Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 29, 2007, 383–401

SCHAUDER'S FIXED POINT AND AMENABILITY OF A GROUP

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Abstract. A criterion for existence of a fixed point for an affine action of a given group on a compact convex space is presented. From this we derive that a discrete countable group is amenable if and only if there exists an invariant probability measure for any action of the group on a Hilbert cube. Amenable properties of the group of all isometries of the Urysohn universal homogeneous metric space are also discussed.

1. The Schauder fixed point theorem [42], [43] is a generalization of the well-known Brouwer fixed point theorem, and in its turn is a subject of various generalizations.

Theorem A (Schauder). *Any continuous mapping $f: X \to X$ of a convex compact subset of a Banach space into itself has a fixed point $f(x) = x$.*

Schauder obtained it at first for spaces with a Schauder basis and made it the starting point of a novel existence, uniqueness, and regularity theory for solutions of partial differential equations [42], [43]. A. Tychonoff proved the fixed point property for convex compact subsets of topological linear locally convex spaces [45]. In what follows, by convex space we mean a convex subset of some topological linear locally convex space.

2000 *Mathematics Subject Classification.* 43A07, 47H20, 47H10.
Key words and phrases. Amenable groups, fixed points, invariant measures.

In Polish topological school much attention is being given to more general case of multivalued mappings f and ANR-compacta X [26]. This case has important applications in functional analysis. We consider two other possible generalizations which are related to each other:

(1) a family of mappings $\{f_\alpha\}$ is considered;
(2) a compactum X is an arbitrary one, but an absent fixed point is changed by more complicated invariant object — an invariant measure.

The first step along this direction was made in 1936 by A. A. Markov and by S. Kakutani [32], who considered a commuting family of affine mappings on a convex compact subset of a topological linear (locally convex) space. In 1998 J. R. Jachymski published a simple proof of the Markov–Kakutani theorem [31]. We consider only the case when the family $\{f_\alpha\}$ forms a group. The most important step was done by N. N. Bogoljubov in 1939 [3], [10], [11], who understood that the amenability property of J. von Neumann [39] plays the main role in the existence of a fixed point (invariant measure). N. N. Bogoljubov called such groups "Banach groups", since they are characterized by the existence of invariant Banach mean. The work of J. von Neumann was inspired by investigations of F. Hausdorff [29], S. Banach [4], and A. Tarski [5] in the measure theory and by the famous paradoxis in the measure theory. It was not until 20 years after the work of N. N. Bogoljubov and 30 years after the work of J. von Neumann passed that the importance of amenable groups was realized definitively [15], [16], [27].

For a space X let $C_b(X)$ be the space of all continuous bounded functions $\varphi\colon X \to \mathbb{R}$, equipped with uniform norm $||\varphi||_\infty$. For a compact space X we shall denote the space $C_b(X)$ by $C(X)$. A linear functional $m\colon C_b(X) \to \mathbb{R}$ is called a *mean*, if

$$\inf_{x\in X}\{\varphi(x)\} \le m(\varphi) \le \sup_{x\in X}\{\varphi(x)\} \quad \text{for all } \varphi \in C_b(X).$$

From definition of a mean m we get that $||m|| = 1$ and $m \ge 0$. In particular, every mean is continuous. The set $\Sigma(X)$ of all means is a convex compact subset of topological linear locally convex space $C_b(X)^*$ of all continuous linear functionals on $C_b(X)$ with weak* topology, that is with pointwise convergence topology. If a left action of a group G is given on the space X then there exist generated left actions of G on the spaces $C_b(X)$ and $\Sigma(X)$. In order to get a representation of a group G (instead of antirepresentation), put $\widetilde{g}\varphi = \varphi \circ g^{-1}$ (we consider g^{-1} on the right as a map of X to itself, corresponding to the element $g^{-1} \in G$; however, it is inverse for the map corresponding to the element $g \in G$), i.e. $(\widetilde{g}\varphi)(x) = \varphi(g^{-1}(x))$. Since only discrete groups will be of interest to us, the induced action is indeed the action of topological group G, and no requirements like uniformness of an original action appear. In this case a mean is called left-invariant, if $m(\varphi) = m(\widetilde{g}\varphi)$ for all $g \in G$ and $\varphi \in C_b(X)$.

Considering $X = \delta(X)$ as a subset of $\Sigma(X)$, one can extend the action of G on X onto $\Sigma(X)$ in the following way:

$$(1.1) \qquad \overline{g}m = m \circ \widetilde{g^{-1}}, \quad \text{i.e. } \overline{g}m(\varphi) = (m \circ \widetilde{g^{-1}})(\varphi) = m(\widetilde{g^{-1}}\varphi) = m(\varphi \circ g).$$

We identify an element $g \in G$ with the corresponding map of the space X into itself. But to be more definite, we denote corresponding maps of the spaces $C_b(X)$ and $\Sigma(X)$ by $\widetilde{g}$ and $\overline{g}$, respectively. The map $\overline{g} = \Sigma(g): \Sigma(X) \to \Sigma(X)$ is continuous and affine. It is a homeomorphism because there is an inverse map $(\overline{g})^{-1} = \overline{g^{-1}}$. So, the group G has a left action on $\Sigma(X)$ as a group of affine homeomorphisms.

From definition (1.1) of induced action of G on $\Sigma(X)$ we obtain that *a mean m is left-invariant if and only if it is invariant with respect to the action of G on $\Sigma(X)$, i.e. is a fixed point of the action of G on $\Sigma(X)$: $\overline{g}m = m$.*

Definition 1.1. A discrete group G is called amenable if there is a left-invariant mean on $C_b(G)$.

Group G acts on itself not only from the left, but also from the right. An existence of a left-invariant mean on $C_b(G)$ is equivalent to an existence of a right-invariant mean and is equivalent to an existence of a two-side-invariant mean [27]. It is why they say simply about an invariant mean in this case.

Theorem B (Bogoljubov–Day). *For a discrete group G the following conditions are equivalent.*

(a) *The group G is amenable.*
(b) *For any affine action of the group G on a compact convex space there exists a fixed point.*
(c) *For any action of the group G on any compact space there exists an invariant probability measure.*

A probability measure on a Hausdorff compact space X is a function $\mu: \mathcal{B} \to [0,1]$, defined on σ-algebra $\mathcal{B}$ of all Baire subsets of X, such that:

(1) $\mu(X) = 1$;
(2) μ is countably additive.

An equivalence of this geometric notion of a measure to a functional notion of a mean contains an essential part of the famous Riesz theorem [23]. In our proofs we do not use this equivalence. We need it only for understanding of Proposition 2.9. To formulate our assertions we prefer to use the notion of a measure as a more geometric one.

Theorem C (Riesz). *For any compact space X the integral $\varphi \mapsto \int \varphi\, d\mu$ provides an isomorphism between the space of all probability measures on X and the space $\Sigma(X)$ of all means on $C(X)$.*

R. J. Zimmer has shown that for a countable group in conditions (b) and (c) of Theorem B, the class of all compact spaces can be reduced to the class of all metrizable compact spaces [49]. We give a topological proof of this result for an arbitrary cardinality of an acting group. It is noteworthy to say that an analogue of mentioned Zimmer's result also can be obtained from the Ščepin spectral theorem for any regular cardinal (the cardinality of the group and the weight of compact spaces) and for any cardinal from theorem about compact extensions possessing the extension of a given family of transformations.

R. Grigorchuk has raised a question on a possibility of a further reducing of a class of "test" spaces. T. Giordano and P. de la Harpe [25] got an equivariant version of the Alexandroff–Urysohn theorem: *Every metrizable compact space with an action of a countable discrete group G is an equivariant image of some action of the group G on the Cantor perfect set C.* Since an image of an invariant measure under an equivariant mapping is an invariant measure, for a discrete countable group G in condition (c) one may consider only the Cantor perfect set C. T. Giordano and P. de la Harpe have also shown that it is impossible to substitute the compactum C by a manifold of arbitrary finite dimension, and formulated a problem to consider the n-dimensional Menger compactum μ^n (or some other compactum Y) instead of the Cantor compactum C.

It is evident that the presence of sufficiently many actions of a given group on a zero-dimensional compactum C rejects the possibility of existence of connected equivariant preimages. It shows that the problem of T. Giordano and P. de la Harpe can not be solved by their method of construction of equivariant preimage (with the basic space μ^n for $n \geq 1$).

In our paper we give a criterion of an existence of a fixed point for an affine action of a given group G on a compact convex space. From this theorem we derive that for a discrete countable group G in condition (c) of Theorem B it is possible to consider only the Hilbert cube Q.

We discuss amenable properties of the group of all isometries of the Urysohn universal homogeneous metric space.

2. Since for a finite group G all statements of this paper are out of question, let τ be an infinite cardinal.

Theorem 2.1. *For an affine action of a group G on a compact convex space the following conditions are equivalent.*

(a) *The action has a fixed point.*
(b) *The action has an invariant measure.*

PROOF. Since Dirac's measure with a support at a fixed point is invariant, the implication (a) $\Rightarrow$ (b) is evident. To prove the inverse implication we need some auxiliary facts.

For a compact subset $X \subset E$ of a locally convex space E a *barycenter* of a measure $\mu \in P(X)$ is a linear functional $b(\mu) \in E^{**}$, where E^{**} is the second dual space of E defined by the equality $b(\mu)(\varphi) = \mu(\varphi|X)$. If X is a convex compactum, then $b(\mu) \in X$ (see [12, Chapter IV, §7]).

So, for a convex compactum $X \subset E$ the mapping

$$b = b_X \colon P(X) \to X$$

of barycenter of probability measures is defined. This mapping is continuous (see [12, Chapter III, §3, Corollary to Proposition 9]) and, evidently, affine.

PROPOSITION 2.2. *If $X \subset E$ is a convex compactum and a group G acts on X in affine way, then the mapping $b_X \colon P(X) \to X$ is equivariant with respect to the action of G on to $P(X)$, which is defined by the equality* (1.1).

PROOF. We have to check the commutativity of the following diagram

$$\begin{array}{ccc} P(X) & \xrightarrow{\ \overline{g}\ } & P(X) \\ {\scriptstyle b_X}\downarrow & & \downarrow{\scriptstyle b_X} \\ X & \xrightarrow[\ g\ ]{} & X \end{array}$$

where $g \in G$ is an arbitrary element. It suffices to show that $b_X \circ \overline{g} = g \circ b_X$ on everywhere dense set $Z \subset P(X)$. As such a set Z one can take the set of all measures $\mu \in P(X)$ with finite supports [12, Chapter III, §2, Theorem 1]. So, let $\mu = \alpha_1\delta(x_1) + \ldots + \alpha_n\delta(x_n)$. Then $b_X(\mu) = \alpha_1 x_1 + \ldots + \alpha_n x_n$, because b_X is affine. But the mapping g is also affine. Hence,

$$gb_X(\mu) = \alpha_1 g(x_1) + \ldots + \alpha_n g(x_n). \tag{2.1}$$

Further,

$$\overline{g}(\mu) = \alpha_1\delta(g(x_1)) + \ldots + \alpha_n\delta(g(x_n)). \tag{2.2}$$

In fact, let $\varphi \in C(X)$. Then

$$\begin{aligned} \overline{g}(\mu)(\varphi) &\overset{(1.1)}{=} \mu(\varphi \circ g) = (\alpha_1\delta(x_1) + \ldots + \alpha_n\delta(x_n))(\varphi \circ g) \\ &= \alpha_1\delta(x_1)(\varphi \circ g) + \ldots + \alpha_n\delta(x_n)(\varphi \circ g) \\ &= \alpha_1\varphi(g(x_1)) + \ldots + \alpha_n\varphi(g(x_n)) \\ &= (\alpha_1\delta(g(x_1)) + \ldots + \alpha_n\delta(g(x_n)))(\varphi). \end{aligned}$$

Now the equality (2.2) yields $b_X\overline{g}(\mu) = \alpha_1 g(x_1) + \ldots + \alpha_n g(x_n)$ (since the mapping b_X is affine). Consequently, applying the equality (2.1) we complete the proof of Proposition 2.2. □

End of the Proof of Theorem 2.1. We come back to a proof of the implication (b) ⇒ (a). Let μ be an invariant measure with respect to an affine action of G on a convex compactum X. Then μ is a fixed point of the action (1.1) of G on the compactum $P(X)$, i.e. for arbitrary $g \in G$ we have

$$\overline{g}(\mu) = \mu. \tag{2.3}$$

Let us show that $x = b_X(\mu)$ is a fixed point of the action of G on X. The assertion of Proposition 2.2 consists of the equality

$$b_X \circ \overline{g} = g \circ b_X \tag{2.4}$$

for arbitrary $g \in G$. Hence,

$$g(x) = gb_X(\mu) \overset{(2.4)}{=} b_X\overline{g}(\mu) \overset{(2.3)}{=} b_X(\mu) = x.$$

Theorem is proved. □

In what follows we need some auxiliary results. For a mapping $f: X \to X$, a set $A \subset X$ is called *free*, if its closure doesn't meet the closure of its image, that is $\overline{A} \cap \overline{f(A)} = \emptyset$. Clearly, a closure of a free set is also free. It is easy to verify that every free set has a free neighbourhood. A set $A \subset X$ is said to be *migrating* if $A \cap f(A) = \emptyset$. Every free set is, evidently, migrating. Conversely, if a map f is closed (in particular, if f is a homeomorphism), then every closed migrating set is free. A mapping f is called *free* if every point is free, that is if f has no fixed points.

Theorem 2.3. *For a mapping $f: X \to X$ of a normal space X the following conditions are equivalent.*

(a) *There is a compactification bX of X such that the mapping f can be extended over bX, and this extension bf is free.*
(b) *The mapping $\beta f: \beta X \to \beta X$ is free.*
(c) *The space X has a finite covering consisting of open migrating sets.*
(d) *The space X has a finite covering consisting of free sets.*
(e) *There is a compactification bX of weight wX and of dimension* $\dim X$ (*in a finite-dimensional case*) *such that f can be extended over bX and this extension $bf: bX \to bX$ is free.*

Proof. (a) ⇒ (b) Let $p: \beta X \to bX$ be the natural projection. For an arbitrary point $y \in \beta X$ we have $p(\beta f(y)) = bf(p(y)) \neq p(y)$, i.e. y can not be a fixed point.

(b) ⇒ (c) Since the mapping $\beta f\colon \beta X \to \beta X$ has no fixed point, for every point $y \in \beta X$ there exist neighbourhoods U_y and $U_{\beta f(y)}$ such that $U_y \cap U_{\beta f(y)} = \emptyset$. The mapping βf is continuous. Hence, one may assume that $\beta f(U_y) \subseteq U_{\beta f(y)}$. In an open covering $\{U_y\colon y \in \beta X\}$ we can find a finite subcovering $\{U_{y_i}\colon i = 1, \dots, k\}$. Then $\omega = \{U_{y_i} \cap X\colon i = 1, \dots, k\}$ is the required finite covering of X by open migrating sets.

(c) ⇒ (d) Let $\omega = \{U_i : i = 1, \dots, k\}$ be a finite covering of X consisting of open migrating sets. There is a combinatorial refining of the covering ω consisting of closed sets: $\lambda = \{A_i : i = 1, \dots, k\}$, $A_i \subseteq U_i$. Since $A_i = \overline{A_i} \subseteq U_i$ and $f(A_i) \subseteq f(U_i) \subseteq X \setminus U_i$, then $\overline{f(A_i)} \subseteq X \setminus U_i$ and A_i is a free set.

(d) ⇒ (e) Let $\lambda = \{A_i : i = 1, \dots, k\}$ be a finite covering of X by free sets. As we have pointed out already, one may assume that sets A_i are closed. For every $i = 1, \dots, k$ we fix a continuous bounded function $\varphi_i\colon X \to \mathbb{R}$ such that $\varphi_i(A_i) = 0$ and $\varphi_i(f(A_i)) = 1$. In accordance with Engelking–Sklyarenko–Zarelua theorem [20], [22], [48] there is a compactification bX of weight wX and of dimension $\dim X$ such that f and all mappings φ_i can be extended over bX (in a unique way). Let us show that this extension bf has no fixed point. Since X is dense in bX, for any point $y \in bX$ there is an index i such that $y \in \overline{A_i}^{bX} \subseteq (b\varphi_i)^{-1}(0)$. Since bf is continuous, we have $bf(y) \in \overline{bf(A_i)}^{bX} = \overline{f(A_i)}^{bX} \subseteq (b\varphi_i)^{-1}(1)$, which implies that $y \neq bf(y)$. □

Remark 2.4. Theorem 2.3 holds for an arbitrary Tychonoff space X, but in this case dimension of X is defined by functionally open coverings, i.e. $\dim X = \dim \beta X$; the sets in (c) have to be functionally open, and in (d) in notion of a free set one have to require a functional separateness of a set and its image.

Remark 2.5. An equivalence of conditions (b) and (c) was proven by E. van Douwen [17]. A mapping which satisfies (equivalent) conditions of Theorem 2.3 is called *colorable.* Since every free set has a free neighbourhood, the implication (d) ⇒ (c) can be easily proved directly. In so doing, one can see that minimal cardinalities of coverings in (c) and (d) coincide. This cardinality is called *the coloring number* $LS(f)$ of the mapping f. M. Katětov proved [33] that for every mapping f without fixed point of a discrete set, $LS(f) \leq 3$. E. van Douwen proved [17] that every free homeomorphism f of finite-dimensional paracompact space X is colorable and $LS(f) \leq 2\dim X + 3$. In [1], [6], [28] a non-improvable estimate was obtained: $LS(f) \leq \dim X + 3$. From the Katětov theorem (the more so from van Douwen's theorem) it follows that *the action of a discrete group G on the compactum βG is free.* By the way, R. Ellis gave a direct proof of this statement in 1960 [19, Theorem 3].

For a topological space X we denote by $\mathrm{CO}(X)$ the algebra (with unity) of all clopen subsets of X. For every subset $Y \subset X$ the restriction homomorphism $R\colon \mathrm{CO}(X) \to \mathrm{CO}(Y)$ is defined.

Proposition 2.6. *For every Tychonoff space X, the restriction homomorphism $R\colon \mathrm{CO}(X) \to \mathrm{CO}(Y)$ is an isomorphism.*

Proof. An inverse homomorphism is defined by the extension operator Ex [21, Lemma 7.1.13]. □

Definition 2.7. By finitely additive normed function, defined on algebra with unity $\mathcal{A} \subseteq 2^X$, we mean a function $\mu\colon \mathcal{A} \to [0,1]$ such that:

(a) $\mu(X) = 1$;

(c) for every disjoint covering $A = A_1 \cup \ldots \cup A_k$ of a set $A \in \mathcal{A}$ by sets $A_i \in \mathcal{A}$ we have

$$\mu(A) = \sum_{i=1}^{k} \mu(A_i).$$

Corollary 2.8. *For every Tychonoff space X the following conditions are equivalent.*

(a) *There is a finitely additive normed function with values in* $[0,1]$ *defined on the algebra* $\mathrm{CO}(X)$.

(b) *There is a finitely additive normed function with values in* $[0,1]$ *defined on the algebra* $\mathrm{CO}(\beta X)$.

Proposition 2.9. *For every finitely additive normed function μ with values in $[0,1]$ defined on the algebra $\mathrm{CO}(X)$, where X is a zero-dimensional compact space, there is a unique probability measure m in X such that $\mu(A) = m(\chi_A)$ for all $A \in \mathrm{CO}(X)$.*

Proof. Let V be the linear subspace of $C(X)$ generated by all characteristic functions of clopen subsets. Every function from V is of the form $\varphi = \sum a_i \chi_{A_i}$ for certain finite family of clopen sets $\{A_i\}_{i=1}^k$. Since intersection of a finite family of clopen sets is a clopen set, one may assume that the family $\{A_i\}_{i=1}^k$ is disjoint. One can define a mean m_V on V by the formula:

$$m_V(\varphi) = \sum_{i=1}^{k} a_i \mu(A_i).$$

Since the space X is zero-dimensional, the set V is dense in $C(X)$. The mean m_V is uniformly continuous. Hence, it has a unique extension $m\colon C(X) \to \mathbb{R}$. This mean m is a probability measure according to Theorem C.

If m' is another probability measure satisfying condition of Proposition 2.9, then $m_V = m'|_V$, because m' is a linear functional. Hence, $m = m'$. □

Remark 2.10. The possibility of extension of finitely additive function to countably additive is not surprising and can be obtained in geometric way. Since in compact space X no clopen subset can be represented as an infinite union of non-empty clopen subsets, then every finitely additive normed function μ with values in $[0,1]$, defined on algebra $\mathrm{CO}(X)$, also can be considered as a countably additive. Further one should apply famous theorems about extension of countably additive measures. It can be explained in a different way. Although by Proposition 2.6, algebras $\mathrm{CO}(X)$ and $\mathrm{CO}(\beta X)$ are isomorphic, for infinite discrete space X algebras $\mathrm{CO}(X)$ and $\mathrm{CO}(\beta X)$ are not isomorphic as σ-algebras already. Namely, there is an infinite operation in $\mathrm{CO}(X)$, but in $\mathrm{CO}(\beta X)$ there is no countable union. It implies that probability measure in βX induces in X only a finitely additive normed function.

Proposition 2.11. *If a discrete group G acts on compact spaces X and Y, and $f\colon X \to Y$ is an equivariant mapping, then $P(f)\colon P(X) \to P(Y)$ is an equivariant mapping too.*

Proof. In fact, since P is a functor, for any element $g \in G$ we have $\overline{g_Y} \circ P(f) = P(g_Y) \circ P(f) = P(g_Y \circ f) = P(f \circ g_X) = P(f) \circ P(g_X) = P(f) \circ \overline{g_X}$. □

Theorem 2.12. *For a discrete group G of cardinality τ the following conditions are equivalent.*

(a) *The group G is amenable.*
(b) *The action of the group G on the compact space βG has an invariant probability measure.*
(c) *For any action of the group G on any compact space there exists an invariant probability measure.*
(d) *For any affine action of the group G on a compact convex space there exists a fixed point.*
(e) *For any affine action of the group G on a compact convex space of weight $\leq \tau$ there exists an invariant probability measure.*
(f) *For any action of the group G on any compact space of weight $\leq \tau$ there exists an invariant probability measure.*
(g) *The action of the group G on the compact space βG has an invariant finitely additive normed function, defined on the algebra of all clopen subsets.*
(h) *The group G has an invariant finitely additive normed function, defined on the algebra of all subsets.*
(i) *The action of the group G on any space has an invariant finitely additive normed function, defined on the algebra of all subsets.*
(j) *Every zero-dimensional compact extension bG of weight $\leq \tau$ which admits free extension of the action of the group G has an invariant finitely additive normed function, defined on the algebra of all clopen subsets.*

Proof. We shall prove this theorem by the scheme (a) ⇔ (b) ⇒ (c) ⇒ (d) ⇒ (e) ⇒ (f) ⇒ (j) ⇒ (b) and (b) ⇔ (g) ⇔ (h) ⇔ (i).

(a) ⇔ (b) Let $\varphi \in C_b(G)$ be an arbitrary bounded continuous function on the Tychonoff space G. Then it can be uniquely extended to continuous (bounded) function $\beta\varphi: \beta G \to \mathbb{R}$ on Stone–Čech compactification βG of the space G [21]. In so doing, the correspondence $\varphi \to \beta\varphi$ defines an isomorphism of Banach spaces $C_b(G)$ and $C(\beta G)$. Consequently, each left-invariant mean m on $C_b(G)$ defines a left-invariant mean m_β on $C(\beta G)$ which is a measure in βG. This measure is, evidently, invariant (in geometric sense) with respect to the natural action of G on βG. Conversely, every invariant measure μ in βG defines a left-invariant mean on $C(\beta G)$ and, consequently, on $C_b(G)$.

(b) ⇒ (c). Let G act on a compact space X. Fix some point $x \in X$ and define an equivariant mapping $f: G \to X$ by formula $f(g) = gx$. This continuous mapping has a unique continuous extension $\beta f: \beta G \to X$. A left-invariant mean m on $C(X)$ can be defined by formula $m(\varphi) = m_\beta(\varphi \circ \beta f)$, where m_β is some left-invariant mean on $C(\beta G)$.

The implication (c) ⇒ (d) is a part of implication (b) ⇒ (a) of Theorem 2.1.

The implication (d) ⇒ (e) is evident.

(e) ⇒ (f) Let group G act on a compact space X of weight $\leq \tau$. As was already mentioned, the extension of a mapping g on $P(X)$ is denoted by $\overline{g}$. Since $\overline{g} = P(g)$, the group G acts from the left? on $P(X)$ as a group of affine homeomorphisms. Further, $wP(X) = wX \leq \tau$ [24, Chapter 7, § 3]. By suggestion, the action of the group G on $P(X)$ has an invariant probability measure. Then in accordance with Theorem 2.1, the action of G on $P(X)$ has a fixed point, which, evidently, is an invariant measure in X.

The implication (f) ⇒ (j) is evident.

(j) ⇒ (b) Represent the compactum βG as an inverse limit of zero-dimensional compact extensions bG of weight $\leq \tau$ which admit free extension of the action of the group G. By Remark 2.5, the action of the group G on βG is free. According to Theorem 2.3, for every element $g \in G$ there is a compact extension $b_g G$ of weight τ such that a homeomorphism g can be extended to a continuous free mapping $b_g g$ on $b_g G$. According to the Engelking–Sklyarenko–Zarelua theorem [20], [22], [48], for every function $\varphi \in C_b(G)$ there exists a zero-dimensional compactification $b_\varphi G$ such that:

- the function φ can be extended continuously over $b_\varphi G$;
- the action of G can be extended over $b_\varphi G$;
- $b_\varphi G$ can be mapped onto each compactification $b_g G$, $g \in G$ by a mapping which is identical on G.

Assume that for every family $\varphi_1, \dots, \varphi_k$, $2 \leq k \leq n$, we constructed compactification $b_{\varphi_1,\dots,\varphi_k} G$, and for each $i \in \{1, \dots, k\}$ we constructed a mapping

$$\pi^{\varphi_1,\dots,\varphi_k}_{\varphi_1,\dots,\varphi_{i-1},\varphi_{i+1},\dots,\varphi_k}: b_{\varphi_1,\dots,\varphi_k} G \to b_{\varphi_1,\dots,\varphi_{i-1},\varphi_{i+1},\dots,\varphi_k} G.$$

By the Engelking–Sklyarenko–Zarelua theorem, for every family $\varphi_1, \dots, \varphi_{n+1}$ there exists a zero-dimensional compact extension $b_{\varphi_1,\dots,\varphi_{n+1}}G$ of weight $\leq \tau$ which admits the extension of the action of the group G, and for any $i \in \{1, \dots, n+1\}$ there is a mapping

$$\pi^{\varphi_1,\dots,\varphi_{n+1}}_{\varphi_1,\dots,\varphi_{i-1},\varphi_{i+1},\dots,\varphi_{n+1}} : b_{\varphi_1,\dots,\varphi_{n+1}}G \to b_{\varphi_1,\dots,\varphi_{i-1},\varphi_{i+1},\dots,\varphi_{n+1}}G.$$

An indexed set of our inverse spectrum is the set of all finite subsets of $C_b(G)$. We denote this set by $\mathcal{B}$ and its members by B. Compositions of different projections commute, because there is a unique continuous mapping from one compatification of G onto another one, which is identical on G. From properties of Stone–Čech extension it follows that compactum βG is the limit of spectrum $\mathcal{S} = \{b_B G, \pi^B_{B'}, B' \subseteq B \in \mathcal{B}\}$.

Let us verify that for every $B \in \mathcal{B}$ the action of the group G on a compact extension $b_B G$ is free. Indeed, let $g \in G$ be non-identity element and $x \in b_B G$ be an arbitrary point. An extension of the mapping g over $b_B G$ shall be denoted by g_B. Natural map of $b_B G$ to $b_g G$ will be denoted by q. Then $q(g_B(x)) = (q \circ g_B)(x) = (b_g g \circ q)(x) = b_g g(q(x)) \neq q(x)$, i.e. points x and $g_B(x)$ are distinct. Hence, according to agreement, on compact $b_B G$ there is an invariant finitely additive normed function, defined on the algebra of all clopen subsets. Then by Proposition 2.9, set of all invariant probability measures on $b_B G$ is not empty; denote it by P_B. Clearly, this set is closed in $P(b_B G)$ and, consequently, is a compactum. Now let us verify that

$$P(\pi^B_{B'})(P_B) \subseteq P_{B'}. \tag{2.5}$$

Action of element g on $P(b_B G)$ we will denote by $\overline{g}_B$. Since $gh = (\pi^B_{B'} \circ g_B)(h) = (g_{B'} \circ \pi^B_{B'})(h)$ for any point $h \in G$, and G is dense subset of the space $b_B G$, the equality

$$\pi^B_{B'} \circ g_B = g_{B'} \circ \pi^B_{B'} \tag{2.6}$$

holds. It means that projection $\pi^B_{B'}$ is an equivariant map. Let $\mu \in P_B$ and $\nu = P(\pi^B_{B'})(\mu)$. Then

$$\overline{g}_{B'}(\nu) = \overline{g}_{B'} \circ P(\pi^B_{B'})(\mu)$$

(according to (g) and Proposition 2.11)

$$= P(\pi^B_{B'}) \circ \overline{g}_B(\mu) = P(\pi^B_{B'})(\mu) = \nu.$$

Hence, the family

$$\mathcal{T} = \{P_B, P(\pi^B_{B'})|_{P_B}, B' \subseteq B \in \mathcal{B}\}$$

forms an inverse spectrum of non-empty compacta. By the Kurosh theorem, $P_\beta = \lim \mathcal{T} \neq \emptyset$. Since the functor P is continuous (see [24, Chapter 7, §3]), a non-empty compactum P_β is naturally identified with subset of $P(\beta G)$. Let us show that all measures from P_β are invariant (in fact, it is exactly the set of all invariant probability measures in βG).

So, $\{P(\beta G), P(\pi_B)\}$ is the limit of inverse spectrum $\{P(b_B G), P(\pi^B_{B'}), B' \subseteq B \in \mathcal{B}\}$ (see [24, Chapter 7, §3]). Maps $\pi^B_{B'}$ and π_B are equivariant, therefore in accordance with Proposition 2.11, maps $P(\pi^B_{B'})$ are equivariant, as well as maps $P(\pi_B)$ are. Let $m \in P_\beta$, i.e. $P(\pi_B)(m) \in P_B$ for every $B \in \mathcal{B}$. Then $P(\pi_B)(\overline{g}m) = \overline{g}_B P(\pi_B)(m) = P(\pi_B)(m)$. Since the established equality is valid for all $B \in \mathcal{B}$, it follows that $\overline{g}m = m$.

(b) $\Leftrightarrow$ (g) The implication (b) $\Rightarrow$ (g) is evident. Conversely, if μ is an invariant finitely additive normed function, defined on the algebra of all clopen subsets of βG, then in accordance with Proposition 2.9 there exists a unique extending probability measure m. Probability measure $\overline{g}m$ is an extension for $\widehat{g}\mu$ ($\widehat{g}\mu(A) = \mu(gA)$). Since function μ is invariant, then $\widehat{g}\mu = \mu$, therefore from the uniqueness of extending probability measure it follows that $\overline{g}m = m$, i.e. m is an invariant mean.

(g) $\Leftrightarrow$ (h) For a discrete space X algebra $\mathrm{CO}(X)$ appears to be algebra of all subsets 2^X, therefore everything can be deduced from Corollary 2.8.

(h) $\Leftrightarrow$ (i) The implication (i) $\Rightarrow$ (h) is evident. Let group G act from the left on a space X. Fix some point $x \in X$ and define an equivariant mapping $f\colon G \to X$ by formula $f(g) = gx$. An invariant finitely additive normed function μ on 2^X can be defined by formula $\mu(A) = \mu_G(f^{-1}(A))$, where μ_G is some invariant finitely additive normed function on G. $\square$

Remark 2.13. An equivalence of conditions (a), (c), and (d) is Bogoljubov–Day theorem, formulated above. An equivalence of conditions (a) and (f) (for $\tau = \aleph_0$) is the Zimmer theorem, discussed above. An equivalence of conditions (a), (h), and (i) was shown by J. von Neuman.

Remark 2.14. Since for every compactum X of weight $\leq \tau$, every non-empty open subset of compactum $X \times D^\tau$ has weight τ and cardinality 2^τ, then in condition (f) only the following can be required: for every action of a group G on any compact space in which every non-empty open subset has weight τ and cardinality 2^τ there exists an invariant measure. According to the Alexandroff theorem, each compactum of weight $\aleph_0$ without isolated points is homeomorphic with Cantor perfect set C. It means that for $\tau = \aleph_0$ implications (f) $\Rightarrow$ (j) $\Rightarrow$ (a) give reduction of T. Giordano and P. de la Harpe. For $\tau \geq \aleph_1$ for every non-diadic compactum X of weight τ, the product $X \times D^\tau$ is not homeomorphic with D^τ; moreover, it is not a continuous image of the latter. It follows that for $\tau \geq \aleph_1$ in condition (f) we can not restrict ourselves by the only space D^τ. However, since in condition (b) the only space βG is figured, then in conditions (c) and (d) we can also restrict ourselves to the only space $P(\beta G)$. It shows an expediency of finding a topological description of the space $P(\beta G)$. Besides that, clearly for validity of implication (e) $\Rightarrow$ (j) in condition (e) it is sufficient to consider not all compact convex spaces, but only those, the sets of extreme points of which are zero-dimensional compacta of weight τ with free action of the group G.

Corollary 2.15. *For a countable discrete group G the following conditions are equivalent.*

(a) *The group G is amenable.*
(b) *For any action of the group G on the Hilbert cube Q there exists an invariant measure.*
(c) *For any free action of the group G on the Cantor compactum C there exists an invariant measure.*

Proof. The condition (b) is a part of condition (f) of Theorem 2.12, therefore the implication (a) $\Rightarrow$ (b) is evident.

(b) $\Rightarrow$ (c) The action of G on C induces the action of G on $P(C)$. The space $P(C)$ is a convex metrizable subspace of $\mathbb{R}^{C(C)}$ (see [24]). By the Keller–Klee theorem [24] the space $P(C)$ is homeomorphic to the Hilbert cube Q. Therefore the action of G on $P(C)$ has an invariant measure. The action of G on $P(C)$ is affine. By Theorem 2.1 this action has a fixed point $\mu \in P(C)$, which is an invariant measure of the original action of G on C.

(c) $\Rightarrow$ (a) Let us show that (Corollary 2.18(c) implies Theorem 2.12(j). Let X be a zero-dimensional metrizable compact space with a free action of the group G. Since an infinite group can not act freely on a scattered compactum, then the perfect kernel of compactum X is perfect, hence in accordance with the Alexandroff theorem is homeomorphic to C. It is evident that an invariant measure in subset C (the perfect kernel) defines an invariant measure in the whole X. However, an invariant measure in X also could be constructed applying the method of Remark 2.14 (multiplying by C). □

Remark 2.16. Compact space βG of a discrete space G may be identified with the set of all finitely additive measures with values in $\{0, 1\}$, defined on all subsets of G. The absense of a fixed point of an action of G on βG means that no non-trivial group G has an invariant finitely additive measure with values in $\{0, 1\}$, defined on all subsets of G. According to J. von Neuman's theorem ((a) $\Leftrightarrow$ (h)), for a discrete group G the existence of invariant finitely additive measure with values in $[0, 1]$, defined on all subsets of G, is equivalent to amenability of this group G. The question about existence on a discrete set G of non-atomaric probability measure is the question about measurableness of cardinal $\tau = |G|$. The question about existence of invariant probability measure on a discrete group G has a naive solution.

Theorem 2.17. *No infinite discrete group G admits an invariant countably additive normed function, defined on σ-algebra of all subsets.*

Proof. Assume that such a function μ exists. Let $\{g_i\}_{i=1}^{\infty}$ be a countable family of distinct elements of a group G and H be the generated group. Consider in a group G some section $\{g_\alpha: \alpha \in A\}$ of the set of right cosets $\{Hg: g \in G\}$. Then sets $\{hT: h \in H\}$ do not intersect pairwise and they form a covering of the group G. From the invariance of function μ the equality $\mu(hT) = \mu(T)$ follows. Therefore from contably additiveness follows the equality

$$\mu(G) = \sum_{h \in H} \mu(hT) = \sum_{h \in H} \mu(T),$$

which leads to a contradiction (since μ is normed). □

Remark 2.18. T. Giordano and P. de la Harpe showed that no compact manifold can be a "test" one for amenability of countable groups. However for a countable discrete group the following question seems to remain open: is such a group necessary amenable if every action of it on each compact manifold admits an invariant measure? Besides that, it is unknown whether for a countable discrete group the following conditions are equivalent: group is amenable; every action of this group on hereditarily indecomposable snake-like continuum [35]–[37] admits an invariant measure; every action of the group on the universal dendrite [14] admits an invariant measure. It is known [27] that group G is amenable if and only if some free action of it (on some space X) has an invariant finitely additive normed function, defined on algebra of all subsets of X. Conditions (g) and (h) show that it may well be true that a discrete group G is amenable if and only if for some free action of it on some (compact) space X there exists an invariant finitely additive normed function, defined on algebra of all Baire (Borel) subsets of X. Next Corollary shows fruitlessness of attempts to built an invariant finitely additive normed function, defined on algebra of all subsets of G, by constructing Baire retraction $r: X \to G$.

Let group G act from the left on a space X. Subset $A \subseteq X$ will be called *a section* provided that it contains exactly one point from every orbit, i.e. $|Gx \cap A| = 1$ for every $x \in X$. Axiom of choice implies that every action has a section. However even in the simplest case axiom of choice does not permit us to control properties of a section. From the Palais slice theorem it follows that every action of a compact Lie group on a paracompact space has a section which is F_σ-subset. G. Villalobos showed [47] that in the class of compact groups, the slice theorem is a characteristic property of Lie groups. We show that for a discrete group the existence of Baire section is a rarity, therefore it is impossible to get the solution of the problem formulated in Remark 2.18 by constructing Baire retraction $r: X \to G$.

Corollary 2.19. *No free action of countable discrete amenable group on compact space has a section by Baire set.*

Proof. Let a countable discrete amenable group G act from the left on a compact space X. Assume that Baire subset $A \subseteq X$ is a section. Amenability of the group G implies that on compact space X there is an invariant probability measure μ, defined on σ-algebra of all Baire subsets of X. Countably additive normed function $\widehat{\mu}$ on σ-algebra of all subsets of G is defined by the formula $\widehat{\mu}(B) = \mu(B \cdot A)$ for an arbitrary subset $B \subseteq G$. However, the existence of such function $\widehat{\mu}$ contradicts Theorem 2.12. □

Conjecture. *No free* (*effective*) *action of an infinite* (*countable, amenable*) *discrete group on a compact space has a Baire section.*

Remark 2.20. Theorem 2.3 implies that *every countable discrete group admits free action on Cantor perfect set.* This result also follows from the equivariant version of the Alexandroff–Urysohn theorem, obtained by T. Giordano and P. de la Harpe, and from the existence (a priori non-evident) of free action of a given countable discrete group on some compact metric space. Problem of describing those homogeneous compact spaces which admit free (effective) action of every countable discrete group, seems to be open and, in view of Remark 2.18, is of certain interest. In particular, it would be desirable to show that every countable discrete group has a free action on the Menger compactum μ^n. In works [2], [13], [18], [37] (free) actions of zero-dimensional compact groups are studied. From Dranišnikov's theorem it follows that every finitely-generated abelian group acts freely on every Menger compactum μ^n. However, group of rational numbers (under addition, with discrete topology) already can not be condensed onto a subgroup of a compact zero-dimensional group; therefore the question about free action of general countable discrete group (on μ^n) can not be reduced to the settled question about free action of zero-dimensional compact group. In the work [44] interesting results concerning minimal actions of countable free groups are obtained.

Remark 2.21. Repetition of arguments of T. Giordano and P. de la Harpe [25] allows us to represent every action of a discrete group G of cardinality $\leq \tau$ on a compactum of weight $\leq \tau$ as an equivariant image of G-action on some zero-dimensional compact space of weight $\leq \tau$ (but not necessarily D^τ). It would be of interest to obtain an analogue of the Kulesza theorem [34] about finite-to-one equivariant cover for an arbitrary countable discrete group. Besides that, it would be desirable to construct an analogue of the Dranišnikov map [2], [18] for an arbitrary countable discrete group.

3. A notion of amenability is defined not only for discrete, but also for topological groups. However a sufficiently deep theory is built at present only for locally compact groups. An analogue of Bogoljubov–Day's theorem concerning a characterization of amenability of locally compact groups was proved by Day (implication (a) $\Rightarrow$ (b)) and Rickert [41] (implication (b) $\Rightarrow$ (a)). Every compact topological group has an invariant measure (the Haar measure), and every compact metrizable topological group has an invariant metric. From this it is easily seen that every compact group is amenable and admits a free action on some compact space (on itself), and a compact metrizable group admits free action on some metric compactum by means of isometries. In 1960 Ellis proved that every discrete group admits a free action on a compact space [19] (see Remark 2.5). In 1977 Veech proved that every locally compact group admits a free action on a compact space [46]. It is natural to ask whether a topological group G can behave in the "opposite" way, that is, can it happen so that every action of a topological group G on every compact space has a fixed point? This question appeared in the 1970 paper of Mitchell [38]. The existence of a fixed point in every compact G-space is in fact an extremely strong version of amenability (the existence of invariant measure in every compact G-space is replaced by the existence of a fixed point in every compact G-space, or the existence of a fixed point for every affine G-action on every compact convex space is replaced by the existence of a fixed point in every compact G-space). For that reason topological groups with this property have been called *extremely amenable* [40]. It appears that the first example of such a topological group was constructed in 1975 by Herer and Christensen [30] quite independently of the problematics of topological dynamics, and their paper, in turn, remained till a work of Pestov [40] virtually unknown to topological dynamicists. V. G. Pestov carried out a systematic investigation of extremely amenable groups. He was giving the characterization of extremely amenable groups and exhibited vast classes of extremely amenable topological groups which are very natural. Pestov proved that the topological group $\mathrm{Aut}(\mathbb{Q}, \leq)$, equipped with the topology of pointwise (simple) convergence, is extremely amenable. Among topological groups containing it as a topological subgroup there are the full symmetric group $S(X)$ of an infinite set X endowed with the topology of simple convergence, the unitary group $U(\mathcal{H})$ of an infinite-dimensional Hilbert space equipped with the strong operator topology, and the group $\mathrm{Homeo}(I^\omega)$ of self-homeomorphisms of the Hilbert cube with the compact-open topology. Pestov proved that any action of these groups on any compact space is not free.

P. S. Urysohn in one of his latter works constructed and investigated the universal metric space (U, ϱ). P. S. Urysohn showed that for a complete separable metric space Y the following conditions are equivalent.

(1) The space Y is isometric with (U, ϱ).

(2) The space Y has properties:

(U) Every finite (separable) metric space can be isometrically embedded into Y.

(H) Every isometry of a finite (compact [8]) subset of the space Y can be extended to an isometry of the whole space Y.

Since an isometry of a compact subset of the Urysohn space can be extended over the whole space, and the 2-dimensional sphere can be isometrically embedded into the latter: $S^2 \mapsto U$, it follows that the group O(3) is a subset of the group $\mathrm{Isom}(U,\varrho)$. However it can not be supposed that this embedding is either topological or algebraical (a possibility to extend an isometry does not ensure that "close" isometries are extended to "close" and that an extension of a composition is a composition of extensions). Group O(3) contains (not closely) a free subgroup F_2 with two generators. Since an extension of isometries which had no relations on a subset, moreover can not has relations on the whole space, then the group $\mathrm{Isom}(U,\varrho)$ contains free subgroup F_2 with two generators. It means that the group $\mathrm{Isom}(U,\varrho)$ (regarded as a discrete group) is not amenable.

From the ability to extend an isometry from compact subset of the Urysohn space over the whole space it follows that the group $\mathrm{Isom}(U,\varrho)$ is very large. We suppose that any action of the group $\mathrm{Isom}(U)$ of self-isometries of the Urysohn homogeneous universal metric space (U,ϱ) with the compact open topology (topology of simple convergence) on any compact space is not free. Next hypothesis is closely connected with the formulated hypothesis. An equilateral subset $A \subset U$ will be called maximal, if it is not contained in a larger equilateral subset.

Conjecture. *For every isometry $h: A \to B$ of maximal equilateral subsets of the Urysohn space there exists an* (*unique*) *extending isometry of the Urysohn space onto itself.*

The results of this paper were presented at Kurosh and Schauder Conferences [7], [9]. The first author thanks Lech Górniewicz for hospitality. The authors thank their colleague Olga Frolkina for help, who made completing this work possible. This work was supported by Russian Foundation for Basic Research (No. 00-01-00289).

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Manuscript received November 26, 2001

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