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MULTIPLICITY OF SOLUTIONS FOR ASYMPTOTICALLY LINEAR n-TH ORDER BOUNDARY VALUE PROBLEMS

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Abstract. In this paper we investigate existence and multiplicity of solutions, with prescribed nodal properties, to a two-point boundary value problem of asymptotically linear n-th order equations. The proof follows a shooting approach and it is based on the weighted eigenvalue theory for linear n-th order boundary value problems.

1. Introduction

This paper is devoted to the study of the existence and multiplicity of solutions, characterized by prescribed nodal properties, to an asymptotically linear n-th order equation satisfying two-point boundary conditions. In particular, we are interested in the study of a problem of the form

$$
\begin{cases}
u^{(n)}(t) + f(t, u(t))\, u(t) = 0, \\
u^{(r)}(0) = 0, \quad r \in \{i_2, \ldots, i_{n-1}\}, \\
u(\pi) = u'(\pi) = 0,
\end{cases}
\tag{1.1}
$$

where $\{i_2, \ldots, i_{n-1}\}$ is a fixed set of distinct integers contained in $\{0, \ldots, n-1\}$, $n \geq 3$ and $f\colon [0,\pi] \times \mathbb{R} \to (-\infty, 0)$ is a continuous function.

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We require our problem to be asymptotically linear in a neighbourhood of the origin and at infinity, by assuming that there exist two continuous maps $a_0, a_\infty\colon [0,\pi] \to (-\infty, 0]$ which do not vanish identically on any interval in $[0,\pi]$ and satisfy

$$\lim_{|x|\to\infty} f(t,x) = a_\infty(t), \tag{1.2}$$

$$\lim_{x\to 0} f(t,x) = a_0(t), \tag{1.3}$$

uniformly in $t \in [0,\pi]$.

The second order, asymptotically linear case (obtained by setting $n = 2$ in our equation and by replacing the two-point boundary conditions of (1.1) with Dirichlet boundary conditions) has been widely investigated in the literature. In this particular framework, multiplicity results have been recently achieved under more general assumptions (for instance, less regularity is required on the nonlinearities, the sign conditions are relaxed, so that an indefinite asymptotic behaviour is allowed, and asymmetric asymptotic situations are analyzed). Among the various reference concerning this topic, we wish to quote the paper [32] by Sadyrbaev, the interesting contribution [30] by Rynne, handling with superlinear nonlinearities near $+\infty$, the more recent work [6] and its asymmetric generalization [4] (cf. also references therein).

The aim of this paper consists in extending to the n-th order case some of the well-known results established for second order problems under asymptotically linear assumptions. We are also interested in describing some continuous dependence properties for weighted eigenvalues and eigenfunctions of linear n-th order boundary value problems (cf. Propositions 2.2 and 2.4) and in developing a detailed analysis of the behaviour of the nonlinearity when it is applied to the solutions of prescribed Cauchy problems associated to (1.1) with suitably large and small initial data (cf. Propositions 2.6 and 2.7). Our multiplicity result follows by using a shooting approach which combines these properties with degree theory.

There also exists an extensive literature concerning existence and multiplicity of solutions for fourth order asymptotically linear problems. An interesting contribution in this setting has been provided by Henrard–Sadyrbaev in [16] where an asymmetric situation is treated. In [16] multiplicity is achieved by combining topological techniques with the conjugate point theory, provided that the nonlinearity satisfies some monotonicity conditions. Among many other multiplicity results concerning fourth order boundary value problems, we wish to mention the work [25] by Ma treating the special case of nonlinear terms of the form $f(t,x) = a(t)f(x)$. The article [25] is based on bifurcation arguments. Bifurcation techniques have been also employed in [21] and in [24] to get multiplicity of nodal solutions for fourth order, asymmetric boundary value problems.

Recently, many authors have focused their attention on the search of multiple solutions for asymptotically linear systems of second order ODE's. We refer to the papers [3], [5] and [9] for some contribution in this direction.

The next references we wish to quote rely on general order problems. The list of multiplicity results for n-th order problems available in the literature is shorter. Existence of infinitely many solutions has been established for superlinear problems by De Coster–Gaudenzi in [7], where a shooting method is adopted, and by Rynne in the paper [31], based on a bifurcation theory approach. In [7] the authors impose $(n-1)$ conditions at the initial point $t=0$, while in [31] it is considered an even order, self-adjoint operator.

In the asymptotically jumping context, we refer to the work [29] by Rynne, regarding existence and multiplicity of solutions of even order problems endowed with self-adjoint boundary conditions.

Interesting multiplicity results can be also found in the papers [13] and [28].

We conclude the list of references by quoting some articles providing existence of solutions (which, in many cases, are positive) for general order boundary value problems. In particular, among all the contributions in this widely investigated area, we wish to mention the works [1], [2], [12], [15], [17], [23], [27], [33] and [34] (cf. also references therein). Superlinear and sublinear problems have been mostly studied and periodic solutions are achieved in [23].

Before presenting our main result, let us recall the eigenvalue theory for n-th order linear problems developed by Elias in [11].

PROPOSITION 1.1 (cf. [11]). *For every* $a \in \mathcal{D} := \{a \in C([0,\pi];(-\infty,0]) : a \not\equiv 0$ *on any interval in* $[0,\pi]\}$, *the problem*

$$\begin{cases} u^{(n)}(t) + \lambda a(t)\, u(t) = 0, \\ u^{(r)}(0) = 0, \quad r \in \{i_2, \dots, i_{n-1}\}, \\ u(\pi) = u'(\pi) = 0, \end{cases} \tag{1.4}$$

admits a positive monotone increasing sequence of eigenvalues

$$0 < \lambda_1(a) < \lambda_2(a) < \dots < \lambda_j(a) \to \infty \quad \text{as } j \to \infty.$$

To every eigenvalue there corresponds an essentially unique eigenfunction and the eigenfunction corresponding to $\lambda_j(a)$ *has exactly* $(j-1)$ *zeros on* $(0,\pi)$.

Let us finally introduce one more notation, denoting by i_0 and i_1 the two different integers belonging to the set $\{0,\dots,n-1\} \setminus \{i_2,\dots,i_{n-1}\}$.

We can now state our main multiplicity result.

THEOREM 1.2. *Let* $f\colon [0,\pi] \times \mathbb{R} \to (-\infty,0)$ *be a continuous function satisfying* (1.2) *and* (1.3). *Assume that there exist* N, $M \in \mathbb{N}$ $(M \le N)$ *such that*

$$\text{either} \quad \lambda_N(a_0) < 1 < \lambda_M(a_\infty) \quad \text{or} \quad \lambda_N(a_\infty) < 1 < \lambda_M(a_0). \tag{1.5}$$

Then, for every $h \in \mathbb{N}$ with $M \le h \le N$ problem (1.1) *has at least two solutions u_h and v_h with $u_h^{(i_0)}(0) < 0$ and $v_h^{(i_0)}(0) > 0$ having exactly $(h-1)$ zeros in $(0,\pi)$.*

In order to prove our main result we first use degree theory to find a planar closed connected set chatacterized by properties involving related weighted eigenvalues. Then, a shooting argument developed on a suitable angular function will lead to the required multiplicity result. We point out that a similar procedure has been followed in [5] to achieve existence of multiple solutions for planar systems of second order equations.

We conclude this introductory section with some notation. Denote by $\|\cdot\|_{n-1}$ the $C^{n-1}([0,\pi])$-norm. We set $\mathcal{R} := \{(x,y) : y > 0\} \cup \{(x,0) : x < 0\}$, $\mathcal{Q}_2 := (-\infty,0) \times (0,\infty)$ and $\mathcal{Q}_4 := (0,\infty) \times (-\infty,0)$.

2. Preliminary results

The first part of this section is devoted to present some results concerning the continuity of eigenvalues and eigenfunctions of problem (1.4). We initially recall a classical theorem regarding the continuous dependence of the solutions of n-th order linear ODEs on the data.

Proposition 2.1 (cf. [35]). *Consider $a_0 \in \mathcal{D}$, $\lambda_0 > 0$, $t_0 \in [0,\pi]$ and $\overrightarrow{x}_0 \in \mathbb{R}^n$. Let $u(\,\cdot\,; a_0, \lambda_0, t_0, \overrightarrow{x}_0)$ be the solution of the Cauchy problem*

$$\begin{cases} u^{(n)}(t) + \lambda_0 a_0(t) u(t) = 0, \\ (u(t_0), u'(t_0), \dots, u^{(n-1)}(t_0)) = \overrightarrow{x}_0. \end{cases}$$

For any fixed $\varepsilon > 0$, there exists $\delta > 0$ such that for every $a \in \mathcal{D}$, $\lambda > 0$, $t \in [0,\pi]$ and $\vec{x} \in \mathbb{R}^n$ satisfying $\|a - a_0\|_{L^1} + |\lambda - \lambda_0| + |t - t_0| + |\overrightarrow{x} - \overrightarrow{x}_0| < \delta$, then

$$\|u(\,\cdot\,; a, \lambda, t, \overrightarrow{x}) - u(\,\cdot\,; a_0, \lambda_0, t_0, \overrightarrow{x}_0)\|_{n-1} < \varepsilon.$$

Recalling that 0 represents a lower bound for each eigenvalue, taking into account the above stated Proposition 2.1 and following the same arguments developed by Elias in [11] to prove Corollary 5, we can deduce a continuity result for the eigenvalues.

Proposition 2.2. *For each $j \in \mathbb{N}$, the j-th eigenvalue λ_j depends continuously on the coefficients $a \in \mathcal{D}$ whenever the set $\mathcal{D}$ is endowed with the $L^1([0,\pi])$-norm.*

An analogous result for the case of second order equations can be found in [19].

We are now interested in proving that also the eigenfunctions of problem (1.4) depend continuously on the coefficient a. Proposition 1.1 guarantees the essential unicity of the j-th eigenfunction. By the linearity of the problem, it is not restrictive to define the j-th eigenfunction $\psi_j(\,\cdot\,; a)$ corresponding to problem (1.4) as the *unique* solution of problem (1.4) with $\lambda = \lambda_j(a)$ satisfying $(\psi_j^{(i_0)}(0;a), \psi_j^{(i_1)}(0;a)) \in S^1 \cap \mathcal{R}$. To prove the continuity of $\psi_j(\,\cdot\,;a)$, we need the following remark.

Remark 2.3. Problem (1.1) does not admit nontrivial solutions u satisfying $u^{(i_0)}(0)u^{(i_1)}(0) \geq 0$. This assertion follows from the fact that $f(t,x) < 0$ for every $(t,x) \in [0,\pi] \times \mathbb{R}$. The same observation holds true for the solutions of problem (1.4).

According to this note, we point out that $(\psi_j^{(i_0)}(0;a), \psi_j^{(i_1)}(0;a)) \in S^1 \cap \mathcal{Q}_2$.

Proposition 2.4. *For each $j \in \mathbb{N}$, the function*

$$(\mathcal{D}, \|\cdot\|_{L^1}) \to (C^{n-1}([0,\pi];\mathbb{R}), \|\cdot\|_{n-1}), \qquad a \mapsto \psi_j(\,\cdot\,;a)$$

is continuous.

For similar results we refer to [20] dealing with in the second order case and to [18] concerned with n-th order BVPs.

As an immediate consequence of Proposition 2.4, we observe that $\psi_j(\,\cdot\,;a)$ depends continuously on a when $\mathcal{D}$ is endowed with the $\|\cdot\|_\infty$-norm.

Proof. Fix $j \in \mathbb{N}$. Assume, by contradiction, that there exist $a_0 \in \mathcal{D}$ and $\varepsilon > 0$ such that for every $h \in \mathbb{N}$ there exists $a_h \in \mathcal{D}$ satisfying the following inequalities

$$\|a_h - a_0\|_{L^1} < \frac{1}{h}, \qquad \|\psi_j(\,\cdot\,;a_h) - \psi_j(\,\cdot\,;a_0)\|_{n-1} \geq \varepsilon. \tag{2.1}$$

In particular, for each $h \in \mathbb{N} \cup \{0\}$, the j-th eigenfunction $\psi_j(\,\cdot\,;a_h)$ solves the problem

$$\begin{cases} u^{(n)}(t) + \lambda_j(a_h)a_h(t)\,u(t) = 0, \\ u^{(r)}(0) = 0, \quad r \in \{i_2, \ldots, i_{n-1}\}, \quad (u^{(i_0)}(0), u^{(i_1)}(0)) = \beta_h, \\ u(\pi) = u'(\pi) = 0, \end{cases}$$

where, according to Remark 2.3, $\beta_h \in S^1 \cap \mathcal{Q}_2$. Being β_h bounded, it admits a subsequence (still denoted by β_h) converging to $\gamma \in S^1 \cap \overline{\mathcal{Q}_2}$ as h tends to ∞. Moreover, combining Proposition 2.2 with the first inequality in (2.1), we conclude that $\lim_{h\to\infty} \lambda_j(a_h) = \lambda_j(a_0)$. Let us denote by w_0 the solution of

$$\begin{cases} u^{(n)}(t) + \lambda_j(a_0)\,a_0(t)\,u(t) = 0, \\ u^{(r)}(0) = 0, \quad r \in \{i_2, \ldots, i_{n-1}\}, \\ (u^{(i_0)}(0), u^{(i_1)}(0)) = \gamma \in S^1 \cap \overline{\mathcal{Q}_2} \subset S^1 \cap \mathcal{R}. \end{cases}$$

Thus, Proposition 2.1 guarantees that

$$\lim_{h\to\infty} \|\psi_j(\,\cdot\,;a_h) - w_0(\,\cdot\,)\|_{n-1} = 0. \tag{2.2}$$

Since $\psi_j(\,\cdot\,;a_h)$ is an eigenfunction for each $h \in \mathbb{N}$, we immediately obtain that $w_0(\pi) = w_0'(\pi) = 0$. Recalling the definitions of $\psi_j(\,\cdot\,;a_0)$ and w_0, we note that $w_0(\,\cdot\,) \equiv \psi_j(\,\cdot\,;a_0)$ and, consequently, (2.2) contradicts (2.1). $\square$

Let us now recall some properties of the solutions to linear n-th order boundary value problems, attained by Elias in [10] and [11]. Observe that in the following lemma only $n-1$ boundary conditions are considered.

Lemma 2.5 (cf. [11]). *For every $a \in \mathcal{D}$ and for every two arbitrary sets of indices $\{j_1,\dots,j_k\}$, $\{l_1,\dots,l_{n-k-1}\}$ in $\{0,\dots,n-1\}$, the problem*

$$
\begin{cases}
u^{(n)}(t) + a(t)u(t) = 0, \\
u^{(r)}(0) = 0, \quad r \in \{j_1,\dots,j_k\}, \\
u^{(r)}(\pi) = 0, \quad r \in \{l_1,\dots,l_{n-k-1}\},
\end{cases}
\tag{2.3}
$$

admits an essentially unique solution u. Such a solution and its derivatives $u^{(r)}$, with $r \in \{1,\dots,n-1\}$, may have only simple zeros in $(0,\pi)$ and each $u^{(r)}$ has exactly one (simple) zero between two consecutive zeros of $u^{(r-1)}$ in $[0,\pi]$. Moreover, at most one of $u, u', \dots, u^{(n-1)}$ can have a zero at most at one of the two endpoints 0, π in addition to the zeros posed in (2.3).

In the second part of this section we focus our attention on more general problems.

In particular, for every $i \in \{1,\dots,n\}$, we consider the function $q_i \in L^p([a,b])$, where $p \geq 1$ and $a, b \in \mathbb{R}$ with $a < b$. We denote by D the n-th order differential operator defined by

$$
Du(t) := u^{(n)}(t) + \sum_{i=1}^{n} q_i(t)\, u^{(n-i)}(t),
$$

whenever $u \in \mathcal{F} := \{u \in C^{n-1}([a,b]) :$ there exists $u^{(n)}(t)$ for a.e. $t \in [a,b]\}$. Given $\alpha \in \mathbb{R}^n$, $t_0 \in [a,b]$ and a function $f\colon [a,b] \times \mathbb{R} \to \mathbb{R}$ satisfying the Carathéodory conditions, we denote by u_α a solution of the initial value problem

$$
\begin{cases}
Du(t) + f(t,u(t))u(t) = 0, \\
(u(t_0),\dots,u^{(n-1)}(t_0)) = \alpha.
\end{cases}
\tag{2.4}
$$

Taking into account the asymptotic assumptions (1.2) and (1.3), we determinate a relation between the behaviour of the nonlinearity f applied to the solutions of the Cauchy problems (2.4) and the corresponding initial data, when the data are large enough and sufficiently small. Since these relations might be considered of independent interest, we prefer to present them in this more general context.

Proposition 2.6. *Consider two maps $a_\infty, \mu \in L^p([a,b])$ and a Carathéodory function $f\colon [a,b]\times\mathbb{R}\to\mathbb{R}$ satisfying* (1.2) *uniformly a.e. in $t\in[a,b]$ and such that*

$$|f(t,x)| \le \mu(t) \quad \text{for all } x\in\mathbb{R}, \text{ for a.e. } t\in[a,b]. \tag{2.5}$$

Then, $f(\,\cdot\,,u_\alpha(\,\cdot\,)) \to a_\infty(\,\cdot\,)$ in $L^p([a,b])$ if $|\alpha|\to\infty$.

Analogous results have been obtained in [3] for systems of second order equations.

Proof. The proof is inspired by some techniques adopted in [14] and follows a procedure analogous to the one developed in the proof of Proposition 4.4 in [3].

Consider a sequence $\alpha_k\in\mathbb{R}^n$ with $|\alpha_k|\to\infty$ as $k\to\infty$. Our aim consists in showing that $f(\,\cdot\,,u_{\alpha_k}(\,\cdot\,))\to a_\infty(\,\cdot\,)$ in $L^p([a,b])$ as $k\to\infty$. To this purpose, we prove that every subsequence of α_k (still called α_k to simplify the notation) admits a subsequence α_{l_k} such that

$$\lim_{k\to\infty} \|f(\,\cdot\,,u_{\alpha_{l_k}}(\,\cdot\,)) - a_\infty(\,\cdot\,)\|_{L^p} = 0. \tag{2.6}$$

Let us now define

$$\begin{aligned} \nu_k &:= \frac{\alpha_k}{|\alpha_k|}\in S^{n-1},\\ \xi_k &:= \left(\frac{u_{\alpha_k}}{|\alpha_k|}, \frac{u'_{\alpha_k}}{|\alpha_k|},\dots,\frac{u^{(n-1)}_{\alpha_k}}{|\alpha_k|}\right)\in C([a,b],\mathbb{R}^n). \end{aligned} \tag{2.7}$$

In particular,

$$\xi_k(t) = \nu_k + \int_{t_0}^t A(\tau,u_{\alpha_k}(\tau))\xi_k(\tau)\,d\tau \quad \text{for all } t\in[a,b], \tag{2.8}$$

where for every $(t,x)\in[a,b]\times\mathbb{R}$ we denote by $A(t,x)$ the $n\times n$ matrix defined by

$$\begin{aligned} A(t,x)_{lm} &= A(t)_{lm} := \delta_{(l+1),m} && \text{if } (l,m)\in\{1,\dots,n-1\}\times\{1,\dots,n\},\\ A(t,x)_{nj} &= A(t)_{nj} := -q_{n+1-j}(t) && \text{if } j\in\{2,\dots,n\},\\ A(t,x)_{n1} &:= -f(t,x) - q_n(t). \end{aligned}$$

Let us now introduce the function

$$\varphi(t) := \max\{1,\mu(t)\} + \max_{1\le i\le n}\{|q_i(t)|\}\in L^p([a,b]).$$

Taking into account (2.5), it is easy to verify that

$$\|A(t,x)\| \le \varphi(t) \quad \text{for all } x\in\mathbb{R} \text{ and a.e. } t\in[a,b]. \tag{2.9}$$

We claim that the sequence ξ_k admits a subsequence converging uniformly to a function $\xi\in C([a,b],\mathbb{R}^n)$.

As a consequence of inequalities (2.9), we obtain, for all $t \in [a,b]$,

$$|\xi_k(t)| \le 1 + \left| \int_{t_0}^t \|A(\tau, u_{\alpha_k}(\tau))\| \, |\xi_k(\tau)| \, d\tau \right| \le 1 + \left| \int_{t_0}^t \varphi(\tau) \, |\xi_k(\tau)| \, d\tau \right|.$$

By applying Gronwall Lemma, it turns out that

$$|\xi_k(t)| \le 1 + e^{| \int_{t_0}^t \varphi(\tau) \, d\tau |} \le 1 + e^{\int_a^b \varphi(\tau) \, d\tau} =: C \tag{2.10}$$

for all $t \in [a,b]$, which proves the uniform boundedness of the sequence ξ_k.

Let us now consider an arbitrary $\varepsilon > 0$. Since $\varphi \in L^p([a,b]) \subset L^1([a,b])$, we note that there exists $\delta_\varepsilon > 0$ such that for every interval $I \subset [a,b]$ with $\ell(I) < \delta_\varepsilon$ we have $\int_I \varphi(\tau) \, d\tau < (\varepsilon/C)$. Taking into account (2.9) and (2.10), we infer that for each $t_1, t_2 \in [a,b]$ with $|t_1 - t_2| < \delta_\varepsilon$

$$|\xi_k(t_1) - \xi_k(t_2)| \le \left| \int_{t_1}^{t_2} A(\tau, u_{\alpha_k}(\tau)) \xi_k(\tau) \, d\tau \right| \le C \left| \int_{t_1}^{t_2} \varphi(\tau) \, d\tau \right| < \varepsilon,$$

ensuring the equicontinuity of ξ_k. Hence, Ascoli–Arzelà Theorem guarantees the existence of a subsequence ξ_{l_k} of ξ_k converging uniformly to a function $\xi = (z_0, \dots, z_{n-1}) \in C([a,b], \mathbb{R}^n)$ as $k \to \infty$. This proves the claim.

The following step consists in showing that the number of zeros of z_0 in $[a,b]$ is finite.

According to (2.5), we can apply the Dunford–Pettis theorem to deduce the existence of a subsequence (still denoted with $f(\,\cdot\,, u_{\alpha_k}(\,\cdot\,))$) of $f(\,\cdot\,, u_{\alpha_k}(\,\cdot\,))$ such that

$$f(\,\cdot\,, u_{\alpha_k}(\,\cdot\,)) \underset{k \to \infty}{\rightharpoonup} h(\,\cdot\,) \quad \text{weakly in } L^1([a,b]).$$

Furthermore, since ν_k is bounded, it converges, up to a subsequence, to $\nu \in S^{n-1}$ as $k \to \infty$. Passing to the limit, component by component, on a suitable subsequence in (2.8), we infer that

$$\xi(t) = \nu + \int_{t_0}^t B(\tau) \xi(\tau) \, d\tau,$$

where $B(t)$ is the $n \times n$ matrix defined by $B(t)_{lm} := A(t)_{lm}$ whenever $l, m \in \{1, \dots, n\}$ with $(l,m) \ne (n,1)$ and $B(t)_{n1} := -h(t) - q_n(t)$. In particular, $z_0 \in \mathcal{F}$ and solves

$$\begin{cases} Du(t) + h(t)\, u(t) = 0 & \text{for a.e. } t \in [a,b], \\ (u(t_0), \dots, u^{(n-1)}(t_0)) = \nu. \end{cases} \tag{2.11}$$

Assume now, by contradiction, that the number of zeros of z_0 in $[a,b]$ is not finite. It is immediate to verify that there exists a limit point $t^* \in [a,b]$ of the set of all the zeros of z_0. By applying iteratively the Rolle Theorem on the derivatives of z_0, starting from the first one, we conclude that $z_0^{(j)}(t^*) = 0$ for each $j \in \{0, \dots, n-1\}$. Since the Cauchy problems associated the linear equation in (2.11) admit a unique solution, it follows that $z_0 \equiv 0$, in contradiction with the fact that $(z_0(t_0), \dots, z_0^{(n-1)}(t_0)) = \nu \in S^{n-1}$.

As claimed, z_0 has a finite number of zeros in $[a,b]$.

Denote by $\{s_1, \dots, s_m\}$ the set of the zeros of z_0 in $[a,b]$. Taking into account the definition of ξ_k in (2.7), we infer that

$$\lim_{k\to\infty} |\alpha_{l_k}| = \infty \quad \text{and} \quad \lim_{k\to\infty} \frac{u_{\alpha_{l_k}}(t)}{|\alpha_{l_k}|} = z_0(t) \quad \text{for all } t \in [a,b],$$

whence

$$\lim_{k\to\infty} |u_{\alpha_{l_k}}(\tau)| = \infty \quad \text{for all } \tau \in [a,b] \setminus \{s_1, \dots, s_m\}.$$

Recalling that assumption (1.2) ensures that $f(t,x) \to a_\infty(t)$ as $|x| \to \infty$ uniformly for every $t \in [a,b] \setminus J$, where J is a suitable set of measure zero, we obtain

$$\lim_{k\to\infty} f(t, u_{\alpha_{l_k}}(t)) = a_\infty(t) \quad \text{for all } t \in [a,b] \setminus (J \cup \{s_1, \dots, s_m\}). \tag{2.12}$$

Moreover, from assumptions (2.5) and (1.2), it immediately follows that

$$|f(t, u_{\alpha_{l_k}}(t)) - a_\infty(t)|^p \le 2^p |\mu(t)|^p \quad \text{for a.e. } t \in [a,b].$$

Taking into account that $2^p |\mu(\,\cdot\,)|^p \in L^1([a,b])$ and according to (2.12), we can apply the Lebesgue dominated convergence theorem to prove (2.6). This completes the proof. □

An equivalent result holds when suitably small initial data are considered.

Proposition 2.7. *Consider a map $a_0 \in L^p([a,b])$ and a Carathéodory function $f\colon [a,b] \times \mathbb{R} \to \mathbb{R}$ satisfying* (1.3) *uniformly a.e. in $t \in [a,b]$. Then,*

$$f(\,\cdot\,, u_\alpha(\,\cdot\,)) \to a_0(\,\cdot\,) \quad \textit{in } L^p([a,b]) \quad \textit{if } |\alpha| \to 0.$$

An analogous proposition has been obtained in [3] for systems of second order equations.

Sketch of the Proof. Our aim consists in showing that

(2.13) for all $\delta > 0$ there exists $\eta > 0$ such that

$$|u_\alpha(t)| < \delta \quad \text{for all } \alpha \in \mathbb{R}^n \text{ with } |\alpha| < \eta \text{ and all } t \in [a,b].$$

Indeed, by combining (2.13) with the assumption (1.3), the thesis easily follows.

In order to prove (2.13), let us first take an arbitrary $\delta > 0$. By the Carathéodory conditions, there exists $\mu = \mu_\delta \in L^1([a,b])$ such that

$$|f(t,x)| \le \mu(t) \quad \text{for a.e. } t \in [a,b] \text{ and all } x \in \mathbb{R} \text{ such that } |x| \le \delta. \tag{2.14}$$

Let us define

$$\varphi(t) := \max\{1, \mu(t)\} + \max_{1\le i\le n} \{|q_i(t)|\} \in L^1([a,b]) \quad \text{and} \quad \eta := \frac{\delta}{1 + e^{\int_a^b \varphi(\tau)\, d\tau}}.$$

Consider $\alpha \in \mathbb{R}^n$ with $|\alpha| < \eta$.

Since by definition $|u_\alpha(t_0)| \le |\alpha| < \delta$, we are able to define

$$a_0 := \min\{t \in [a, t_0) : |u_\alpha(s)| < \delta \text{ for all } s \in (t, t_0]\},$$
$$b_0 := \max\{t \in (t_0, b] : |u_\alpha(s)| < \delta \text{ for all } s \in [t_0, t)\}.$$

According to (2.14) and arguing as in the first steps of the proof of Proposition 2.6, we can apply the Gronwall Lemma to obtain

$$|(u_\alpha(t), u'_\alpha(t), \dots, u_\alpha^{(n-1)}(t))| \le |\alpha|(1 + e^{|\int_{t_0}^t \varphi(\tau)\,d\tau|}) < \delta \tag{2.15}$$

for all $t \in [a_0, b_0]$. In particular, $|u_\alpha(a_0)| < \delta$ and $|u_\alpha(b_0)| < \delta$ which, respectively, imply that $a_0 = a$, $b_0 = b$. Hence, from (2.15) it follows that $|u_\alpha(t)| < \delta$ for every $t \in [a, b]$, which proves (2.13). □

3. Main result

In the first part of this section we introduce an additional assumption on the statement of Theorem 1.2 and we exhibit a proof of our main result under this further condition. The final part of the section will be devoted to extend the proof to the general case and to state some remarks concerning more general situations.

Let us now restrict our attention to nonlinearities f for which *uniqueness* of the solutions of initial value problems associated to $u^{(n)}(t) + f(t, u(t))\,u(t) = 0$ is guaranteed. We are interested in handling Cauchy problems of the form

$$\begin{cases} u^{(n)}(t) + f(t, u(t))u(t) = 0, \\ u^{(r)}(0) = 0, \quad r \in \{i_2, \dots, i_{n-1}\}, \\ (u^{(i_0)}(0), u^{(i_1)}(0)) = \alpha, \end{cases} \tag{3.1}$$

with $\alpha \in \mathbb{R}^2$. According to the uniqueness assumptions, problem (3.1) admits a unique solution, which will be denoted by u_α throughout the next part of the paper.

By adding the uniqueness condition, the statement of Theorem 1.2 becomes

Theorem 3.1. *Let $f\colon [0, \pi] \times \mathbb{R} \to (-\infty, 0)$ be a continuous function satisfying* (1.2) *and* (1.3). *Assume that the solutions of $u^{(n)}(t) + f(t, u(t))u(t) = 0$ are unique with respect to the initial data and that there exist $N, M \in \mathbb{N}$ ($M \le N$) satisfying* (1.5). *Then, for every $h \in \mathbb{N}$ with $M \le h \le N$ problem* (1.1) *has at least two solutions u_h and v_h with $u_h^{(i_0)}(0) < 0$ and $v_h^{(i_0)}(0) > 0$ having exactly $(h-1)$ zeros in $(0, \pi)$.*

We wish to remark that the proof we are going to exhibit follows a procedure analogue to the one developed in [5], where planar systems of second order ODEs are studied.

Proof. Let us concentrate on the study of the case $\lambda_N(a_0) < 1 < \lambda_M(a_\infty)$. The case $\lambda_N(a_\infty) < 1 < \lambda_M(a_0)$ can be treated analogously. Fix $h \in \mathbb{N}$ with $M \leq h \leq N$. From the monotonicity of the eigenvalues, we obtain that

$$\lambda_h(a_0) < 1 < \lambda_h(a_\infty). \tag{3.2}$$

According to the continuity of the nonlinear term and to condition (1.2), we notice that

there exists $\sigma > 0$ such that $|f(t,x)| \leq \sigma$ for all $x \in \mathbb{R}$ and all $t \in [0,\pi]$.

As a consequence, the assumption (2.5) of Proposition 2.6 is satisfied. Thus, by combining Propositions 2.6 and 2.7 (restricted to the case $p = 1$) with Proposition 2.2, we infer that

$$\lim_{|\alpha|\to 0} \lambda_h(f(\,\cdot\,, u_\alpha(\,\cdot\,))) = \lambda_h(a_0) \quad \text{and} \quad \lim_{|\alpha|\to\infty} \lambda_h(f(\,\cdot\,, u_\alpha(\,\cdot\,))) = \lambda_h(a_\infty). \tag{3.3}$$

From (3.2) and (3.3), we attain the existence of two positive constants R_1, R_2 with $R_1 < R_2$ such that $\lambda_h(f(\,\cdot\,, u_\alpha(\,\cdot\,))) < 1$ for every $\alpha \in \mathbb{R}^2$ with $|\alpha| = R_1$ and $\lambda_h(f(\,\cdot\,, u_\alpha(\,\cdot\,))) > 1$ for every $\alpha \in \mathbb{R}^2$ with $|\alpha| = R_2$.

Let us now introduce the map $F\colon \mathbb{R}^2 \to \mathcal{D}$ by setting $F(\alpha)(\,\cdot\,) := f(\,\cdot\,, u_\alpha(\,\cdot\,))$. This map is well defined, due to the continuity of f. By the continuous dependence from initial data, it follows that F is continuous, whenever $\mathcal{D} \subset C([0,\pi])$ is endowed with the $\|\cdot\|_\infty$-norm.

Our next aim consists in proving the existence of a solution u to problem (1.1) with $(u^{(i_0)}(0), u^{(i_1)}(0)) \in \mathcal{Q}_2$, having exactly $(h-1)$ zeros in $(0,\pi)$. This occurs if there exist $\alpha_* \in \mathcal{Q}_2$ and a positive constant K such that

$$K u_{\alpha_*}(\,\cdot\,) \equiv \psi_h(\,\cdot\,; F(\alpha_*)), \tag{3.4}$$

where $\psi_h(\,\cdot\,; F(\alpha_*))$ represents the eigenfunction corresponding to $\lambda_h(F(\alpha_*)) = \lambda_h(f(\,\cdot\,, u_{\alpha_*}(\,\cdot\,)))$. In this case, u_{α_*} is the required solution.

To achieve our goal, we initially provide a closed connected set $\mathcal{C} \subset \overline{\mathcal{Q}_2} \setminus \{(0,0)\}$ such that $\lambda_h(F(\alpha)) = 1$ for every $\alpha \in \mathcal{C}$. Finally, by using a shooting approach, we find $\alpha_* \in \mathcal{C} \cap \mathcal{Q}_2$ and $K > 0$ satisfying

$$(\psi_h^{(i_0)}(0; F(\alpha_*)), \psi_h^{(i_1)}(0; F(\alpha_*))) = K\,\alpha_*.$$

Define the map $g\colon [R_1, R_2] \times [\pi/2, \pi] \to \mathbb{R}$ by setting

$$g(\rho, \vartheta) := \lambda_h(F(\rho\cos\vartheta, \rho\sin\vartheta)) - 1.$$

Note that the function g satisfies $g(R_1, \vartheta) < 0 < g(R_2, \vartheta)$ for every $\vartheta \in [\pi/2, \pi]$ and it is continuous by Proposition 2.2. Moreover, denoting by deg the Brower degree, we can deduce that $\deg(g(\,\cdot\,, (\pi/2)), (R_1, R_2)\,, 0) \neq 0$. Hence, we can apply the Leray–Schauder continuation theorem (cf. [22, Théorème Fondamental] and [26]), which ensures the existence of a closed connected set $\mathcal{C}^* \subset \{(\rho, \vartheta) \in$

$(R_1, R_2) \times [(\pi/2), \pi] : g(\rho, \vartheta) = 0\}$ such that $\mathcal{C}^* \cap ((R_1, R_2) \times \{(\pi/2)\}) \neq \emptyset$ and $\mathcal{C}^* \cap ((R_1, R_2) \times \{\pi\}) \neq \emptyset$. Reformulating this in cartesian coordinates, we can assert that there exist two constants $a, b \in \mathbb{R}$ with $b < 0 < a$ and a closed connected set $\mathcal{C} \subset \overline{\mathcal{Q}_2} \cap \{\alpha \in \mathbb{R}^2 : R_1 < |\alpha| < R_2\}$ such that $(0, a), (b, 0) \in \mathcal{C}$ and

$$\lambda_h(f(\,\cdot\,, u_\alpha(\,\cdot\,))) = \lambda_h(F(\alpha)) = 1 \quad \text{for all } \alpha \in \mathcal{C}.$$

It remains to prove the existence of $\alpha_* \in \mathcal{C} \cap \mathcal{Q}_2$ satisfying (3.4). To this aim, we set

$$\text{(3.5)} \quad \widetilde{\alpha} := (0, a), \quad \widehat{\alpha} := (b, 0) \quad \text{and} \quad \beta(\alpha) := (\psi_h^{(i_0)}(0; F(\alpha)), \psi_h^{(i_1)}(0; F(\alpha)))$$

for all $\alpha \in \mathbb{R}^2$. Furthermore, given $\gamma = (\gamma_1, \gamma_2) \in \overline{\mathcal{Q}_2} \setminus \{(0,0)\}$, let us consider its polar coordinates $(\vartheta(\gamma), \rho(\gamma)) \in [(\pi/2), \pi] \times (0, \infty)$, defined by $\gamma_1 = \rho(\gamma) \cos\vartheta(\gamma)$, $\gamma_2 = \rho(\gamma) \sin\vartheta(\gamma)$.

According to Remark 2.3 and to the definitions (3.5), we can deduce that $\beta(\alpha) \in S^1 \cap \mathcal{Q}_2$,

$$\text{(3.6)} \quad \vartheta(\beta(\widetilde{\alpha})) - \vartheta(\widetilde{\alpha}) = \vartheta(\beta(\widetilde{\alpha})) - \frac{\pi}{2} > 0, \quad \vartheta(\beta(\widehat{\alpha})) - \vartheta(\widehat{\alpha}) = \vartheta(\beta(\widehat{\alpha})) - \pi < 0.$$

If we combine the continuity of the map F with Proposition 2.4, we infer that $\psi_h(\,\cdot\,; F(\alpha))$ and, consequently, $\beta(\alpha)$ depend continuously on $\alpha \in \mathbb{R}^2$. Therefore, the function

$$g: \mathcal{C} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \qquad \alpha \mapsto \vartheta(\beta(\alpha)) - \vartheta(\alpha)$$

is continuous as well. Hence, recalling that $\mathcal{C}$ is a connected set, from (3.6) it follows the existence of $\alpha_* \in \mathcal{C}$ such that $\vartheta(\beta(\alpha_*)) = \vartheta(\alpha_*)$. In particular, $\alpha_* \in \mathcal{C} \cap \mathcal{Q}_2$ and $\beta(\alpha_*) = K\,\alpha_*$ for a suitable positive constant K. Thus, since both $\psi_h(\,\cdot\,; F(\alpha_*))$ and $K u_{\alpha_*}$ satisfy the linear problem

$$\begin{cases} u^{(n)} + f(t, u_{\alpha_*}(t)) u(t) = 0, \\ u^{(r)}(0) = 0, \quad r \in \{i_2, \dots, i_{n-1}\}, \\ (u^{(i_0)}(0), u^{(i_1)}(0)) = \beta(\alpha_*), \end{cases}$$

we obtain the claimed (3.4), which ensures that u_{α_*} solves problem (1.1) and has exactly $(h-1)$ zeros in $(0, \pi)$.

Let us recall that, from Remark 2.3, every solution u to problem (1.1) satisfies the initial condition $(u^{(i_0)}(0), u^{(i_1)}(0)) \in \mathcal{Q}_2 \cup \mathcal{Q}_4$. To complete the proof, it remains to prove that for every $h \in \mathbb{N}$ with $M \leq h \leq N$ there exists a second solution v to (1.1) having exactly $(h-1)$ zeros in $(0, \pi)$ and satisfying $(v^{(i_0)}(0), v^{(i_1)}(0)) \in \mathcal{Q}_4$.

Proceeding exactly as in the previous steps, we easily find a closed connected set $\widetilde{\mathcal{C}} \subset \overline{\mathcal{Q}_4} \cap \{\alpha \in \mathbb{R}^2 : R_1 < |\alpha| < R_2\}$ such that $\widetilde{\mathcal{C}} \cap (\{0\} \times (-\infty, 0)) \neq \emptyset$, $\widetilde{\mathcal{C}} \cap ((0, \infty) \times \{0\}) \neq \emptyset$ and $\lambda_h(F(\alpha)) = 1$ for every $\alpha \in \widetilde{\mathcal{C}}$. Then, we consider, for each $\gamma \in \overline{\mathcal{Q}_4} \setminus \{(0,0)\}$, the corresponding angular coordinate $\vartheta(\gamma) \in [3\pi/2, 2\pi]$ in order to study the behaviour of the continuous function $\alpha \mapsto \vartheta(-\beta(\alpha)) - \vartheta(\alpha)$ as α varies in $\widetilde{\mathcal{C}}$. With argument analogous to the ones used above, we deduce the existence of $\alpha_* \in \widetilde{\mathcal{C}} \cap \mathcal{Q}_4$ and a negative constant K verifying (3.4), which yields the conclusion. $\square$

We are now in position to prove our main result. Approximation techniques based on an application of the Stone–Weierstrass Theorem (similar to the one adopted in [8]) combined with Lemma 2.5 allow us to extend the results of Theorem 3.1 to the general case where no uniqueness assumptions are required on the nonlinearity.

Proof of Theorem 1.2. As in the proof of the previous theorem, let us concentrate on the study of the case $\lambda_N(a_0) < 1 < \lambda_M(a_\infty)$ and fix $h \in \mathbb{N}$ with $M \leq h \leq N$. According to the continuity of f and to condition (1.2), we deduce the existence of a constant $\sigma > 1$ satisfying

$$-\sigma < f(t,x) < 0 \quad \text{for all } t \in [0,\pi] \text{ and all } x \in \mathbb{R}. \tag{3.7}$$

Arguing exactly as in the proof of Theorem 3.1, we also prove that there exist two positive constants R_1, R_2 with $R_1 < R_2$ such that

$$\begin{aligned} &\lambda_h(f(\,\cdot\,,u_\alpha(\,\cdot\,))) < 1 \quad \text{for all } \alpha \in \mathbb{R}^2 \text{ such that } |\alpha| = R_1,\\ &\lambda_h(f(\,\cdot\,,u_\alpha(\,\cdot\,))) > 1 \quad \text{for all } \alpha \in \mathbb{R}^2 \text{ such that } |\alpha| = R_2, \end{aligned} \tag{3.8}$$

where u_α is a solution of (3.1). Let us define $K := R_2(1 + e^{\sigma\pi})$.

The Stone–Weierstrass Theorem ensures the existence of a sequence of continuous functions $a_l\colon [0,\pi] \times [-K,K] \to \mathbb{R}$ which are lipschitzian in the second variable and verify

$$a_l(\,\cdot\,) \underset{l\to\infty}{\longrightarrow} f(\,\cdot\,) \quad \text{uniformly in } [0,\pi] \times [-K,K]. \tag{3.9}$$

According to (3.7) and (3.9), it is not restrictive to assume that $-\sigma \leq a_l < 0$ in $[0,\pi]\times[-K,K]$. Let us now extend the function a_l to a continuous function (still denoted by a_l) which is lipschitzian in the second variable and whose domain is the whole set $[0,\pi]\times\mathbb{R}$, by setting $a_l(t,x) := a_l(t,K)$ for every $(t,x) \in [0,\pi] \times (K,\infty)$ and $a_l(t,x) := a_l(t,-K)$ for every $(t,x) \in [0,\pi] \times (-\infty,K)$. According to this definition, it is immediate to verify that

$$-\sigma \leq a_l(t,x) < 0 \quad \text{for all } (t,x) \in [0,\pi] \times \mathbb{R} \text{ and all } l \in \mathbb{N}. \tag{3.10}$$

Fix $\alpha \in \mathbb{R}^2$ with $|\alpha| \leq R_2$ and denote by $u_{\alpha,l}$ the unique solution of the Cauchy problem

$$\begin{cases} u^{(n)}(t) + a_l(t,u(t))u(t) = 0,\\ u^{(r)}(0) = 0, \quad r \in \{i_2,\dots,i_{n-1}\},\\ (u^{(i_0)}(0), u^{(i_1)}(0)) = \alpha. \end{cases}$$

Our first aim consists in proving that

(3.11) there exists $l_0 \in \mathbb{N}$ such that

$$\lambda_h(a_l(\,\cdot\,, u_{\alpha,l}(\,\cdot\,))) < 1 \quad \text{for all } l \geq l_0 \text{ and all } \alpha \in \mathbb{R}^2 \text{ with } |\alpha| = R_1.$$

Let us argue by contradiction, assuming that for every $l \in \mathbb{N}$ there exist $k_l \geq l$ and $\alpha_l \in \mathbb{R}^2$ with $|\alpha_l| = R_1$ such that

$$\lambda_h(a_{k_l}(\,\cdot\,, u_{\alpha_l,k_l}(\,\cdot\,))) \geq 1. \tag{3.12}$$

Taking into account (3.10) and the definition of K, we apply the Gronwall Lemma (as in the proof of Proposition 2.7) to show that

$$|(u_{\alpha_l,k_l}(t), u'_{\alpha_l,k_l}(t), \dots, u^{(n-1)}_{\alpha_l,k_l}(t))| \leq |\alpha_l|(1 + e^{\sigma\pi}) < K, \tag{3.13}$$

for all $t \in [0,\pi]$ and all $l \in \mathbb{N}$, which proves the uniform boundedness of $\xi_{\alpha_l,k_l} := (u_{\alpha_l,k_l}, u'_{\alpha_l,k_l}, \dots, u^{(n-1)}_{\alpha_l,k_l})$. Proceeding as in the proof of Proposition 2.6, we easily deduce the equicontinuity of ξ_{α_l,k_l} and, consequently, from an application of the Ascoli–Arzelà Theorem, we conclude that ξ_{α_l,k_l} converges (up to a subsequence) uniformly to a function $\xi = (\xi_0, \dots, \xi_{n-1}) \in C([a,b], \mathbb{R}^n)$ as $l \to \infty$. From (3.13), we immediately note that $u_{\alpha_l,k_l}(t)$ and $\xi_0(t)$ belong to the interval $[-K, K]$ when $t \in [0,\pi]$. Taking into account (3.9) and the equicontinuity of f on $[0,\pi] \times [-K,K]$, we infer that

$$\lim_{l\to\infty} \|a_{k_l}(\,\cdot\,, u_{\alpha_l,k_l}(\,\cdot\,)) - f(\,\cdot\,, \xi_0(\,\cdot\,))\|_\infty = 0. \tag{3.14}$$

Observe also that, up to a subsequence, α_l converges to a certain α when $l \to \infty$. Arguing as in the proof of Proposition 2.6, it is easy to check that ξ_0 solves (3.1). Moreover, $|(\xi_0^{(i_0)}(0), \xi_0^{(i_1)}(0))| = R_1$ and $a_{k_l} \in \mathcal{D}$. Thus, by combining Proposition 2.2 with (3.14) and (3.8), we finally obtain that, up to a subsequence, $\lambda_h(a_{k_l}(\,\cdot\,, u_{\alpha_l,k_l}(\,\cdot\,))) < 1$. This contradicts (3.12) and, consequently, proves the assertion (3.11).

Analogously, one can show that

(3.15) there exists $l_* \in \mathbb{N}$ such that

$$\lambda_h(a_l(\,\cdot\,, u_{\alpha,l}(\,\cdot\,))) > 1 \quad \text{for all } l \geq l_* \text{ and all } \alpha \in \mathbb{R}^2 \text{ with } |\alpha| = R_2.$$

For each $l \in \mathbb{N}$ with $l \geq \min\{l_0, l_*\}$, we are now able to argue exactly as in the proof of Theorem 3.1 and to achieve two solutions u_l and v_l to

$$\begin{cases} u^{(n)}(t) + a_l(t, u(t))u(t) = 0, \\ u^{(r)}(0) = 0, \quad r \in \{i_2, \dots, i_{n-1}\}, \quad R_1 < |(u^{(i_0)}(0), u^{(i_1)}(0))| < R_2, \\ u(\pi) = u'(\pi) = 0, \end{cases}$$

having exactly $(h-1)$ zeros in $(0,\pi)$ and satisfying respectively $(u_l^{(i_0)}(0), u_l^{(i_1)}(0)) \in \mathcal{Q}_2$, $(v_l^{(i_0)}(0), v_l^{(i_1)}(0)) \in \mathcal{Q}_4$.

Let us focus our attention on the sequence u_l. With arguments analogous to the ones used above, we can easily verify that it admits a subsequence (still called u_l) converging in the $\|\cdot\|_{n-1}$-norm to a solution u_* of the boundary value problem (1.1). Note that $(u_*^{(i_0)}(0), u_*^{(i_1)}(0)) \in \overline{\mathcal{Q}_2} \cap \{\alpha \in \mathbb{R}^2 : R_1 \le |\alpha| \le R_2\}$. Being $R_1 > 0$, we infer that u_* is not trivial and, hence, according to Proposition 1.1, it is the eigenfunction corresponding to the eigenvalue $\lambda = 1$ of the problem

$$\begin{cases} u^{(n)}(t) + \lambda f(t, u_*(t))u(t) = 0, \\ u^{(r)}(0) = 0, \quad r \in \{i_2, \dots, i_{n-1}\}, \\ u(\pi) = u'(\pi) = 0. \end{cases} \tag{3.16}$$

Lemma 2.5 guarantees that u_* and its derivatives $u_*^{(r)}$ (with $r \in \{1, \dots, n-1\}$) have only simples zeros in $(0, \pi)$ and cannot have other zeros at the two endpoints 0, π in addiction to the ones posed in (3.16). As a first consequence, $(u_*^{(i_0)}(0), u_*^{(i_1)}(0)) \in \mathcal{Q}_2$.

It remains only to prove that u_* has exactly $(h-1)$ zeros in $(0, \pi)$.

Let us denote by t_l^j the zeros of u_l in $(0, \pi)$ with $j \in \{1, \dots, h-1\}$, arranged in such a way that $t_l^1 < \dots < t_l^{h-1}$. For each $j \in \{1, \dots, h-1\}$, we know that, up to a subsequence, $t_l^j \to t_j \in [0, \pi]$ as $l \to \infty$. Notice that $t_1 \le \dots \le t_{h-1}$ and t_j are simples zeros of u_*.

We now claim that $t_1 > 0$. Without loss of generality, assume that $i_0 < i_1$. If $i_0 = 0$, the goal is immediately achieved since 0 cannot be a zero of $u_* = u_*^{(i_0)}$. Consider now the case $i_0 > 0$. Since u_l is the eigenfunction corresponding to $\lambda_h(a_l(\,\cdot\,, u_l(\,\cdot\,))) = 1$, we can proceed iteratively, by applying i_0 times Lemma 2.5, to deduce the existence of $\tau_l \in (0, t_l^1)$ satisfying $u_l^{(i_0)}(\tau_l) = 0$. If, by contradiction, $t_1 = 0$, then $\tau_l \to 0$ as $l \to \infty$ and, consequently, $u_*^{(i_0)}(0) = 0$. This contradicts the result regarding the maximum number of zeros of u_* at its endpoints and ensures the positivity of t_1.

Analogous arguments can be used to show that $t_{h-1} < \pi$. Thus, we can conclude that $t_j \in (0, \pi)$ for every $j \in \{1, \dots, h-1\}$.

Furthermore, by an application of Lemma 2.5, we deduce the existence of $s_l^j \in (t_l^j, t_l^{j+1})$ such that $u_l'(s_l^j) = 0$ for every $j \in \{1, \dots, h-2\}$. This implies that $t_j < t_{j+1}$. Indeed, the existence of $t_* := t_j = t_{j+1}$ would imply that $u_*'(t_*) = 0$, a contradiction with the simplicity of the zeros of u_*. We have so proved that u_* has at least $(h-1)$ simples zeros in $(0, \pi)$.

To complete the proof, we need to verify that u_* does not admit other zeros in $(0, \pi)$ besides $t_1, \dots, t_{h-1}$. For every $l \in \mathbb{N}$, let us define $t_l^0 = t_0 = 0$ and $t_l^h = t_h = \pi$ and assume, by contradiction, that there exists one more zero s of u_* in (t_j, t_{j+1}), with $j \in \{0, \dots, h-1\}$. Being s a simple zero, we can find $\delta > 0$

such that $[s-\delta, s+\delta] \subset (t_j, t_{j+1})$ and

$$u_*(s-\delta)u_*(s+\delta) < 0. \tag{3.17}$$

Note that there exists $l_0 \in \mathbb{N}$ such that $[s-\delta, s+\delta] \subset (t_l^j, t_l^{j+1})$ for every $l \geq l_0$, whence it follows that $u_l(s-\delta)\, u_l(s+\delta) > 0$. Passing to the limit as $l \to \infty$, we obtain a contradiction with (3.17). This means that u_* is the required solution to problem (1.1) with $u_*^{(i_0)}(0) < 0$, having exactly $(h-1)$ zeros in $(0,\pi)$.

Analogous considerations can be repeated for the sequence v_l and lead to the existence of a second solution v_* of problem (1.1) having exactly $(h-1)$ zeros in $(0,\pi)$ and satisfying $v_*^{(i_0)}(0) > 0$. This completes the proof of Theorem 1.2.□

Remark 3.2. Under the assumptions of Theorem 1.2, let us fix $h \in \mathbb{N}$ with $M \leq h \leq N$. Taking into account Remark 2.3, we observe that the two solutions u_h and v_h to problem (1.1) provided by Theorem 1.2 satisfy $u_h(t)\, v_h(t) < 0$ for every $t \in (0, \delta_h)$, where δ_h is a suitably small positive constant.

We conclude this work by observing that more general two-point boundary conditions than the ones exhibited in (1.1) can be considered in the statement of our main theorem.

Remark 3.3. Theorem 1.2 also holds if we generalize the two-point boundary conditions of problem (1.1) into the following

$$\begin{cases} u^{(r)}(0) = 0 & \text{for } r \in \{i_2, \dots, i_{n-1}\}, \\ u^{(r)}(\pi) = 0 & \text{for } r \in \{j_0, j_1\}, \end{cases} \tag{3.18}$$

where $\{i_2, \dots, i_{n-1}\}$ and $\{j_0, j_1\}$ are fixed subsets of $\{0, \dots, n-1\}$ such that

$$\begin{gathered} \text{for all } q \in \{1, \dots, n-1\} \\ \text{at least } q \text{ boundary conditions are imposed on } u, u', \dots, u^{(q-1)}. \end{gathered} \tag{3.19}$$

According to Corollary 3 in [11], this last condition is equivalent to excluding the existence of zero eigenvalues to the problem (1.4) endowed with boundary conditions of type (3.18).

Our main multiplicity result holds true by imposing only one condition at the final point $t = \pi$ and $(n-1)$ boundary conditions at the initial point $t = 0$, whenever $n \geq 2$. In this framework, the multiplicity result follows from an easy application of the shooting method.

Remark 3.4. Theorem 1.2 also holds for the problem

$$\begin{cases} u^{(n)}(t) + f(t, u(t))u(t) = 0, \\ u^{(r)}(0) = 0, \quad r \in \{i_1, \dots, i_{n-1}\}, \\ u^{(j_0)}(\pi) = 0, \end{cases} \tag{3.20}$$

where $\{i_1, \dots , i_{n-1}\}$ and $\{j_0\}$ are fixed subsets of $\{0, \dots , n-1\}$ ensuring the validity of condition (3.19), $f(t, x) > 0$ for every $(t, x) \in [0, \pi] \times \mathbb{R}$, the asymptotic functions $a_0(\,\cdot\,)$, $a_\infty(\,\cdot\,)$ introduced in (1.2)–(1.3) are non negative in $[0, \pi]$ and do not vanish identically on any interval in $[0, \pi]$. In this setting, the eigenvalues $\lambda_j(a)$ of assumption (1.5) correspond to problem (1.4) endowed with the boundary conditions in (3.20). (The eigenvalue theory developed in [11] guarantees the existence of a positive, unbounded from above, monotone increasing sequence of these eigenvalues $\lambda_j(a)$ whenever $a \in C([0, \pi]; [0, \infty))$ and does not vanish identically on any interval in $[0, \pi]$). Finally, the number i_0 which appears in the thesis of Theorem 1.2 is now the only integer belonging to $\{0, \dots , n-1\}\setminus\{i_1, \dots , i_{n-1}\}$.

The next few lines are devoted to sketch the proof of Remark 3.4. We first restrict ourselves to the case in which uniqueness assumptions on the solutions of Cauchy problems associated to the given equation are guaranteed. In what follows, the expression $\lambda_j(a)$ will be used to denote exclusively the eigenvalues introduced in the previous remark. Let us first recall that Propositions 1.1 and 2.2 can be rewritten in terms of these eigenvalues. Moreover, for every $\alpha \in \mathbb{R}$, one can introduce the notation of u_α to denote $u_{(\alpha,0)}$. According to this notation, one can argue as in the first steps of the proof of Theorem 3.1 and prove (3.3). Hence, taking into account the assumption (1.5), it is immediate to deduce the existence of $\alpha_1 < 0$ and $\alpha_2 > 0$ such that $\lambda_h(f(\,\cdot\,, u_{\alpha_i}(\,\cdot\,))) = 1$ for every $i \in \{1, 2\}$. By the linearity of the eigenvalue problems combined with the assumption about uniqueness of solutions of Cauchy problems, it turns out that u_{α_i} is a nontrivial multiple of the eigenfunction corresponding to $\lambda_h(f(\,\cdot\,, u_{\alpha_i}(\,\cdot\,)))$ for every $i \in \{1, 2\}$. Consequently, u_{α_1} and u_{α_2} are the required solutions of problem (3.20) having $(h - 1)$ zeros in $(0, \pi)$. This proves Remark 3.4 under uniqueness assumptions. Approximation techniques analogous to the ones developed in the proof of Theorem 1.2 allow us to complete the proof of the remark.

Remark 3.5. If we restrict problem (3.20) to the second order case (by setting $n = 2$) and if we endow it with Dirichlet boundary conditions, we note that the result stated in Remark 3.4 can be seen as a corollary of Theorem 1.1 in [6]. Let us recall that in [6] and in other papers concerning Dirichlet second order problems more general nonlinearities have been treated.

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