We will apply and amplify previous methods and results regarding the Arnold bifurcation pattern in the context of the family (1.1).

We will in particular make systematic use of the fact that the nonautonomous part of (1.1) is small, i.e. proportional to ρ. Specifically, we will use a method of Mel'nikov type to study the behavior of solutions of (1.1) when $\varepsilon_1 < \varepsilon < \varepsilon_2$, and we will use the continuation properties of the Conley index to study invariant sets and control sets for (1.1) when $\varepsilon > \varepsilon_2$. These methods seem particularly suited to the study of nonautonomous equations which are perturbations of autonomous equations.

After we have studied the solutions of (1.1) for small $\rho \neq 0$, we will analyze the corresponding control sets [8]. These sets provide basic information about the local controllability properties of the nonlinear control system (1.1). We will use results of Colonius–Kliemann [8]; see also Gayer [15] and Grünvogel [17].

The present paper is organized as follows. In Section 2 we discuss some basic facts concerning nonautonomous differential systems. We review the theory of the dynamical spectrum and an appropriate version of integral manifold theory. We then discuss the concept of control flow and that of control set.

In Section 3 we discuss the intermediate regime when $\varepsilon_1 < \varepsilon < \varepsilon_2$, where B_ε has one positive and one negative eigenvalue. When $\varepsilon > \varepsilon_2$ and ε is close to ε_1, certain general statements can be made about the control flow ([8]; see Section 2) defined by (1.1). We use results and methods presented in [24]. We use properties of the control flow to deduce statements about the control sets. Next, we conside values $\varepsilon \in (\varepsilon_1, \varepsilon_2)$ which are not "near" ε_1: in particular we allow the possibility that equation (1.2) admits an orbit which is homoclinic to the origin $x = 0$. In this case, one can use a Mel'nikov-type method to make a systematic study of the control flow defined by (1.1) when ρ is small. We observe phenomena which may be related to those mentioned by Arnold [3]; he applied techniques due to Dellnitz and his collaborators (e.g. [11]).

The case $\varepsilon > \varepsilon_1$ is treated in Section 4. We make use of the fact that the Poincaré–Bendixson theory is valid for equation (1.2). Of the various possible phase portraits which (1.2) may exhibit, we choose to study one which seems related to a phenomenon observed by L. Arnold in a forced Duffing–van der Pol operator and studied by Schenk–Hoppé and others [32], [33], [24]. Namely, we assume that the phase flow of (1.2) admits two saddles and two sinks together with certain heteroclinic orbits. We will use the perturbation properties of the Conley index to study the control flow of (1.1) when ρ is small. Once again, we then use the properties of the control flow to discuss control sets.

We finish this Introduction by indicating some notation which will be used throughout the paper, and repeating some basic definitions.

For each integer $d \geq 1$, let $\mathbb{M}_d$ denote the set of $d \times d$ real matrices. Let $\langle \cdot, \cdot \rangle$ denote the Euclidean inner product and $|\cdot|$ the corresponding norm.

Let X be a metric space. For each $t \in \mathbb{R}$, let $\tau_t : X \to X$ be a homeomorphism. The couple $(X, \{\tau_t\})$ is said to define a *flow* or *dynamical system* on X if the following conditions are satisfied:

(i) $\tau_0(x) = x$ for all $x \in X$;
(ii) $\tau_{t+s}(x) = \tau_t \circ \tau_s(x)$ for all $t, s \in \mathbb{R}$ and all $x \in X$;
(iii) the map $T : X \times \mathbb{R} \to X : T(x,t) = \tau_t(x)$ is continuous.

A subset $Y \subset X$ is said to be *invariant* if for each $y \in Y$, the *orbit* (or *trajectory*) $\{\tau_t(y) | t \in \mathbb{R}\}$ is contained in Y. A compact subset $M \subset X$ is said to be *minimal* if it is invariant and if, for each $x \in M$, the orbit $\{\tau_t(x) | t \in \mathbb{R}\}$ is dense in M. Let $x \in X$; the ω*-limit set* of x is

$$\{y \in X \mid \text{there exists a sequence } t_n \to +\infty \text{ such that } \tau_{t_n} \to y\}.$$

The α*-limit set* is defined analogously by considering sequences $t_n \to -\infty$.

2. Preliminaries

In this section, we review some facts concerning nonautonomous control systems. In particular we review the integral manifold theory for nonautonomous differential systems. We also repeat some basic material concerning the control flows of Colonius and Kliemann.

Let P be a compact metric space, and let $(P, \{\tau_t\})$ be a flow on P. We will study a family of time-dependent ordinary differential equations which is parametrized in a certain way by the points of P. Specifically, let $f\colon P \times \mathbb{R}^d \to \mathbb{R}^d$ be a continuous map. Assume that the derivatives $D_x^k f\colon P \times \mathbb{R}^d \to \mathbb{R}^d$ of f with respect to x exist and are continuous on $P \times \mathbb{R}^d$ for each $k = 1, 2, 3$. We consider the family of differential equations

$$x' = f(\tau_t(p), x) \quad (x \in \mathbb{R}^d,\ p \in P). \tag{2.1}$$

Generally speaking, the right-hand side of (2.1) varies with the time t for fixed $p \in P$; that is, (2.1) is a nonautonomous differential equation.

Let us assume that, for each $p \in P$ and each $x_0 \in \mathbb{R}^d$, equation (2.1) admits a unique solution $\varphi(t; p, x_0)$, which exists for all $t \in \mathbb{R}$. For each $t \in \mathbb{R}$ define $N_t\colon P \times \mathbb{R}^d \to P \times \mathbb{R}^d$ as follows: $N_t(p, x_0) = (\tau_t(p), \varphi(t; p, x_0))$. It is easily seen that the pair $(P \times \mathbb{R}^d, \{N_t\})$ is a flow. It is typical of the theory of nonautonomous differential systems that one makes use of the recurrence properties of the trajectories of this flow to study the solutions of equation (2.1).

A family of the type (2.1) can be obtained beginning with a single differential equation

$$x' = \widetilde{f}(t, x) \quad (t \in \mathbb{R},\ x \in \mathbb{R}^d) \tag{2.1$'$}$$

when $\widetilde{f}$ satisfies the following condition: $\widetilde{f}$ together with its x-derivatives $D_x^k \widetilde{f}$ of orders $k = 1, 2, 3$ are uniformly continuous on $\mathbb{R} \times K$ for each compact subset $K \subset \mathbb{R}^d$. In fact, view $\widetilde{f}$ as a point in the space $\mathcal{F}$ of all functions $\widehat{f}\colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ which satisfy this condition. Topologize $\mathcal{F}$ by stating that a sequence $\{\widehat{f}_k\} \subset \mathcal{F}$ converges to $\widehat{f} \in \mathcal{F}$ just when $\widehat{f}_n \to \widehat{f}$ uniformly on each compact subset of $\mathbb{R} \times \mathbb{R}^d$. Introduce a Bebutov-type flow [5] on $\mathcal{F}$ by setting $\tau_t(\widehat{f})(s, x) = \widehat{f}(t+s, x)$ for each $\widehat{f} \in \mathcal{F}$ and $t \in \mathbb{R}$. Then the pair $(\mathcal{F}, \{\tau_t\})$ is indeed a flow on $\mathcal{F}$. Let $P = \mathrm{cl}\{\tau_t(\widetilde{f}) | t \in \mathbb{R}\}$, and set

$$f(p, x) = p(0, x) \quad (p \in P,\ x \in \mathbb{R}^d).$$

Then $\widetilde{f} \in P$, and equation (2.1$'$) is one equation in the family (2.1) defined by f and P.

Assume from now on that $f(p, 0) = 0$ for all $p \in P$. Let us write

$$\ell(p) = D_x f(p, 0), \quad n(p, x) = f(p, x) - \ell(p)x.$$

Then equation (2.1) takes the equivalent form

$$x' = \ell(\tau_t(p))x + n(\tau_t(p), x). \tag{2.2}$$

Clearly $n(p, x) = O(|x|^2)$ as $x \to 0$, uniformly with respect to $p \in P$.

We review some basic concepts and facts concerning the family of linear systems

$$x' = \ell(\tau_t(p))x \quad (p \in P). \tag{2.3}$$

Let $\Phi_p(t)$ be the fundamental matrix solution of (2.3) ($p \in P$).

Definition 2.1 ([10]). The family (2.3) is said to have an *exponential dichotomy* (ED) over P if there are positive constants k, γ together with a continuous projection-valued function $Q\colon P \to \mathbb{M}_d$ such that:

$$\begin{aligned} \|\Phi_p(t)Q(p)\Phi_p(s)^{-1}\| &\le ke^{-\gamma(t-s)} \quad \text{for } t \ge s, \\ \|\Phi_p(t)(I - Q(p))\Phi_p(s)^{-1}\| &\le ke^{\gamma(t-s)} \quad \text{for } t \le s. \end{aligned}$$

We define the important concept of dynamical spectrum [29], [34] of equation (2.3) in the following way.

Definition 2.2. Say that $\lambda \in \mathbb{R}$ belongs to the *resolvent* of the family (2.3) just when the translated family

$$x' = (-\lambda I + \ell(\tau_t(p)))x$$

admits an exponential dichotomy over P. If λ does not belong to the resolvent of the family (2.3), we say that λ belongs to the *dynamical spectrum* Σ of (2.3).

It was shown in [23] that λ lies in the dynamical spectrum of (2.3) if and only if λ lies in the spectrum of the differential operator $(d/dt) - \ell(\tau_t(p))$ acting on an appropriate Banach space. This point of view has been systematically exploited by Latushkin and his co-workers; see [6].

We now formulate an important result regarding the dynamical spectrum. To do this, we must first note that the pair $(P \times \mathbb{R}^d, \{L_t\})$ is a flow where

$$L_t(p, x_0) = (\tau_t(p), \Phi_p(t)x_0) \quad (t \in \mathbb{R},\ p \in P,\ x_0 \in \mathbb{R}^d).$$

This is an example of a so called skew-product flow.

Theorem 2.3 ([29]). *The dynamical system Σ of the family* (2.3) *is the union of at most d compact pairwise disjoint subintervals of $\mathbb{R}$:*

$$\Sigma = [a_1, b_1] \cup [a_2, b_2] \cup \ldots \cup [a_k, b_k] \quad (1 \le k \le d)$$

where $-\infty < a_1 \le b_1 < a_2 \le b_2 < \ldots < a_k \le b_k < \infty$. To each spectral interval $[a_r, b_r] \subset \Sigma$ there corresponds a topological (*hence closed*) *vector subbundle $V_r \subset P \times \mathbb{R}^d$, with base space P, such that the following properties hold:*

(a) *V_r is invariant for each $r = 1, 2, \ldots, d$. Explicitly, if $(p, x_0) \in V_r$ and if $t \in \mathbb{R}$, then $L_t(p, x_0) \in V_r$;*

(b) *if* $(p,x_0)\in V_r$ *and* $x_0\neq 0$, *then*

$$a_r\le \liminf_{t\to\pm\infty}\frac{1}{t}\ln\|\Phi_p(t)x_0\|\le \limsup_{t\to\pm\infty}\frac{1}{t}\ln\|\Phi_p(t)x_0\|\le b_r;$$

(c) $P\times\mathbb{R}^d=V_1\oplus\ldots\oplus V_k$ (*Whitney sum*).

If P *reduces to a single point, i.e. if the family* (2.3) *reduces to a single autonomous linear equation* $x'=\ell x$, *then the subbundles* V_r *are vector subspaces of* $\mathbb{R}^d$. *The dynamical spectrum* Σ *is equal to the set*

$$\{\lambda\in\mathbb{R}\mid \lambda \textit{ is the real part of an eigenvalue of } \ell\}.$$

If $\lambda_r=a_r=b_r\in\Sigma$, *then* V_r *is the real part of the direct sum of the generalized eigenspace of* ℓ *corresponding to those eigenvalues* μ *of* ℓ *satisfying* $\operatorname{Re}\mu=\lambda_r$.

Next we sketch a version of the classical integral manifold theory [18] for nonautonomous differential systems (2.2). We refer to [22], which makes use of techniques of Hirsch–Pugh–Shub [19], Irwin [20], [21], and Foster [12]. See also [36].

Let $\eta>0$ be a positive number. Let $\psi_\eta:\mathbb{R}^d\to\mathbb{R}$ be a C^∞ function which takes values in the interval $[0,1]$ and which satisfies the conditions $\psi_\eta(x)=1$ if $|x|\le\eta$, $\psi_\eta(x)=0$ if $|x|\ge 2\eta$ and $|D\psi_\eta|\le k/\eta$ where k is a constant which does not depend on η.

Consider the family of equations

$$x'=\ell(\tau_t(p))x+\psi_\eta(x)n(\tau_t(p),x)\quad (p\in P) \tag{2.4}$$

obtained from (2.3) by multiplying $n(\,\cdot\,,\,\cdot\,)$ by $\psi_\eta(\,\cdot\,)$. We abuse notation and write N_t for the homeomorphism of $P\times\mathbb{R}^d$ defined by

$$N_t(p,x_0)=(\tau_t(p),\varphi(t,p,x_0)),$$

where now $\varphi(t,p,x_0)$ denotes the solution of (2.4) with initial value x_0. The flow $(P\times\mathbb{R}^d,\{N_t\})$ depends on η, but we do not explicitly indicate the dependence.

Now let $\Delta>0$ be given. We can choose $\eta>0$ so that

$$|(N_t(p,x_1)-L_t(p,x_1))-(N_t(p,x_0)-L_t(p,x_0))|\le\Delta|x_1-x_0| \tag{2.5}$$

for all $p\in P$ and all $x_1,x_2\in\mathbb{R}^d$. This relation is fundamental in proving the existence of the so-called ρ-stable and ρ-unstable invariant foliations of $P\times\mathbb{R}^d$, for appropriate values of ρ ([19]–[21]). We refer to [22] for a proof of the existence of and a discussion of the properties of these foliations. Here we describe them in a manner sufficient for the purposes of this paper.

Let λ_0 be a real number which does not lie in the dynamical spectrum Σ of the family (2.3). Let $\rho=e^{\lambda_0}$. Set

$$\widetilde{V}^s=\bigoplus\{V_r\mid b_r<\lambda_0\},\quad \widetilde{V}^u=\bigoplus\{V_r\mid a_r>\lambda_0\}$$

where the $\oplus$ indicates the Whitney sum of vector subbundles of $P \times \mathbb{R}^d$. Then $\widetilde{V}^s$ and $\widetilde{V}^u$ are closed vector subbundles of $P \times \mathbb{R}^d$. We indicate the fibers of these subbundles at a given point $p \in P$ by

$$\widetilde{V}^s_p = (\{p\} \times \mathbb{R}^d) \cap \widetilde{V}^s, \quad \widetilde{V}^u_p = (\{p\} \times \mathbb{R}^d) \cap \widetilde{V}^u.$$

Using the theory of ρ-hyperbolic linear maps and Lipschitz small perturbations of these ([19], [21]) one can determine a number $\Delta = \Delta(\rho) > 0$ with the following properties. Let η be chosen so that (2.5) holds, and let $\{N_t\}$ correspond to such a value of η.

(1) There are maps $h^s_p: \widetilde{V}^s_p \to \widetilde{V}^u_p$ and $h^u_p: \widetilde{V}^u_p \to \widetilde{V}^s_p$ which are of class C^1, such that the sets

$$W^s = \{(p,x) \in P \times \mathbb{R}^d \mid x = v \oplus h^s_p(v), v \in \widetilde{V}^s_p\},$$
$$W^u = \{(p,x) \in P \times \mathbb{R}^d \mid x = v \oplus h^u_p(v),\ v \in \widetilde{V}^u_p\}$$

are N_t-invariant. If $\lambda_0 \le 0$, the maps h^s_p are of class C^3; if $\lambda_0 \ge 0$, the maps h^u_p are of class C^3 $(p \in P)$.

(2) For each $p \in P$, the fiber $W^s_p = W^s \cap (\{p\} \times \mathbb{R}^d)$ is tangent at $x = 0$ to $\widetilde{V}^s_p$, and the fiber $W^u_p = W^u \cap (\{p\} \times \mathbb{R}^d)$ is tangent at $x = 0$ to $\widetilde{V}^u_p$.

(3) The sets W^s and W^u can be characterized as follows: for each $p \in P$, $x_0 \in W^s_p$ if and only if $\rho^{-t}|N_t(p,x_0)| \to 0$ as $t \to \infty$; $x_0 \in W^u_p$ if and only if $\rho^{-t}|N_t(p,x_0)| \to 0$ as $t \to -\infty$.

(4) The fibers W^s_p and W^u_p $(p \in P)$ fit together smoothly in the following sense. Define $h^s: \widetilde{V}^s \to \widetilde{V}^u$: $h^s(p,v) = h^s_p(v)$ and $h^u: \widetilde{V}^u \to \widetilde{V}^s$: $h^u(p,v) = h^u_p(v)$. Then h^s and h^u are F^1 in the sense of Foster [12]. That is, if $(p_n, v_n) \to (p,v)$ in $\widetilde{V}^s$, if $x_n \in \widetilde{V}^s_{p_n}$, and if $x_n \to x$ in $\mathbb{R}^d$, then $Dh^s_{p_n} \cdot x_n \to Dh^s_p \cdot x$. The analogous condition holds for h^u. If $\lambda_0 \le 0$, then h^s is F^3 in the sense of Foster, while if $\lambda_0 \ge 0$ then h^u is F^3 in the sense of Foster (see [12]).

(5) If $\lambda_0 \le 0$, then there exists $\eta_* > 0$ such that, if $|x_0| \le \eta_*$ and if $(p,x_0) \in W^s$, then $|N_t(p,x_0)| \le \eta$ for all $t \ge 0$. Thus $\varphi(t;p,x_0) \in W^s$ for all $t \ge 0$ where now $\varphi(\,\cdot\,;\,\cdot\,,\,\cdot\,)$ refers to the original equation (2.1).

Next we review some terminology and facts concerning control flows; the standard reference for this material is [8]. Let $U \subset \mathbb{R}$ be a compact interval containing 0 in its interior. Let $f(\,\cdot\,)$ and $g(\,\cdot\,)$ be C^3 vector fields on $\mathbb{R}^d$. Consider the control system

$$x' = f(x) + u(t)g(x) \quad (x \in \mathbb{R}^d) \tag{2.6}$$

where $u: \mathbb{R} \to U$ is a measurable function. This is of course not the most general form of control system considered in [8], but the control systems dealt with in this paper have the form (2.6).

Let $\mathcal{U} = \{u:\mathbb{R} \to U \mid u \text{ is measurable}\}$. Observe that $\mathcal{U} \subset L^\infty(\mathbb{R})$ and that $\mathcal{U}$ is closed under translation: if $u \in \mathcal{U}$, $t \in \mathbb{R}$ and $\tau_t(u)(s) =: u(s+t)$ $(s \in \mathbb{R})$, then $\tau_t(u) \in \mathcal{U}$. We give $\mathcal{U}$ the weak-* topology defined as follows: a sequence $\{u_n\} \subset \mathcal{U}$ converges to a point $u \in \mathcal{U}$ just when, for each $\varphi \in L^1(\mathbb{R})$, there holds

$$\int_{-\infty}^{\infty} u_n(t)\varphi(t)\,dt \to \int_{-\infty}^{\infty} u(t)\varphi(t)\,dt.$$

It is easy to see that the pair $(\mathcal{U}, \{\tau_t\})$ is a flow.

Next let $u \in \mathcal{U}$, $x_0 \in \mathbb{R}^d$, and consider the solution $\varphi(t; u, x_0)$ of (2.6) satisfying $\varphi(0; u, x_0) = x_0$. One can prove a local existence and uniqueness result regarding $\varphi(t; u, x_0)$. Assume that $\varphi(t; u, x_0)$ is defined for all $t \in \mathbb{R}$, for each $u \in \mathcal{U}$ and $x_0 \in \mathbb{R}^d$. This holds if, for example, $f(\,\cdot\,)$ and $g(\,\cdot\,)$ are bounded on $\mathbb{R}^d$. Set $N_t(u, x_0) = (\tau_t(u), \varphi(t; u, x_0))$ for each $t \in \mathbb{R}$, $u \in \mathcal{U}$, $x_0 \in \mathbb{R}^d$. Note that, formally, this does not fit into the framework sketched for equations (2.1), since the evaluation map $u \to u(0)$ is not well defined on $\mathcal{U}$. Nevertheless one proves [8] that $(\mathcal{U} \times \mathbb{R}^d, \{N_t\})$ is a flow.

Assume from now on that $f(0) = g(0) = 0$, so that $x = 0$ is an equilibrium of (2.6) for each $u \in \mathcal{U}$. We will also assume from now on that f and g satisfy a controllability rank condition which can be formulated as follows. Let $M = \mathbb{R}^d \setminus \{0\}$. Define $\mathrm{ad}_f(g)$ to be the Lie bracket $[f, g]$ of the vector fields f, g on M. Further define $\mathrm{ad}_f^{k+1}(g) = \mathrm{ad}_f(\mathrm{ad}_f^k(g))$ for $k = 1, 2, \ldots$. The condition we impose is this: for each $x \in M$ it is required that

$$\mathrm{span}\{f(x), \mathrm{ad}_f^k(g)(x) \mid k = 1, 2, 3\} = \mathbb{R}^2.$$

If f and g are of class C^r for some $r > 3$, then the condition can be relaxed to $\mathrm{span}\{f(x), \mathrm{ad}_f^k(g)(x) \mid 1 \leq k \leq r\} = \mathbb{R}^2$ for each $x \in M$.

Let $\pi:\mathcal{U} \times \mathbb{R}^d \to \mathbb{R}^d$ be the projection onto the second factor.If $x_0 \in \mathbb{R}^d$, let $\mathcal{O}^+(x_0)$ be the reachable set from x_0:

$$\mathcal{O}^+(x_0) = \{y \in \mathbb{R}^d \mid \text{there exist } u \in \mathcal{U} \text{ and } t \geq 0 \text{ such that } \pi(N_t(u, x_0)) = y\}.$$

Definition 2.4. A set $D \subset \mathbb{R}^d$ is called a *control set* of (2.6) if

(a) D has nonempty interior;
(b) for all $x_0 \in D$ there holds $D \subset \mathrm{cl}\mathcal{O}^+(x_0)$ where cl means closure;
(c) D is maximal with respect to the properties (a) and (b), i.e. if $D \subset D'$ and if D' satisfies (a) and (b) then $D = D'$.

According to this definition, approximate controllability holds in D: if $x_0, y \in D$ and O is a neighbourhood of y, then there exists $u \in \mathcal{U}$ and $t \geq 0$ such that $\pi(N_t(u, x_0)) \in O$.

There is a useful relation between control sets and ω-limit sets in the control flow. To explain it we introduce the following concept.

DEFINITION 2.5. A pair $(u, x_0) \in \mathcal{U} \times \mathbb{R}^d$ is called an *inner pair* if there exists $T > 0$ such that $N_T(u, x_0) \in \mathrm{Int}\mathcal{O}^+(x_0)$.

There are several results to the effect that, if a point $(u, x_0) \in \mathcal{U} \times \mathbb{R}^d$ has a bounded positive semiorbit $\{N_t(u, x_0) \mid t \geq 0\}$, and if every point (v, y) in the ω-limit set Ω of (u, x_0) is an inner pair, then the projection of Ω to $\mathbb{R}^d$ is contained in the interior of a control set. We give one statement of this sort. It can be proved by combining Corollary 4.5.12 and Proposition 4.5.19 of [8].

THEOREM 2.6. *Let $M = \mathbb{R}^d \setminus \{0\}$, and suppose that f and g satisfy the controllability rank condition on M. Let $K \subset M$ be a compact set such that, if $u \in \mathcal{U}$ and $x_0 \in K$, then there exists $T = T(u, x_0) > 0$ with the property that $\varphi(t; u, x_0) \in K$ for all $t \geq T$. Then there exists a positive number ρ_0 such that: if $|u|_\infty \leq \rho_0$, if $x_0 \in K$, and if Ω is the ω-limit set of (u, x_0), then $\pi(\Omega)$ is contained in some control set D.*

PROOF. First one uses ([8, Proposition 4.5.19]) to prove that there exists $\rho_0 > 0$ such that, if $|u|_\infty \leq \rho_0$ and $x_0 \in K$, then (u, x_0) is an inner pair. Then one applies ([8, Proposition 4.5.12]) to show that, if $|u|_\infty \leq \rho_0$ and $x_0 \in K$, then the ω-limit set $\Omega \subset \mathcal{U} \times K$ of (u, x_0) has the property that $\pi(\Omega)$ is contained in a control set D. $\square$

3. The intermediate regime

Consider the two-parameter family of control systems

$$x' = f(x, \varepsilon) + \rho u(t) g(x, \varepsilon) \quad (x \in \mathbb{R}^2) \tag{3.1}$$

where ε lies in an open interval $I \subset \mathbb{R}$ and $-\rho_0 < \rho < \rho_0$ where $\rho_0 > 0$. We will often assume that u is uniformly continuous; we let $\mathcal{U}_c = \{u\colon \mathbb{R} \to U \mid u(\cdot) \text{ is uniformly continuous}\}$. Then $\mathcal{U}_c \subset \mathcal{U}$. We give $\mathcal{U}_c$ the standard compact-open topology. Then $\mathcal{U}_c$ is not compact; however the set $\{\tau_t \mid t \in \mathbb{R}\}$ of translations defines a flow $(\mathcal{U}_c, \{\tau_t\})$. Moreover, if $u \in \mathcal{U}_c$, then the orbit closure $\mathrm{cl}\{\tau_t(u) \mid t \in \mathbb{R}\} = P_u$ is compact.

Let $B_\varepsilon = D_x f(0, \varepsilon)$. We suppose in this section that there exist parameter values $\varepsilon_1, \varepsilon_2 \in I$ with $\varepsilon_1 < \varepsilon_2$ such that, if $\varepsilon \in (\varepsilon_1, \varepsilon_2)$, then B_ε admits two real eigenvalues $\lambda_1(\varepsilon)$ and $\lambda_2(\varepsilon)$ such that $\lambda_1(\varepsilon) < 0 < \lambda_2(\varepsilon)$. We further suppose that $\lambda_1(\varepsilon_1) < \lambda_2(\varepsilon_1) = 0$, and that $\lambda_1(\varepsilon) < \lambda_2(\varepsilon) < 0$ if $\varepsilon < \varepsilon_1$.

Clearly the origin $x = 0$ defines an asymptotically stable solution of (3.1) when $\rho = 0$ and when $\varepsilon < \varepsilon_1$. For each fixed $\varepsilon < \varepsilon_1$, there exists a number $\rho(\varepsilon) > 0$ such that $x = 0$ defines an asymptotically stable solution of (3.1) for $|\rho| \leq \rho(\varepsilon)$. We propose to analyze the behavior of the solutions of (3.1) when $\rho \neq 0$ is small and when ε lies in the interval $(\varepsilon_1, \varepsilon_2)$.

First suppose that $\rho = 0$. If the derivative $(d\lambda_2/d\varepsilon)(\varepsilon_1) > 0$ and if the higher-order terms of the vector field f satisfy appropriate conditions, then the equation

$$x' = f(x, \varepsilon) \tag{3.2}$$

exhibits a transcritical bifurcation at $x = 0$, $\varepsilon = \varepsilon_1$. We impose a sufficient condition for the existence of a transcritical bifurcation with an eye to studying equation (3.1) for x near zero, ρ near zero, and ε near ε_1.

Hypothesis 3.1. *Let e be a unit vector lying in the one-dimensional eigenspace E of B_{ε_1} which corresponds to $\lambda_2(\varepsilon_1) = 0$. Suppose that there exist positive constants c, η such that*

$$\langle f(\psi e, \varepsilon_1), e\rangle \leq -c\psi^2 \quad (0 \leq \psi \leq 2\eta).$$

In this case, a standard analysis (e.g. [7]) shows that a transcritical bifurcation takes place in (3.2) as ε passes through ε_1. More precisely, there is an equilibrium point $x(\varepsilon) \neq 0$ of (3.2) for values of $\varepsilon > \varepsilon_1$ such that $\varepsilon - \varepsilon_1$ is small. This equilibrium point defines an exponentially asymptotically stable solution of (3.2). Moreover, $x(\varepsilon) \to 0$ as $\varepsilon \to \varepsilon_1^+$. For each $\varepsilon > \varepsilon_1$ with $\varepsilon - \varepsilon_1$ sufficiently small, $x(\varepsilon)$ lies on the ε-slice of each center manifold of the augmented system

$$x' = f(x, \varepsilon), \quad \varepsilon' = 0.$$

Now let $\varepsilon > \varepsilon_1$ be a point near ε_1, and let $x(\varepsilon) \neq 0$ be the equilibrium of (3.2) discussed above. Let $u_0 \in \mathcal{U}$ be a non-zero constant control. There exists a positive number $\rho_0 = \rho_0(\varepsilon)$ such that, if $|\rho| \leq \rho_0$, then the system

$$x' = f(x, \varepsilon) + \rho u_0 g(x, \varepsilon)$$

admits an exponentially asymptotically stable equilibrium $x(\varepsilon, \rho) \neq 0$; moreover $x(\varepsilon, \rho) \to x(\varepsilon)$ as $\rho \to 0$.

Adopting the notation of Section 2, let $M = \mathbb{R}^2 \setminus \{0\}$. By hypothesis, the controllability rank condition holds on M for each $\varepsilon \in I$. It follows from ([8, Proposition 4.5.19]) that $(u_0, x(\varepsilon, \rho))$ satisfies the inner pair condition for each $\rho \in [-\rho_0, \rho_0]$. We can interpret the singleton set $\Omega = \{(u_0, x(\varepsilon, \rho))\} \subset \mathcal{U} \times M$ as an ω-limit set in the control flow. Therefore, by ([8, Proposition 4.5.12]), the projection $\pi(D) = \{x(\varepsilon, \rho)\}$ lies in the interior of a control set $D \subset \mathbb{R}^2$.

We summarize this discussion as follows.

Theorem 3.2. *Suppose that Hypothesis* 3.1 *holds. Then if $\varepsilon > \varepsilon_1$ is sufficiently close to ε_1, there exists a positive number $\rho_0 = \rho_0(\varepsilon)$ such that, if $|\rho| \leq \rho_0(\varepsilon)$, then the control system* (3.1) *admits a control set $D \subset \mathbb{R}^2$.*

The proof of Theorem 3.2 requires only very limited information concerning the nature of the control flow $(\mathcal{U} \times \mathbb{R}^2, \{N_t\})$. We point out that, if the non-linear terms of f and g satisfy certain conditions, then the control flow can be investigated in more detail for small positive values of $\varepsilon - \varepsilon_1$ and small values of ρ. For example, if $P \subset \mathcal{U}_c$ is a minimal set, then methods from [24] can be used to give fairly natural sufficient conditions for the existence of a compact invariant subset $\Omega \subset P \times \mathbb{R}^2 \subset \mathcal{U}_c \times \mathbb{R}^2$ which is near the zero-section $P \times \{0\}$ in the Hausdorff sense, and which has the property that $\Omega \cap (P \times \{0\})$ is empty. Roughly speaking, Ω "bifurcates" from $P \times \{0\}$ as ε increases through ε_1. If ρ is sufficiently small, then the projection $\pi(\Omega)$ of such a compact invariant set is contained in a control set D.

Let us now turn to the situation when $\varepsilon_1 < \varepsilon < \varepsilon_2$ but $\varepsilon - \varepsilon_1$ is not "small". We examine the question of the existence of control sets D for the system (3.1) when ρ is small and when the uncontrolled system (3.2) admits an orbit which is homoclinic to $x = 0$. We will give a condition sufficient for the existence of an ω-limit set $\Omega \subset \mathcal{U} \times \mathbb{R}^2$ which is disjoint from $\mathcal{U} \times \{0\}$ and whose projection to $\mathbb{R}^2$ lies in a control set D. We note that Grünvogel [17] takes a different approach: in the situation he considers, he is able to construct a non-trivial control which leads to an ω-limit set which, in addition to the origin, contains other points. This allows him to infer the existence of a control set containing the origin as a boundary point.

Continuing the discussion, let $\varepsilon_* \in (\varepsilon_1, \varepsilon_2)$ be a parameter value such that the equation

$$x' = f(x, \varepsilon_*)$$

admits a non-zero solution $y(t)$ satisfying $\lim_{t\to\infty} y(t) = \lim_{t\to-\infty} y(t) = 0$. Let P be a compact translation invariant subset of $\mathcal{U}_c = \{u\colon \mathbb{R} \to U \mid u(\cdot) \text{ is uniformly continuous}\}$ where we recall that $\mathcal{U}_c$ has the compact-open topology. We indicate a generic element of P by p; of course $p(\cdot)$ is to be interpreted as a control function. We will study the control system (3.1) when $\varepsilon = \varepsilon_*$, ρ is small, and the control function take values in P.

For general values $\varepsilon \in I$ and $\rho \in \mathbb{R}$, we can write

$$\ell(p) = Df(0, \varepsilon) + \rho p(0) Dg(0, \varepsilon)$$

where we suppress the dependence of ℓ on ε and ρ. Then (3.1) takes the form

$$x' = \ell(\tau_t(p))x + n(\tau_t(p), x))$$

where $n(p, x) = f(x, \varepsilon) + \rho p(0) g(x, \varepsilon) - \ell(p)x$. Let $\Sigma = \Sigma_{\rho,\varepsilon}$ be the dynamical spectrum of the family of linear systems

$$x' = \ell(\tau_t(p))x.$$

By a basic perturbation result [29] one has that, for small $\rho \neq 0$, $\Sigma = [a_1, b_1] \cup [a_2, b_2]$ where $[a_1, b_1]$ is close in the Hausdorff sense to $\lambda_1(\varepsilon)$, and $[a_2, b_2]$ is close in the Hausdorff sense to $\lambda_2(\varepsilon)$.

Now let $\varepsilon = \varepsilon_*$, and recall that $\lambda_1(\varepsilon_*) < 0 < \lambda_2(\varepsilon_*)$. There exists a positive number ρ_* such that, if $|\rho| \leq \rho_*$, then the dynamical spectrum $\Sigma_{\rho_*, \varepsilon_*} = [a_1, b_1] \cup [a_2, b_2]$ where $[a_1, b_1] \subset (-\infty, 0)$ and $[a_2, b_2] \subset (0, \infty)$. For $\varepsilon = \varepsilon_*$ and $\rho \in [-\rho_*, \rho_*]$, consider the integral manifolds $W^s = W^s(p) \subset P \times \mathbb{R}^2$ and $W^u = W^u(p) \subset P \times \mathbb{R}^2$ introduced in Section 2. Referring to the discussion in Section 2, we see that λ_0 can be chosen to be zero. Letting $\eta_* > 0$ be as in point (5) of that discussion, define

$$\begin{aligned} W^{s,\mathrm{loc}} &= \{(p, x_0) \in W^s \mid |x| \leq \eta_*\}, \\ W^{u,\mathrm{loc}} &= \{(p, x_0) \in W^u \mid |x| \leq \eta_*\}. \end{aligned}$$

It can be shown that these local integral manifolds generate the global integral manifolds as follows:

$$W^s = \bigcup_{t \leq 0} N_t\left(W^{s,\mathrm{loc}}\right), \quad W^u = \bigcup_{t \geq 0} N_t\left(W^{u,\mathrm{loc}}\right).$$

Finally we note that, according to Point 4 in the discussion in Section 2, the fibers

$$W^s_p = W^s \cap (\{p\} \times P), \quad W^u_p = W^u \cap (\{p\} \times P)$$

are C^3-submanifolds of $\mathbb{R}^2$ for each $p \in P$.

Write $Y = \{y(t) | t \in \mathbb{R}\}$ for the homoclinic orbit of (3.2) at $\varepsilon = \varepsilon_*$. Let L be a line segment in $\mathbb{R}^2$ which contains $y(0)$ in its interior and which is transversal to Y. Using the smoothness of the fibers, one can show that, for small $\rho \neq 0$, there are "first" points of intersection $y^s_p(\rho)$ (resp. $y^u_p(\rho)$) of $W^s_p(\rho)$ (resp. $W^u_p(\rho)$) with L, and that these are points of transversal intersection. One can further show that the fibers $W^{s,u}_p(\rho)$ are C^3 as functions of ρ, and from this deduce that the maps $\rho \mapsto y^s_p(\rho)$ and $\rho \mapsto y^u_p(\rho)$ are C^3 in ρ for each $p \in P$. In fact, one can show that the derivatives $D^k_\rho y^{s,u}_p$ are continuous with respect to p $(0 \leq k \leq 3)$.

For each fixed $p \in P$, we introduce a function which measures the distance along L between $y^s_{\tau_r(p)}(\rho)$ and $y^u_{\tau_r(p)}(\rho)$ for translates $\tau_r(p)$ $(r \in \mathbb{R})$. Let us write $y^s_p(\rho, r) = y^s_{\tau_r(p)}(\rho)$, $y^u_p(\rho, r) = y^u_{\tau_r(p)}(\rho)$, then define

$$h(\rho, r) = \begin{cases} \dfrac{y^u_p(\rho, r) - y^s_p(\rho, r)}{\rho} \wedge f(y(0), \varepsilon_*) & \text{for } \rho \neq 0, \\ M(r) & \text{for } \rho = 0, \end{cases}$$

where the *Mel'nikov function* $M(r)$ is defined to be

$$M(r) = \frac{\partial}{\partial \rho}(y^u_p(\rho, r) - y^s_p(\rho, r))_{\rho = 0} \wedge f(y(0), \varepsilon_*).$$

Here "$\wedge$" denotes the wedge product in $\mathbb{R}^2$. Since the difference $y_p^u(\rho, r) - y_p^s(\rho, r)$ lies on L and since L is transversal to $W^s_{\tau_r(p)}(\rho)$ and $W^u_{\tau_r(p)}(\rho)$ for all $p \in P$ and all small values of ρ, the function M measures the velocity with which these manifolds cross each other at $\rho = 0$.

We now use a standard method to derive a formula for $M(r)$. For this, let $y_p^s(t; \rho, r)$ (resp. $y_p^u(t; \rho, r)$) denote the solution of (3.1) with $u(t) = p(t+r)$ which has initial condition $y_p^s(\rho, r)$ (resp. $y_p^u(\rho, r)$). These solutions decay exponentially as $t \to \infty$ (resp. $t \to -\infty$). We write

$$\Delta_\pm(t, r) = \frac{\partial}{\partial \rho} y_p^{s,u}(t; \rho, r)\bigg|_{\rho=0} \wedge f(y(t), \varepsilon_*).$$

One observes that $z(t) = (\partial/\partial\rho) y_p^{s,u}(t; \rho, r)|_{\rho=0}$ satisfies

$$z' = D_x f(y(t), \varepsilon_*) z + p(t+r) g(y(t), \varepsilon_*).$$

Using the formula $(\mathrm{tr} B)(q_1 \wedge q_2) = Bq_1 \wedge q_2 + q_1 \wedge Bq_2$, valid for 2×2 matrices B and vectors $q_1, q_2 \in \mathbb{R}^2$, we see that

$$\frac{d}{dt} \Delta_\pm(t, r) = \mathrm{tr} D_x f(y(t), \varepsilon_*) \Delta_\pm + p(t+r) g(y(t), \varepsilon_*) \wedge f(y(t), \varepsilon_*).$$

One can show that $\lim_{t\to\infty} \Delta_+(t, r) = \lim_{t\to-\infty} \Delta_-(t, r) = 0$; see, e.g. ([25], [4]). It follows that

$$\Delta_\pm(t, r) = \int_{\pm\infty}^{t} e^{\int_0^s \mathrm{tr} D_x f(y(\sigma), \varepsilon_*) d\sigma} p(s+r) g(y(s), \varepsilon_*) \wedge f(y(s), \varepsilon_*)\, ds.$$

Setting $t = 0$ and using the fact that $M(r) = \Delta_-(0, r) - \Delta_+(0, r)$, we obtain

Proposition 3.3.

$$M(r) = \int_{-\infty}^{\infty} e^{\int_0^s \mathrm{tr} D_x f(y(\sigma), \varepsilon_*)\, d\sigma} p(s+r) g(y(s), \varepsilon_*) \wedge f(y(s), \varepsilon_*)\, ds.$$

Next we use the Mel'nikov function to determine values of (ρ, r) for which the fibers $W^{s,u}_{\tau_r(p)}(\rho)$ cross transversally.

Proposition 3.4. *Let $r_0 \in \mathbb{R}$ be a number such that $M(r_0) = 0$ and $M'(r_0) \neq 0$. Then the stable and unstable fibers $W^s_{\tau_r(p)}(\rho)$ and $W^u_{\tau_r(p)}(\rho)$ cross transversally along a curve $r = r(\rho)$ for small values of ρ, where $r(0) = r_0$.*

Proof. We apply the implicit function theorem to the relation $g(\rho, r) = 0$ near the solution $\rho = 0$, $r = r_0$. Since $(\partial g/\partial r)(0, r) \neq 0$ we see that there is a smooth curve $r = r(\rho)$, defined for small values of ρ, such that $r(0) = r_0$ and $g(\rho, r(\rho)) = 0$.

We now show that the fibers $W^s_{\tau_r(p)}(\rho)$ and $W^u_{\tau_r(p)}(\rho)$ cross transversally at $(\rho, r(\rho))$ for small ρ. To do this, let $a \in (-1, 1)$ be a number and let $L(a)$ be a segment containing $y(a)$ in its interior which is parallel to L. There are "first" points of intersection $y_p^s(\rho, r, a)$ (resp. $y_p^u(\rho, r, a)$) of $W^s_{\tau_r(p)}(\rho)$ (resp.

$W^u_{\tau_r(p)}(\rho)$) with $L(a)$ and these functions are of class C^3 in (ρ, r, a). The derivatives $(\partial/\partial a)y^{s,u}_p(\rho, r, a)|_{a=0}$ are tangent to $W^{s,u}_p(\rho)$ at $y^{s,u}_p(\rho, r)$. Moreover, these derivatives are non-zero for small ρ because they are, up to an $O(\rho)$ term, equal to $f(y(0), \varepsilon_*)$.

Let

$$\widetilde{M}(r, a) = \frac{\partial}{\partial \rho}\big(y^s_p(\rho, r, a) - y^u_p(\rho, r, a)\big)\Big|_{\rho=0} \wedge f(y(a), \varepsilon_*).$$

Applying Proposition 3.3 we obtain

$$\widetilde{M}(r, a) = \int_{-\infty}^{\infty} e^{\int_0^s \operatorname{tr} D_x f(y(\sigma+a), \varepsilon_*)\, d\sigma} p(s+r) g(y(s+a), \varepsilon_*) \wedge f(y(s+a), \varepsilon_*)\, ds.$$

Hence

$$\widetilde{M}(r, a) = e^{-\int_0^a \operatorname{tr} D_x f(y(\sigma), \varepsilon_*)\, d\sigma} M(r - a).$$

It follows that $(\partial\widetilde{M}/\partial a)(r(\rho), 0) = -M'(r(\rho)) \neq 0$. It follows that the non-zero tangent vectors $(\partial/\partial a)y^{s,u}_p(\rho, r(\rho), a)|_{a=0}$ have the property that their difference has a component normal to $f(y(0), \varepsilon_*)$ which is nonzero. Hence the fibers $W^s_{\tau_r(p)}(\rho)$ and $W^u_{\tau_r(p)}(\rho)$ indeed cross transversally at $r = r(\rho)$. □

We can apply Proposition 3.4 to the control flow and the control sets of (3.1) when ρ is small. Let $p(\,\cdot\,)$ be a periodic control function of class C^2. Suppose that

$$\int_{-\infty}^{\infty} e^{\int_0^s \operatorname{tr} D_x f(y(\sigma), \varepsilon_*)\, d\sigma} p(s) g(y(s), \varepsilon_*) \wedge f(y(s), \varepsilon_*)\, ds = 0,$$

$$\int_{-\infty}^{\infty} e^{\int_0^s \operatorname{tr} D_x f(y(\sigma), \varepsilon_*)\, d\sigma} p'(s) g(y(s), \varepsilon_*) \wedge f(y(s), \varepsilon_*)\, ds \neq 0.$$

Let T be the period of $p(\,\cdot\,)$, and set $\mathcal{P} = \{\tau_t(p) \mid 0 \le t \le T\} \subset \mathcal{U}$. Let $r = r(\rho)$ be the curve whose existence is guaranteed by Proposition 3.4 (with $r(0) = 0$).

Let $Q\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the period map defined by following the solutions of

$$x' = f(x, \varepsilon_*) + \rho p(t + r(\rho)) g(x, \varepsilon_*)$$

for time T. Then the point of intersection q of $W^s_{\tau_r(p)}(\rho)$ and $W^u_{\tau_r(p)}(\rho)$ at $r = r(\rho)$ is transversal, and defines a transversal homoclinic point of the map Q. It then follows ([35], [28]) that each neighbourhood of q contains a set S which is invariant under an iterate Q^N of Q, and (S, Q^N) is isomorphic to a shift. Let $\Omega = \{N_t(\tau_{r(\rho)}(p), s) \mid s \in S,\ 0 \le t \le NT\}$; then $\Omega \subset \mathcal{U} \times \mathbb{R}^2$ is a compact invariant set which is disjoint from $\mathcal{U} \times \{0\}$. The projection $\pi(\Omega)$ of Ω to $\mathbb{R}^2$ lies in the interior of a control set D (use Theorem 2.6). This uses the fact that Ω admits dense positive semitrajectories, hence is an ω-limit set.

As an example we mention the control system

$$x' = y, \quad y' = x - x^2 + \rho p(t) x$$

where $p(\,\cdot\,) \in \mathcal{U}_c$. The unperturbed system

$$x' = y, \quad y' = x - x^2$$

admits the homoclinic orbit

$$x_0(t) = \frac{3}{2}\mathrm{sech}^2\frac{t}{2}, \quad y_0(t) = -\frac{3}{2}\mathrm{sech}^2\frac{t}{2}\tanh\frac{t}{2}.$$

The Mel'nikov function at $r = 0$ is

$$\begin{aligned} M(0) &= \int_{-\infty}^{\infty} p(s)\left[\begin{pmatrix} 0 \\ x_0(s) \end{pmatrix} \wedge \begin{pmatrix} y_0(s) \\ 4x_0(s) - x_0^2(s) \end{pmatrix}\right] ds \\ &= \int_{-\infty}^{\infty} p(s)\frac{d}{ds}(x_0(s)^2)\, ds = -\int_{-\infty}^{\infty} x_0(s)^2 \frac{dp}{ds}(s)\, ds. \end{aligned}$$

If $M(0) = 0$ and

$$M'(0) = -\int_{-\infty}^{\infty} x_0(s)^2 \frac{d^2p}{ds^2}(s)\, ds \neq 0,$$

then Proposition 3.4 can be applied.

We summarize the preceding discussion in the following

Proposition 3.5. *Let $\varepsilon \in (\varepsilon_1, \varepsilon_2)$ be a parameter value such that the uncontrolled equation admits a solution homoclinic to the origin. Let $p(\,\cdot\,)$ be a periodic control function of class C^2, and suppose that the Mel'nikov function satisfies $M(0) = 0$ and $M'(0) \neq 0$. Then for all small $\rho > 0$ there is a control set $D = D(\rho, \varepsilon_*)$ containing a subset which may be described as the projection of a set S which* (*for the N-fold concatenation of the control p*) *is isomorphic to a shift.*

4. The unstable regime

In this section, we continue our study of the control system

$$x' = f(x, \varepsilon) + \rho u(t) g(x, \varepsilon) \tag{4.1}$$

where now $\varepsilon > \varepsilon_2$ and ρ is small. Our starting point is again the unperturbed system

$$x' = f(x, \varepsilon). \tag{4.2}$$

Of course there are many possibilities for the phase portrait of such a system. We will pick out a particular phase portrait by imposing certain conditions on f and g, and will then show that for small $\rho \neq 0$, this phase portrait "induces" a bifurcation pattern of Arnold type on system (4.1) as ε passes through ε_2.

The first condition we impose is the following. We assume that there is an interval $I_1 \subset I$, containing ε_2 in its interior, such that, for each $\varepsilon \in I_1$, the flow on $\mathbb{R}^2$ defined by (4.2) has a dissipative structure. That is, there exist positive numbers δ, R, and T, which do not depend on $\varepsilon \in I_1$, such that, if $|x_0| = R$, then the corresponding solution $\varphi(t,x_0)$ of (4.2) satisfies $|\varphi(t,x_0)| \le R-\delta$ for all $t \ge T$. In this case, the system (4.2) admits an attractor $S = S_\varepsilon$ defined as follows:

$$S_\varepsilon = \bigcap_{t\ge 0}\{\varphi(t,x_0) \mid |x_0| \le R\} \quad (\varepsilon \in I_1).$$

Clearly $S_\varepsilon \subset B_R = \{x \in \mathbb{R}^2 \mid |x_0| \le R\}$. We refer to S_ε as the "global attractor" for (4.2) even though, strictly speaking, the flow on $\mathbb{R}^2$ defined by (4.2) may admit other compact attracting invariant sets in $\mathbb{R}^2 \setminus B_R$.

Let us note that a sufficient condition for the existence of such a dissipative structure is the following: there exist constants $R > 0$, $c > 0$, and $\alpha > 1$ such that for each $x \in \mathbb{R}^2$ with $|x| \ge R$, each $\varepsilon \in I_1$, and each unit vector $e \in \mathbb{R}^2$ there holds

$$\langle f(x,\varepsilon), e\rangle \le -c|x|^\alpha.$$

Let $\varepsilon \in I_1$ and $\rho \in \mathbb{R}$ be given. For each $x_0 \in \mathbb{R}^2$ and each $u \in \mathcal{U}$, let $\varphi(t,u,x_0)$ denote the solution of (4.1) satisfying $\varphi(0,u,x_0) = x_0$; the dependence of φ on ε and ρ is suppressed. We state and prove an elementary result.

Proposition 4.1. *Let δ, R, and T be as above. There exists $\rho_1 > 0$ such that, if $|\rho| \le \rho_1$, $\varepsilon \in I_1$, and $|x_0| = R$, then $|\varphi(t,u,x_0)| \le R$ for all $t \ge T$.*

Proof. By continuity of φ and by compactness of $\mathcal{U}$, we can determine $\rho_0 > 0$ such that, if $\varepsilon \in I_1$ and $|\rho| \le \rho_0$, then for each $u \in \mathcal{U}$ and each $x_0 \in \mathbb{R}^2$ satisfying $|x_0| = R$ there holds

$$|\varphi(T,u,x_0)| \le R - \frac{\delta}{2}.$$

It follows that, for each $n = 1,2,\ldots$ there holds

$$|\varphi(nT,u,x_0)| \le R - \frac{\delta}{2}.$$

Using again the compactness of $\mathcal{U}$, we can determine a positive number $\rho_1 \le \rho_0$ such that the statement of the proposition holds. □

Next fix $\varepsilon \in I_1$ and $\rho \in [-\rho_1,\rho_1]$, and set

$$S = S_{\varepsilon,\rho} = \bigcap_{t\ge 0}\{(\tau_t(u),\varphi(t,u,x_0)) \mid u \in \mathcal{U},\ |x_0| \le R\}.$$

Then $S_{\varepsilon,\rho}$ is an attractor for the control flow on $\mathcal{U} \times \mathbb{R}^2$ determined by (4.1). As before, we abuse language slightly and refer to $S_{\varepsilon,\rho}$ as the "global attractor" of the control flow determined by (4.1). Note that $(u,0) \in S_{\varepsilon,\rho}$ for all $\varepsilon \in I_1$, $|\rho| \le \rho_1$.

Now we show that, if $\varepsilon \in I_1$ and $\varepsilon > \varepsilon_2$, and if ρ is sufficiently small, then $S_{\varepsilon,\rho}$ contains a subattractor $A_{\varepsilon,\rho}$ which lies in the product of $\mathcal{U}$ with an annulus $A \subset \mathbb{R}^2$ which is centered at the origin $x = 0$.

PROPOSITION 4.2. *Let $\varepsilon \in I_1 \cap (\varepsilon_2, \infty)$. Then there exist positive numbers ρ_0, r, T, which may depend on ε, such that, if $|\rho| \le \rho_0$, $|x_0| = r$, and $u \in \mathcal{U}$, then $|\varphi(t, u, x_0)| \ge r$ for all $t \ge T$.*

PROOF. We only sketch the argument. By assumption, the eigenvalues $\lambda_1(\varepsilon)$, $\lambda_2(\varepsilon)$ of the matrix $B_\varepsilon = D_x f(0, \varepsilon)$ lie in the right half-plane. Recall that the dynamical spectrum Σ of the family (4.1) is upper semi-continuous in the Hausdorff sense with respect to ρ. Therefore we can find a number $\rho_0 = \rho_0(\varepsilon) > 0$ such that, if $|\rho| \le \rho_0(\varepsilon)$, then $\Sigma \subset (0, \infty)$. The existence of r and T now follows by standard arguments. $\square$

Next let I' be a compact subinterval of $I_1 \cap (\varepsilon_2, \infty)$. One can determine positive numbers ρ_1, r, T which do not depend on ε, for which the conclusion of Proposition 4.2 holds for all $\varepsilon \in I'$. Let $\rho' = \min\{\rho_0, \rho_1\}$, let $|\rho| \le \rho'$, and let A be the annulus $\{x \in \mathbb{R}^2 \mid r \le |x| \le R\}$. Define

$$A_{\varepsilon,\rho} = \bigcap_{t \ge 0} \{(\tau_t(u), \varphi(t, u, x_0)) \mid u \in \mathcal{U},\ r \le |x_0| \le R\}.$$

Then $A_{\varepsilon,\rho}$ is an attractor contained in $\mathcal{U} \times A$. It is strictly contained in $S_{\varepsilon,\rho}$ because $(u, 0) \notin A_{\varepsilon,\rho}$ whenever $u \in \mathcal{U}$, $\varepsilon \in I'$, $|\rho| \le \rho'$.

We will study in more detail the global attractor $S_{\varepsilon,\rho}$ and the subattractor $A_{\varepsilon,\rho} \subset S_{\varepsilon,\rho}$. The next result is a corollary of Proposition 7 of [24], the proof of which is based on the continuity properties of the Čech cohomology theory [16].

PPROPOSITION 4.3. *Fix $\varepsilon \in I'$ and let $\rho \in [-\rho', \rho']$. Write $A = A_{\varepsilon,\rho}$. For each $u \in \mathcal{U}$, consider the fiber*

$$A_u = \{x \in \mathbb{R}^2 \mid (u, x) \in A\}.$$

Then the Čech cohomology (with coefficient group $\mathbb{Z}$) of A_u is that of a circle:

$$\check{H}^k(A_u, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } k = 1, \\ 0 & \text{for } k \neq 1. \end{cases}$$

Thus in a general sense "the fibers of A are circles". To obtain more detailed information about the global attractors $S_{\varepsilon,\rho}$ and its subattractor $A_{\varepsilon,\rho}$, we will impose certain elements of structure on the global attractor $S_\varepsilon = S_{\varepsilon,0}$ of the unperturbed system (4.2), then use the continuation properties of the Conley index to prove facts about $S_{\varepsilon,\rho}$ and $A_{\varepsilon,\rho}$ for small values of ρ.

We assume, then, that for each $\varepsilon \in I'$, the attractor A_ε contains exactly four equilibrium points a_1, a_2, s_1, s_2, of which s_1 and s_2 are saddles, while a_1 and a_2 are attractors (sinks), see Figure 1. In more detail, we assume that the Jacobian matrix $D_x f(s_i, \varepsilon)$ has one positive and one negative eigenvalue ($i = 1, 2$). Let us

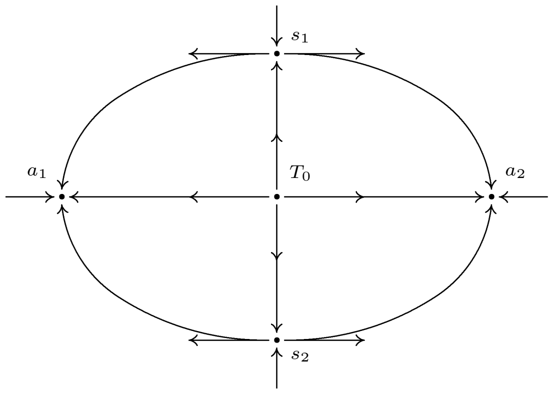

Figure 1. The phase portrait

denote the origin $x = 0$ by r_0; it follows from the condition $\lambda_2(\varepsilon) > 0$ that r_0 is a repelling equilibrium point for (4.2).

We impose still more conditions on S_ε. Namely, we assume that there are heteroclinic orbits joining r_0 with each of the equilibria a_1, a_2, s_1, s_2, and that there are heteroclinic orbits joining s_i to a_1 and a_2 $(i = 1, 2)$, again see Figure 1. Under the conditions we have imposed, one can show using Poincaré–Bendixson theory that A_ε contains $\{a_1, a_2, s_1, s_2\}$ together with heteroclinic orbits connecting these four equilibria. These heteroclinic orbits are unique because the equilibria are hyperbolic. For the same reason, there are no orbits homoclinic to s_i $(i = 1, 2)$.

A simple example which exhibits equilibria together with heteroclinic orbits as above is [24]

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} \varepsilon - 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - (x_1^2 + x_2^2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (\varepsilon > \varepsilon_2 = 1).$$

In fact, in this example the heteroclinic orbits are unique in the sense that there is just one heteroclinic orbit joining r_0 to s_1, r_0 to s_2 etc. Moreover, A_ε consists exactly of $\{a_1, a_2, s_1, s_2\}$ together with the heteroclinic orbits joining s_i to a_1 and a_2 $(i = 1, 2)$.

Now suppose that $\varepsilon \in I'$ and that $\rho \in [-\rho', \rho']$ is non-zero. We look for elements of the structure of S_ε and A_ε which can carry over to $S_{\varepsilon,\rho}$ and $A_{\varepsilon,\rho}$. We will use the concepts of Morse decomposition and connection matrix to find sets $a_1(\rho)$, $a_2(\rho)$, $s_1(\rho)$, and $s_2(\rho)$ which are contained in $\mathcal{U} \times \mathbb{R}^2$, and which are continuations of a_1, a_2, s_1, and s_2 for small $\rho \neq 0$. We will also show that there exist connecting orbits joining certain of these sets. We will freely use basic concepts and facts from the theory of the Conley index; see ([9], [13], [14], [26], [27], [30], [31]).

Let m denote one of the elements of $M = \{\{a_1\}, \{a_2\}, \{s_1\}, \{s_2\}, \{r_0\}\}$. (In what follows we will sometimes not distinguish between a_1 and $\{a_1\}$, etc.) Then m is a non-degenerate zero of the vector field $f(\,\cdot\,, \varepsilon)$. Let N_m be an isolating neighbourhood of m, and let $L_m \subset N_m$ be an exit set. The Conley index $h(m)$ is by definition the homotopy type of the pointed topological space N_m/L_m obtained by collapsing L_m to a point in N_m. For each $m \in M$, introduce the graded module $H_*(h(m), \mathbb{Z}_2)$ of singular homology groups with coefficients in $\mathbb{Z}_2$. We write $CH_*(m)$ for this graded module. It is well-known (see, e.g. [9] or [27, Theorem 3.1]) that

$$\begin{aligned} CH_k(r_0) &= \begin{cases} \mathbb{Z}_2 & \text{for } k = 2, \\ 0 & \text{for } k \neq 2, \end{cases} \\ CH_k(s_1) = CH_k(s_2) &= \begin{cases} \mathbb{Z}_2 & \text{for } k = 1, \\ 0 & \text{for } k \neq 1, \end{cases} \\ CH_k(a_1) = CH_k(a_2) &= \begin{cases} \mathbb{Z}_2 & \text{for } k = 0, \\ 0 & \text{for } k \neq 0. \end{cases} \end{aligned}$$

Note now that the global attractor $S_\varepsilon \subset \mathbb{R}^2$ of the unperturbed system (4.2) is an isolated invariant set with isolating neighbourhood B_R. Using the Poincaré-Bendixson theory, one can show that the collection $M = \{a_1, a_2, s_1, s_2, r_0\}$ of equilibria is a Morse decomposition of S_ε in the sense that, if $x \in S_\varepsilon \setminus M$, then the α-limit set $\alpha(x)$ and the ω-limit set $\omega(x)$ of x are equilibria of $f(\,\cdot\,, \varepsilon)$; i.e. they lie in M.

Consider the total order

$$a_1 < a_2 < s_2 < s_1 < r_0 \tag{4.4}$$

on M. Then M is a $<$-ordered Morse decomposition of S_ε in the sense that, if $x \in S_\varepsilon \setminus M$, then there exist $m_1 < m_2$ in M such that $\omega(x) = m_1$ and $\alpha(x) = m_2$. The order $<$ is an example of an *admissible* order on M [13]. Another admissible order on M is given by the flow order $<_F$, defined as follows: $a_i <_F s_i$ $(i, j = 1, 2)$, $a_i, s_i <_F r_0$ $(i, j = 1, 2)$. The flow order is the "extremal" admissible order on M [13]. We will not use the flow order in the following.

We proceed to study the ordered Morse decomposition $(M, <)$ defined by (4.4). A *connection matrix* c for M compatible with the order $<$ is a homomorphism of degree -1 of the sum of graded modules

$$CH_*(a_1) \oplus CH_*(a_2) \oplus CH_*(s_2) \oplus CH_*(s_1) \oplus CH_*(r_0)$$

with the following properties [13]. First, $c^2 = 0$. Second, c is upper triangular in the following sense. Represent c as a matrix $(c_i^j)_{i,j=1}^5$ where, e.g. $c_1^3\colon CH_*(a_1) \to CH_*(s_2)$. Then $c_i^j = 0$ if $j \le i$.

A basic result of the Conley theory is that a $<$-compatible connection matrix exists. In fact, a connection matrix can be defined by making use of the existence of an index triple for each pair $\{m_i, m_j\} \subset M$ with $m_i < m_j$. In a bit more detail, let us write $\partial_k(m_i, m_j)$ for the homomorphism $CH_k(m_j) \to CH_{k-1}(m_i)$ of level k in c_i^j $(k \geq 1)$. Then the $\partial_k(m_i, m_j)$ are boundary operators in a long exact homology sequence defined by the index triple. If there are no connecting orbits joining m_i and m_j (that is, if there is no $x \in S_\varepsilon \setminus M$ satisfying $\omega(x) = m_i$ and $\alpha(x) = m_j$) then $\partial_k(m_i, m_j) = 0$ for all $k \geq 1$. See ([13], [31]).

It follows from the above discussion that a connection matrix compatible with the order $<$ is

$$c = \begin{pmatrix} 0 & 0 & \partial_1(A_1, S_2) & \partial_1(A_1, S_1) & 0 \\ 0 & 0 & \partial_1(A_2, S_2) & \partial_1(A_2, S_1) & 0 \\ 0 & 0 & 0 & 0 & \partial_2(S_2, R_0) \\ 0 & 0 & 0 & 0 & \partial_2(S_1, R_0) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is possible to compute all the homomorphisms in c. We indicate how this can be done; again we make free use of results concerning the Conley index.

In order to compute $\partial_1(a_1, s_2)$ we consider the exact sequence

$$CH_1(a_1 s_2) \longrightarrow CH_1(s_2) \xrightarrow{\partial_1(a_1, s_2)} CH_0(a_1). \tag{4.5}$$

Here by $a_1 s_2$ we mean the union of $\{a_1\}$, of $\{s_2\}$, and all connecting orbits joining these points. Since $CH_1(a_1 s_2) = 0$, (4.5) becomes

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{\partial_1(a_1, s_2)} \mathbb{Z}_2$$

which implies that $\partial_1(a_1, s_2)$ is injective, thus equal to the homomorphism 1 (recall that we are dealing with maps $\mathbb{Z}_2 \to \mathbb{Z}_2$). The same argument can be used to compute $\partial_1(a_2, s_2)$, $\partial_1(a_1, s_1)$, and $\partial_1(a_2, s_1)$. In order to compute $\partial_1(s_2, r_0)$ we consider the exact sequence

$$CH_2(r_0) \xrightarrow{\partial_1(s_2, r_0)} CH_1(s_2) \longrightarrow CH_1(s_2 r_0) \tag{4.6}$$

where $s_2 r_0$ indicates the union of s_2, r_0, and all connecting orbits joining these points. Since $\partial_1(s_2, r_0) = 0$, (4.6) becomes

$$\mathbb{Z}_2 \xrightarrow{\partial_2(s_2, r_0)} \mathbb{Z}_2 \longrightarrow 0$$

which implies that $\partial_2(s_2, r_0)$ is onto, thus equal to 1. The same argument can be used to compute $\partial_1(s_1, r_0)$. In conclusion, a connection matrix c which is compatible with the order $<$ is

$$c = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The constructions above actually show that c is the unique connection matrix which is compatible with the order $<$. This is of interest because connection matrices are not in general uniquely defined. It is also important in our study of (4.1) for $\rho \neq 0$.

Next, let $\varepsilon \in I'$. Let $m \in M$, and let (N_m, L_m) be an index pair relative to the flow on $\mathbb{R}^2$ defined by (4.2). Let $\mathcal{U}_0$ be a weak-* compact, convex, translation invariant subset of $\mathcal{U}$. Then $(\mathcal{U}_0 \times N_m, \mathcal{U}_0 \times L_m)$ is an index pair for the isolated invariant set $\mathcal{U}_0 \times \{m\} \subset \mathcal{U}_0 \times \mathbb{R}^2$ relative to the product flow on $\mathcal{U}_0 \times \mathbb{R}^2$ defined by $\{\tau_t\}$ and (4.2). We claim that the Conley index $h(\mathcal{U}_0 \times \{m\})$ equals $h(\{m\})$. To prove this, we must exhibit a homotopy equivalence between the pointed topological spaces N_m/L_m and $(\mathcal{U}_0 \times N_m)/(\mathcal{U}_0 \times L_m)$. So fix $\bar{u} \in \mathcal{U}_0$ and define $i: N_m \to \mathcal{U}_0 \times N_m : x \mapsto (\bar{u}, x)$. Further define $j: \mathcal{U}_0 \times N_m \to N_m : (u,x) \mapsto x$. Then $i \circ j(u,x) = (\bar{u}, x)$, so $i \circ j$ is homotopic to the identity on $(\mathcal{U}_0 \times N_m)/(\mathcal{U}_0 \times L_m)$ via the homotopy

$$\theta_s : \mathcal{U}_0 \times N_m \to \mathcal{U}_0 \times N_m, \quad \theta_s(u,x) = (s\bar{u} + (1-s)u, x).$$

Moreover, $j \circ i$ equals the identity on N_m/L_m. This proves that $h(\mathcal{U}_0 \times \{m\}) = h(\{m\})$.

Of course one can take $\mathcal{U}_0 = \mathcal{U}$ in the above discussion. However, it is more interesting in the present context to consider weak-* compact, convex, translation-invariant subsets $\mathcal{U}_0 \subset \mathcal{U}$ which do not contain $u = 0$. We will return to this point below. Such sets are easily obtained. For example, let $u_0 \in \mathcal{U}$ and define

$$\mathcal{U}_0 = \operatorname{cl}\left\{ \sum_{i=1}^{N} s_i \tau_{t_i}(u_0) \;\middle|\; s_i \geq 0,\ \sum_{i=1}^{N} s_i = 1,\ t_i \in \mathbb{R},\ N \geq 1 \right\}.$$

One can certainly choose u_0 such that $0 \notin \mathcal{U}_0$.

We now apply the continuation theory of the Conley index for small $\rho \neq 0$ ([13], [14], [27]). Let $\mathcal{U}_0$ be a weak-* compact, translation-invariant subset of $\mathcal{U}$. We abuse notation and write $S_{\varepsilon,\rho}$ instead of $S_{\varepsilon,\rho} \cap (\mathcal{U}_0 \times \mathbb{R}^2)$. First of all, there are isolated invariant sets $a_1(\rho)$, $a_2(\rho)$, $s_1(\rho)$, $s_2(\rho)$ which are near a_1, a_2, s_1, s_2 in the following sense: if e.g. N_{a_1} is an isolating neighbourhood for a_1, then for small enough ρ, $\mathcal{U}_0 \times N_{a_1}$ is an isolating neighbourhood for $a_1(\rho)$. Further, using Proposition 4.1 and the definition of $S_{\varepsilon,\rho}$, one sees that, if $\rho \in [-\rho', \rho']$, then $\mathcal{U}_0 \times B_R$ is an isolating neighbourhood of $S_{\varepsilon,\rho}$. Write $r_0(\rho) = \mathcal{U}_0 \times \{0\} \subset \mathcal{U}_0 \times \mathbb{R}^2$ for $\rho \neq 0$. Using ([13, Proposition 4.11]), one sees that the family $M(\rho) = \{r_0(\rho), a_1(\rho), a_2(\rho), s_1(\rho), s_2(\rho)\}$ of compact invariant subsets of $S_{\varepsilon,\rho}$ is a Morse decomposition of $S_{\varepsilon,\rho}$ for sufficiently small $\rho \neq 0$. Moreover, an admissible order for this Morse decomposition is given by

$$a_1(\rho) < a_2(\rho) < s_2(\rho) < s_1(\rho) < r_0(\rho).$$

Still more, by decreasing ρ' if necessary, we can assume that the above conditions hold for all $\varepsilon \in I'$, $\rho \in [-\rho', \rho']$.

We now determine a connection matrix for $M(\rho)$ which is compatible with the order $<$. For this, we need only apply Theorem 5.7 of [13] together with the uniqueness of the $<$-compatible connection matrix c for M. We see that, if $\varepsilon \in I'$ is fixed and $\rho \neq 0$ is small:

$$c_\rho = c = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is a $<$-compatible connection matrix for $M(\rho)$. Of course the equality "$c_\rho = c$" is formal since the entries of c_ρ indicate homomorphisms between graded modules which are, generally speaking, different from those used to define c. Decreasing ρ' if necessary, we can assume that, if $\varepsilon \in I'$, then c_ρ is a $<$-compatible connection matrix for all $\rho \in [-\rho', \rho']$.

The fact that c_ρ is a $<$-compatible connection matrix for $M(\rho)$ implies that there are connecting orbits in $\mathcal{U}_0 \times \mathbb{R}^2$ joining:

(i) $a_1(\rho)$ and $s_2(\rho)$;
(ii) $a_1(\rho)$ and $s_1(\rho)$;
(iii) $a_2(\rho)$ and $s_2(\rho)$;
(iv) $a_2(\rho)$ and $s_1(\rho)$;
(v) $s_1(\rho)$ and $r_0(\rho)$;
(vi) $s_2(\rho)$ and $r_0(\rho)$.

In particular, if $\mathcal{U}_0$ is the compact convex hull of the orbit of any fixed $u_0 \in \mathcal{U}$, then such connecting orbits exist in $\mathcal{U}_0 \times \mathbb{R}^2$. Note however, that the existence of orbits joining $r_0(\rho)$ with $a_i(\rho)$ cannot be concluded ($i = 1, 2$).

We can use the information contained above to discuss the existence of control sets for values $\varepsilon \in I'$, $\rho \in [-\rho', \rho']$. In fact, there are control sets D_1, D_2, D_3, D_4 whose existence is due to the fact that there are connecting orbits in the control flow $\mathcal{U} \times \mathbb{R}^2$ defined by (4.1) with ω-limit sets contained in $s_1(\rho)$, $s_2(\rho)$, $a_1(\rho)$, $a_2(\rho)$. Here we have used Theorem 2.6. It is not clear if these control sets are all distinct. It is also not clear if the closure of any of these control sets contains $x = 0$.

This completes our discussion of the case $\varepsilon > \varepsilon_2$. We now give a brief summary of what we have proved.

First, if $\rho \neq 0$ is small, the control flow defined by (4.1) may experience a bifurcation of transcritical type as ε passes through ε_1. In this case, control sets appear. Moreover, if $\varepsilon_* \in (\varepsilon_1, \varepsilon_2)$ is a point where the unperturbed system (4.2) admits a homoclinic orbit, then the control flow may exhibit transversal homoclinic points. In this case also, control sets are present.

Second, if $\rho \neq 0$ is small, then under mild assumptions, for some values $\varepsilon > \varepsilon_2$ the control flow admits an attractor $A_{\varepsilon,\rho}$ which is a "random invariant circle". Moreover, the Conley index can be used to study the Morse-type structure of the isolated invariant set $S_{\varepsilon,\rho}$. Again the existence of control sets can be deduced.

We end the paper by mentioning the relation to results of Ludwig Arnold [3] and others ([32], [33]) for the Duffing–van der Pol oscillator. Although the unperturbed second order equation of the oscillator does not exhibit a phase portrait like that discussed above, it does seem that the randomly perturbed oscillator has properties which can be understood in terms of a two-step bifurcation pattern.

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