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GLOBAL AXIALLY SYMMETRIC SOLUTIONS WITH LARGE SWIRL TO THE NAVIER–STOKES EQUATIONS

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Abstract. Long time existence of axially symmetric solutions to the Navier–Stokes equations in a bounded cylinder and with boundary slip conditions is proved. The axially symmetric solutions with nonvanishing azimuthal component of velocity (swirl) are examined. The solutions are such that swirl is small in a neighbourhood close to the axis of symmetry but it is large in some positive distance from it. There is a great difference between the proofs of global axially symmetric solutions with vanishing and nonvanishing swirl. In the first case global estimate follows at once but in the second case we need a lot of considerations in weighted spaces to show it.

The existence is proved by the Leray–Schauder fixed point theorem.

1. Introduction

We consider the motion of a viscous incompressible fluid described by Navier–Stokes equations in a bounded cylinder Ω and with boundary slip conditions

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(see [18]):

$$
\begin{aligned}
&v_{,t} + v\cdot\nabla v - \operatorname{div}\mathbb{T}(v,p) = f && \text{in } \Omega^T \equiv \Omega\times(0,T),\\
&\operatorname{div} v = 0 && \text{in } \Omega^T,\\
&v\cdot\overline{n} = 0 && \text{on } S^T = S\times(0,T),\\
&\nu\overline{n}\cdot\mathbb{D}(v)\cdot\overline{\tau}_\alpha = 0,\ \alpha=1,2, && \text{on } S^T,\\
&v|_{t=0} = v(0) && \text{in } \Omega,
\end{aligned}
\tag{1.1}
$$

where $v = v(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t)) \in \mathbb{R}^3$ is the velocity of the fluid, $p = p(x,t) \in \mathbb{R}$ the pressure, $f = f(x,t) = (f_1(x,t), f_2(x,t), f_3(x,t)) \in \mathbb{R}^3$ the external force field, $\overline{n}$ is the unit outward normal vector to S, $\overline{\tau}_\alpha$, $\alpha = 1,2$, are tangent vectors to S, $\nu > 0$ is the constant viscosity coefficient.

By $\mathbb{T}(v,p)$ we denote the stress tensor of the form

$$
\mathbb{T}(v,p) = \nu\mathbb{D}(v) - p\mathbb{I},
\tag{1.2}
$$

where $\mathbb{D}(v) = \{v_{i,x_j} + v_{j,x_i}\}_{i,j=1,2,3}$ is the dilatation tensor and $\mathbb{I}$ is the unit matrix. Finally, the dot describes the scalar product in $\mathbb{R}^3$.

To describe the domain $\Omega\subset\mathbb{R}^3$ and the considered motion we introduce the cylindrical coordinates r, φ, z by the relations $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$ $x_3 = z$, where x_1, x_2, x_3 are the Cartesian coordinates.

We assume that

$$
\Omega = \{x\in\mathbb{R}^3 : r<R,\ -a<z<a,\ \varphi\in[0,2\pi]\}.
$$

Then $\partial\Omega = S = S_1\cup S_2$ where

$$
\begin{aligned}
S_1 &= \{x\in\mathbb{R}^3 : r=R,\ -a<z<a,\ \varphi\in[0,2\pi]\},\\
S_2 &= \{x\in\mathbb{R}^3 : r<R\ z=-a \text{ or } z=a,\ \varphi\in[0,2\pi]\}.
\end{aligned}
$$

Let u be any vector. We introduce the cylindrical coordinates of u by: $u_r = u\cdot\overline{e}_r$, $u_\varphi = u\cdot\overline{e}_\varphi$, $u_z = u\cdot\overline{e}_z$, where $\overline{e}_r = (\cos\varphi, \sin\varphi, 0)$, $\overline{e}_\varphi = (-\sin\varphi, \cos\varphi, 0)$, $\overline{e}_z = (0,0,1)$.

Definition 1.1. By an *axially symmetric solution* to problem (1.1) we mean such a solution that the cylindrical components of v, f, $v(0)$ and p do not depend on φ.

The aim of this paper is to prove the existence of global and regular axially symmetric solutions to problem (1.1). To this purpose we follow the considerations from [18], [19]. Hence we need the following notation. We distinguish the angular component of velocity by denoting $v_\varphi = w$. Let $\alpha = \operatorname{rot} v$ be the vorticity vector. Its cylindrical coordinates in the axially symmetric case assume the form

$$
\alpha_r = -w_{,z},\quad \alpha_\varphi = v_{r,z} - v_{z,r} \equiv \chi,\quad \alpha_z = \frac{w}{r} + w_{,r}.
\tag{1.3}
$$

From [19] we have

Lemma 1.2. *Let v, w, $F_\varphi = (\operatorname{rot} f)_\varphi$ be given. Then χ is a solution to the problem*

$$
\begin{aligned}
&\chi_{,t} + v\cdot\nabla\chi + (v_{r,r} + v_{z,z})\chi \\
&\quad - \nu\bigg[\bigg(r\Big(\frac{\chi}{r}\Big)_{,r}\bigg)_{,r} + \chi_{,zz} + 2\Big(\frac{\chi}{r}\Big)_{,r}\bigg] = \frac{2}{r}ww_{,z} + F_\varphi && \text{in } \Omega^T,\\
&\chi = 0 && \text{on } S^T,\\
&\chi|_{t=0} = \chi(0) && \text{in } \Omega.
\end{aligned}
\tag{1.4}
$$

From [19] we also have

Lemma 1.3. *Let v and f_φ be given. Then w is a solution to the problem*

$$
\begin{aligned}
&w_{,t} + v\cdot\nabla w + \frac{v_r}{r}w - \nu\Delta w + \nu\frac{w}{r^2} = f_\varphi && \text{in } \Omega^T,\\
&w_{,r} = \frac{1}{R}w && \text{on } S_1^T,\\
&w_{,z} = 0 && \text{on } S_2^T,\\
&w|_{t=0} = w(0) && \text{in } \Omega.
\end{aligned}
\tag{1.5}
$$

To show existence of global regular axially symmetric solutions to problem (1.1) we follow the ideas from [9], [17]. In these papers axially symmetric solutions with $v_\varphi = 0$, $f_\varphi = 0$ are considered. Therefore, problem (1.5) disappears and also the first term on the r.h.s. of $(1.4)_1$. Moreover, vector α takes the form $\chi\overline{e}_\varphi$, so it has only one nonvanishing component. Having χ, vector $v = v_r\overline{e}_r + v_z\overline{e}_z$ is calculated from the elliptic problem

$$
\begin{aligned}
&v_{r,z} - v_{z,r} = \chi && \text{in } \Omega,\\
&v_{r,r} + v_{z,z} = -\frac{v_r}{r} && \text{in } \Omega,\\
&v_r|_{S_1} = 0, \quad v_z|_{S_2} = 0.
\end{aligned}
\tag{1.6$'$}
$$

Utilizing problems (1.4), (1.6′) Ladyzhenskaya in [9] was able to get global estimates guaranteeing existence of global regular solutions. Existence follows from the Galerkin method applied to the equation for the stream function ψ described by the relations: $\psi_{,z} = rv_r$, $\psi_{,r} = -rv_z$.

We have to underline that in this case the crucial global estimate for $I = \|\chi/r\|_{V_2^0(\Omega^T)}$ is at once derived, where $V_2^0(\Omega^T)$ is the energy norm for (1.1) (see notation). In reality, it follows from multiplying $(1.4)_1$ by χ/r^2, integrating over Ω^T and utilizing initial and boundary conditions for χ.

The axially symmetric case with $v_\varphi \neq 0$ is totally different because we are not able to obtain the above estimate for I because the norm $J \equiv \|w/r\|_{L_4(\Omega^T)}$

appears, which follows from

$$\int\limits_{\Omega^T} \frac{2}{r} w w_{,z} \frac{\chi}{r^2}\, dx\, dt = \int\limits_{\Omega^T} \frac{1}{r^2} (w^2)_{,z} \frac{\chi}{r}\, dx\, dt = -\int\limits_{\Omega^T} \frac{w^2}{r^2} \left(\frac{\chi}{r}\right)_{,z} dx\, dt \equiv I_1,$$

so

$$|I_1| \le \varepsilon_1 \left| \left(\frac{\chi}{r}\right)_{,z} \right|_{2,\Omega^T}^2 + c\left(\frac{1}{\varepsilon_1}\right) \left| \frac{w}{r} \right|_{4,\Omega^T}^4.$$

Further, we do not know how to estimate J without smallness assumptions on data (see assumptions of Theorems 1 and 2, see also [18]).

Considering cylinder Ω with cutted of the axis of symmetry we have

$$\left\| \frac{w}{r} \right\|_{L_4(\Omega^T)} \le c\|w\|_{L_4(\Omega^T)} \le c\|v\|_{V_2^0(\Omega^T)},$$

so it is estimated in terms of the energy estimate for the weak solutions of (1.1) and the fact that such domain can be treated as two-dimensional (for more details see [19]).

We have to underline that axially symmetric weak solution to (1.1) behaves as three-dimensional near the axis of symmetry (see Lemma 2.2). This is connected with the fact that Jacobian $rdrdz$ appears in the energy norm, so weighted Sobolev spaces must be utilized.

For the axially symmetric case examined in this paper, vorticity vector has three components (see (1.3)), so instead of (1.6′) we have

$$\begin{aligned} \operatorname{rot} v &= \alpha && \text{in } \Omega, \\ \operatorname{div} v &= 0 && \text{in } \Omega, \\ v \cdot \overline{n} &= 0 && \text{on } S, \end{aligned} \tag{1.6$''$}$$

where α satisfies the compatibility condition $\operatorname{div} \alpha = 0$.

Since in our case we have three components of vorticity we need additional problems for vorticity (comparing with (1.4)) to obtain regularity of v in a neighbourhood of the axis of symmetry (see [18]). The above considerations suggest that problem (1.1) should be examined in a different way in a neighbourhood of the axis of symmetry and in a positive distance from it. Hence it is natural to introduce a partition of unity connected with this separation (see Section 2).

In view of the above remarks we have to consider additional problems in a neighbourhood of the axis of symmetry. First we have the problem for α_r,

$$\begin{aligned} & \alpha_{r,t} + v \cdot \nabla \alpha_r - (\alpha_r v_{r,r} + \alpha_z v_{r,z}) - \nu \Delta \alpha_r + \nu \frac{\alpha_r}{r^2} = F_r && \text{in } \Omega^T, \\ & \alpha_{r,r} = -\frac{1}{R} w_{,z} && \text{on } S_1^T, \\ & \alpha_r = 0 && \text{on } S_2^T, \\ & \alpha_r|_{t=0} = \alpha_r(0) && \text{in } \Omega. \end{aligned} \tag{1.7}$$

Next the problem for α_z,

$$
(1.8)\qquad
\begin{aligned}
&\alpha_{z,t} + v\cdot\nabla\alpha_z - (\alpha_r v_{z,r} + \alpha_z v_{z,z}) - \nu\Delta\alpha_z = F_z && \text{in } \Omega^T,\\
&\alpha_z = \frac{2}{R}w && \text{on } S_1^T,\\
&\alpha_{z,z} = 0 && \text{on } S_2^T,\\
&\alpha_z|_{t=0} = \alpha_z(0) && \text{in } \Omega.
\end{aligned}
$$

Finally, we consider problem for $u = w_{,z}$,

$$
(1.9)\qquad
\begin{aligned}
&u_{,t} + v\cdot\nabla u + \frac{v_r}{r}u - \nu\Delta u + \nu\frac{u}{r^2}\\
&\qquad = -v_{,z}\cdot\nabla w - \frac{v_{r,z}}{r}w + f_{\varphi,z} && \text{in } \Omega^T,\\
&u_{,r} = \frac{1}{R}u && \text{on } S_1^T,\\
&u = 0 && \text{on } S_2^T,\\
&u|_{t=0} = u(0) && \text{in } \Omega.
\end{aligned}
$$

Now, we formulate the main results and outline their proofs.

Let us introduce smooth functions $\zeta^{(i)} = \zeta^{(i)}(r)$, $i = 1,2,3,4$, which are such that:

$$
\begin{aligned}
&\zeta^{(1)}(r) = 1 \quad \text{for } r \le r_0, && \zeta^{(1)}(r) = 0 \quad \text{for } r \ge r_0 + \delta_1,\\
&\zeta^{(3)}(r) = 1 \quad \text{for } r \le r_0 + \delta_1, && \zeta^{(3)}(r) = 0 \quad \text{for } r \ge r_0 + 2\delta_1,\\
&\zeta^{(2)}(r) = 1 \quad \text{for } r \ge r_0, && \zeta^{(2)}(r) = 0 \quad \text{for } r \le r_0 - \delta_2,\\
&\zeta^{(4)}(r) = 1 \quad \text{for } r \ge r_0 - \delta_2, && \zeta^{(4)}(r) = 0 \quad \text{for } r \le r_0 - 2\delta_2.
\end{aligned}
$$

We assume that $r_0 > 2\delta_2$ and $r_0 + 2\delta_1 < R$. We denote that $u^{(i)} = u\zeta^{(i)}$, $i = 1,\dots,4$, where u replaces any vector or function used in this paper.

Theorem 1.

(a) *Assume that*

$$
\begin{gathered}
f_\varphi^{(1)} \in L_{2,-1}(\Omega^T), \quad f_\varphi^{(3)} \in L_2(\Omega^T), \quad (\operatorname{rot} f)_r^{(1)}, (\operatorname{rot} f)_z^{(1)} \in L_2(\Omega^T),\\
v_{\varphi,z}^{(1)} \in L_{2,-\delta}(\Omega^T), \quad v_\varphi^{(3)}(0) \in H_0^1(\Omega), \quad (\operatorname{rot} v)_r^{(1)}(0), \quad (\operatorname{rot} v)_z^{(1)}(0) \in L_2(\Omega),\\
(\operatorname{rot} f)_\varphi^{(1)} \in L_2(0,T;L_{2,-1}(\Omega)), \quad (\operatorname{rot} f)_\varphi^{(4)} \in L_2(\Omega^T), \quad f_\varphi^{(2)} \in L_2(\Omega^T),\\
f_\varphi^{(4)} \in L_4(0,T;L_{4/3}(\Omega)), \quad (\operatorname{rot} v)_\varphi^{(1)}(0) \in L_{2,-1}(\Omega), \quad (\operatorname{rot} v)_\varphi^{(4)}(0) \in L_2(\Omega),\\
v_\varphi^{(4)}(0) \in L_4(\Omega), \quad v_\varphi^{(2)}(0) \in H^1(\Omega), \quad \delta = \frac{1}{2} + \varepsilon_0,
\end{gathered}
$$

where $\varepsilon_0 > 0$ is an arbitrary small number.

(b) *Let us introduce the quantities*

$$
\begin{aligned}
X(T) &= \|f_\varphi^{(1)}\|_{L_{2,-1}(\Omega^T)} + \|f_\varphi^{(3)}\|_{L_2(\Omega^T)} + \|(\operatorname{rot} f)_r^{(1)}\|_{L_2(\Omega^T)} \\
&\quad + \|(\operatorname{rot} f)_z^{(1)}\|_{L_2(\Omega^T)} + \|v_{\varphi,z}^{(1)}(0)\|_{L_{2,-\delta}(\Omega)} + \|v_\varphi^{(3)}(0)\|_{H_0^1(\Omega)} \\
&\quad + \|(\operatorname{rot} v)_r^{(1)}(0)\|_{L_2(\Omega)} + \|(\operatorname{rot} v)_z^{(1)}(0)\|_{L_2(\Omega)}, \\
Y(T) &= \|(\operatorname{rot} f)_\varphi^{(1)}\|_{L_2(0,T;L_{2,-1}(\Omega))} + \|(\operatorname{rot} f)_\varphi^{(4)}\|_{L_2(\Omega^T)} \\
&\quad + \|f_\varphi^{(4)}\|_{L_4(0,T;L_{4/3}(\Omega))} + \|f_\varphi^{(2)}\|_{L_2(\Omega^T)} + \|(\operatorname{rot} v)_\varphi^{(1)}(0)\|_{L_{2,-1}(\Omega)} \\
&\quad + \|(\operatorname{rot} v)_\varphi^{(4)}(0)\|_{L_2(\Omega)} + \|v_\varphi^{(2)}(0)\|_{H^1(\Omega)} + \|v_\varphi^{(4)}(0)\|_{L_4(\Omega)},
\end{aligned}
$$

$$
\frac{1}{\delta_*} = \frac{1}{\delta_1} + \frac{1}{\delta_2} + X(T).
$$

(c) *Assume that* $v \in W_{5/2}^{2,1}(\Omega^T)$.

(d) *Assume that there exists a weak solution* (*see Lemma* 2.1) *such that*

$$
\|v(t)\|_{L_2(\Omega)} \le d_1, \quad \|v(t)\|_{L_2(\Omega)} + \|\nabla v\|_{L_2(\Omega^T)} \le d_2, \quad t \in \mathbb{R}_+.
$$

Then there exists a positive increasing function φ_* *such that*

$$
\|v\|_{L_{10}(\Omega^T)} + \|\nabla v\|_{L_{10/3}(\Omega^T)} \le \varphi_*\biggl(\frac{1}{\delta_*}\|v\|_{W_{5/2}^{2,1}(\Omega^T)}, d_1, d_2, Y\biggr). \tag{1.10}
$$

THEOREM 2. *Let the assumptions* (a), (b), (d) *of Theorem* 1 *be satisfied.*

(a) *Assume that* $v(0) \in W_{5/2}^{6/5}(\Omega)$, $f \in L_{5/2}(\Omega^T)$, $v_{r,\varphi}(0) = 0$, $v_{z,\varphi}(0) = 0$, $f_{r,\varphi} = 0$, $f_{z,\varphi} = 0$.

(b) *Assume that* δ_* *is so large that there exists a positive constant* A *such that*

$$
c\biggl[\varphi_*\biggl(\frac{1}{\delta_*}A, d_1, d_2, Y(T)\biggr) + Y\biggr]^2 + c(\|f\|_{L_{5/2}(\Omega^T)} + \|v(0)\|_{W_{5/2}^{6/5}(\Omega)}) \le A. \tag{1.11}
$$

Then there exists a unique axially symmetric solution to problem (1.1) *such that* $v \in W_{5/2}^{2,1}(\Omega^T)$ *and*

$$
\|v\|_{W_{5/2}^{2,1}(\Omega^T)} \le A.
$$

Moreover, the solution is such that

$$
\begin{gathered}
v_{\varphi,z}^{(1)} \in L_\infty(0,T;L_{2,-\delta}(\Omega)) \cap L_2(0,T;H_{-\delta}^1(\Omega)), \\
\frac{1}{r}(\operatorname{rot} v)_\varphi^{(1)} \in L_\infty(0,T;L_2(\Omega)) \cap L_2(0,T;H^1(\Omega)), \\
v_\varphi^{(1)} \in L_\infty(0,T;H_0^1(\Omega)), \quad (\operatorname{rot} v)_r^{(1)} \in L_2(0,T;L_{2,-1}(\Omega)).
\end{gathered}
$$

Theorem 1 is proved by a series of lemmas from Sections 3 and 4. Having estimate (1.10) Theorem 2 follows from the Leray–Schauder fixed point theorem in Section 5.

The solution from Theorem 2 behaves in a different way in a neighbourhood of the axis of symmetry and in a sufficiently large distance from it.

We have proved existence of such axially symmetric solutions that azimuthal component of initial velocity must be sufficiently small in a neighbourhood of the axis of symmetry. However, this neighbourhood must be sufficiently large.

We have shown the existence of solutions without restrictions on the existence time T. However, we are not able to pass with T to infinity.

Since time integral norms of f, $\operatorname{rot} f$, v appear in formulations of Theorems 1 and 2 we have strong restrictions on these functions for large T. To relax the restrictions we tried to prove global existence step by step having local existence on a fixed time interval $[0, T]$. However, we have not been able to obtain the same estimates on $[kT, (k+1)T]$, $k > 0$, $k \in \mathbb{N}$, as we have had on $[0, T]$.

Finally Lemma 2.2 implies that the axially symmetric solution behaves as three-dimensional near the axis of symmetry.

At the end we recall results on existence of global regular solutions to $3d$-Navier–Stokes equations. They base on less-dimensional global regular solutions to Navier–Stokes equations:

(1.12)

1. two-dimensional [10],
2. axially-symmetric [9], [17],
3. helically symmetric [12].

Global regular solutions close to $(1.12)_1$ are shown in thin domain [14]–[16], [2], [3], [7] and in cylindrical type domains [21], [23].

In [14]–[16] Raugel and Sell proved existence of global regular solutions to Navier–Stokes equations in a thin domain $\Omega_\varepsilon = \Omega' \times ((0, \varepsilon)$, $\Omega' \subset \mathbb{R}^2$, ε — small and with periodic boundary conditions by using semigroup technique. The result was generalized by Avrin [2], [3], who proved also existence of global regular solutions in the thin domain Ω_ε but with Dirichlet boundary conditions on $\partial\Omega'$ and periodic conditions in the third direction. In his considerations the smallness of ε was replaced by large first eigenvalue of — $P\Delta$, where P is the projection operator on the divergence free vector fields. He used a fixed point argument. A generalization of the above results was done by Iftimie and Raugel in [7] who relax the conditions on the magnitude and regularity of $v(0)$ and f.

In [21] global existence of regular solutions is proved by the Leray–Schauder fixed point theorem in Besov spaces in the case of slip conditions on a cylindrical boundary. In [23] there is considered Navier–Stokes motion in non-axially symmetric cylinder with inflow and outflow on S_2. The methods and spaces are similar as in [21], however the proof is much more complicated because nonhomogeneous Dirichlet boundary conditions are considered. Results in [21], [23] are such that derivatives of v and p in the direction x_3 are sufficiently small.

Existence of different global regular solutions close to $(1.12)_2$ are shown by the author (see [18]–[20], [22]). In [18] global existence of solutions to Navier–Stokes equations in an axially symmetric cylinder with slip boundary conditions is proved. The solution is close to the axially symmetric solution because derivatives of cylindrical coordinates of velocity and pressure with respect to φ and v_φ are small. Global existence follows because a time decay on the external force is imposed.

In [20] global existence of solutions with slip boundary conditions and in an arbitrary axially symmetric domain is proved. The solution is close to the axially symmetric solutions in the similar way as in [18]. The energy estimate for the azimuthal component of vorticity (α_φ) is the main estimate in [18] and [20] (see also [9], [17]). To obtain such estimate we need homogeneous boundary conditions for α_φ. Such boundary conditions are in [18] and also in [9]. However in [20] we have nonhomogenous boundary conditions for α_φ because curvature of the boundary is different from zero. This makes proof in [20] much more complicated than in [18].

In [19] and [22] global existence of solutions is proved in a cylinder without the axis of symmetry. Solutions in [19] and [22] have large swirl. In [19] we proved existence of axially symmetric solutions but in [22] existence of solutions which are close to axially symmetric.

Finally, in [4], [5] existence of solutions, which are some generalizations of solutions from $(1.12)_3$, is shown.

2. Notation and auxiliary results

To simplify considerations we introduce

$$
\begin{aligned}
|u|_{p,Q} &= \|u\|_{L_p(Q)}, & Q &\in \{\Omega, S, \Omega^T, S^T\}, & p &\in [1,\infty],\\
\|u\|_{s,Q} &= \|u\|_{H^s(Q)}, & Q &\in \{\Omega, S\}, & s &\in \mathbb{R}_+ \cup \{0\},\\
\|u\|_{s,Q} &= \|u\|_{W_2^{s,s/2}(Q)}, & Q &\in \{\Omega^T, S^T\}, & s &\in \mathbb{R}_+ \cup \{0\},
\end{aligned}
$$

where $\|u\|_{0,Q} = |u|_{2,Q}$, $H^s(Q) = W_2^s(Q)$.

We use weighted spaces $L_{p,\mu}(Q)$, $H^s_\mu(Q)$, $W^s_\mu(Q)$, $Q \in \{\Omega, S\}$ with the norms

$$
\begin{aligned}
|u|_{p,\mu,Q} \equiv \|u\|_{L_{p,\mu}(Q)} &= \bigg(\int_Q |u|^p r^{p\mu} dQ\bigg)^{1/p},\\
\|u\|_{s,\mu,Q} = \|u\|_{H^s_\mu(Q)} &= \bigg(\sum_{|\alpha|\le s}\int_Q |D^\alpha_x u|^2 r^{2(\mu-s+|\alpha|)} dQ\bigg)^{1/2},\\
|||u|||_{s,p,\mu,Q} \equiv \|u\|_{W^s_{p,\mu}(Q)} &= \bigg(\sum_{|\alpha|\le s}\int_Q |D^\alpha_x u|^p r^{p\mu} dQ\bigg)^{1/p},
\end{aligned}
$$

where $\mu \in \mathbb{R}$, $p \in [1,\infty]$, $s \in \mathbb{N} \cup \{0\}$ and $W^s_{p,0}(Q) = W^s_p(Q)$, so $|||u|||_{s,p,0,Q} = |||u|||_{s,p,Q}$.

We need also anisotropic Sobolev spaces

$$\begin{aligned} \|u\|_{s,\mu,Q^T} &= \|u\|_{H^{s,s/2}_\mu(Q^T)}, \quad Q \in \{\Omega, S\}, \\ |||u|||_{s,p,\mu,Q^T} &= \|u\|_{W^{s,s/2}_{p,\mu}(Q^T)}, \quad Q \in \{\Omega, S\}, \end{aligned}$$

where $|||u|||_{s,p,0,Q^T} = |||u|||_{s,p,Q^T}$ and

$$\begin{aligned} \|u\|_{H^{s,s/2}_\mu(Q^T)} &= \bigg(\sum_{|\alpha|+2a\le s} \int_0^T \int_Q |D^\alpha_x \partial^a_t u|^2 r^{2(\mu-s+|\alpha|+2a)}\, dx\, dt \bigg)^{1/2}, \\ \|u\|_{W^{s,s/2}_{p,\mu}(Q^T)} &= \bigg(\sum_{|\alpha|+2a\le s} \int_0^T \int_Q |D^\alpha_x \partial^a_t u|^p r^{p\mu}\, dx\, dt \bigg)^{1/p}. \end{aligned}$$

Moreover, we introduce anisotropic Lebesque spaces

$$\begin{aligned} |u|_{p_1,p_2,Q^T} &= \|u\|_{L_{p_2}(0,T;L_{p_1}(Q))}, \quad Q \in \{\Omega, S\}, \\ |u|_{p_1,p_2,\mu,Q^T} &= \|u\|_{L_{p_2}(0,T;L_{p_1,\mu}(Q))}, \quad Q \in \{\Omega, S\},\ \mu \in \mathbb{R}, \end{aligned}$$

and $p_1, p_2 \in [1,\infty]$.

In the above definitions Ω is a cylinder with the axis of symmetry, $S = \partial\Omega$ and r is the distance from the axis.

Let $r_0 \in (0,R)$, $0 < \delta_i$, $i = 1,2$, such that $r_0 + 2\delta_1 < R$ and $r_0 - 2\delta_2 > 0$ be given. Then we introduce a partition of unity $\{\zeta^{(i)}(r)\}$, $i = 1,2$, such that $\zeta^{(1)}(r) = 1$ for $r \le r_0$ and $\zeta^{(1)}(r) = 0$ for $r \ge r_0 + \delta_1$, $\zeta^{(2)}(r) = 1$ for $r \ge r_0$ and $\zeta^{(2)}(r) = 0$ for $r \le r_0 - \delta_2$.

Moreover, $\Omega_{r_0} = \{x \in \Omega : r < r_0\}$, $\widehat{\Omega}_{r_0} = \{x \in \Omega : r > r_0\}$, $\Omega'_{(k)} = \Omega_{r_0+k\delta_1} \setminus \Omega_{r_0+(k-1)\delta_1}$, $\widehat{\Omega}'_{(k)} = \widehat{\Omega}_{r_0-k\delta_2} \setminus \widehat{\Omega}_{r_0-(k-1)\delta_2}$, $k \in \mathbb{N}$.

Introduce also a function $\zeta^{(3)} = \zeta^{(3)}(r)$ such that

$$\begin{aligned} \zeta^{(3)}(r) &= 1 \quad \text{for } r \le r_0 + \delta_1, \\ \zeta^{(3)}(r) &= 0 \quad \text{for } r \ge r_0 + 2\delta_1. \end{aligned}$$

Finally we need a smooth function $\zeta^{(4)} = \zeta^{(4)}(r)$ such that $\zeta^{(4)}(r) = 1$ for $r \ge r_0 - \delta_2$ and $\zeta^{(4)}(r) = 0$ for $r \le r_0 - 2\delta_2$.

We use also the notation

$$\begin{aligned}\|u\|_{L_{p,\alpha}(\widetilde{\Omega})} &= \bigg(\int\limits_{\widetilde{\Omega}} |u|^p r^{p\alpha}\,dr\,dz\bigg)^{1/p},\\ \|u\|_{L_p(\Omega)} &= \bigg(\int\limits_{\Omega} |u|^p\,dx\bigg)^{1/p},\\ \|u\|_{L_{p,\alpha}(\Omega)} &= \bigg(\int\limits_{\Omega} |u|^p r^{p\alpha}\,dx\bigg)^{1/p}, \quad \text{where } dx = r\,dr\,dz.\end{aligned}$$

We need also,

$$\begin{aligned}\Omega_\varepsilon &= \{x\in\mathbb{R}^3 : 0<\varepsilon<r<R,\ -a<z<a\},\\ \Omega_{\varepsilon(r_0)} &= \{x\in\mathbb{R}^3 : 0<\varepsilon<r<r_0<R,\ -a<z<a\}\\ S_\varepsilon &= \{x\in S : 0<\varepsilon<r\}.\end{aligned}$$

Considering problem (1.1) in Ω_ε we need additional boundary conditions

$$v\cdot\overline{n}|_{r=\varepsilon} = v_r|_{r=\varepsilon} = 0.$$

Let u be any scalar or vector function. We introduce $u^{(i)} = u\zeta^{(i)}$, $i=1,2,3,4$.

By c we denote generic constants. To distinguish a certain constant we denote it by c_k, $k\in\mathbb{N}$. By φ we denote also generic increasing positive functions.

By l.h.s. (r.h.s.) we mean the left-hand side (right-hand side), respectively.

Finally, we do not distinguish scalar and vector-valued functions.

For the reader convenience we recall imbeddings and results utilized in this paper.

Let $V_2^k(\Omega^T)$ be defined by the norm

$$\|u\|_{V_2^k(\Omega^T)} = \sup_{t\le T}\|u(t)\|_{k,\Omega} + \bigg(\int\limits_0^T \|\nabla u(t)\|_{k,\Omega}^2\,dt\bigg)^{1/2}, \quad k\in\mathbb{N}\cup\{0\}.$$

From [11, Chapter 2, Section 3] we have

$$\bigg(\int\limits_0^T \|u(t)\|_{W_q^k(\Omega)}^r dt\bigg)^{1/r} \le c\|u\|_{V_2^k(\Omega^T)}, \quad \Omega\subset\mathbb{R}^n,$$

where $2/r + n/q = n/2$. Then imbedding $W_q^k(\Omega)\subset L_\sigma(\Omega)$ implies

$$\bigg(\int\limits_0^T \|u(t)\|_{L_\sigma(\Omega)}^r\,dt\bigg)^{1/r} \le c\|u\|_{V_2^k(\Omega^T)}, \quad \Omega\subset\mathbb{R}^n$$

for $2/r + 3/\sigma \ge 3/2 - k$. From [13] the following imbedding

(2.1) $$\|u\|_{V^s_{q,\beta+s-l+n/p-n/q}(\Omega)} \le c\|u\|_{V^l_{p,\beta}(\Omega)}, \quad \Omega\subset\mathbb{R}^n,$$

holds for $s-l+n/p-n/q\le 0$, where

$$\|u\|_{V^s_{p,\mu}(\Omega)}=\bigg(\sum_{|\alpha|\le s}\int_\Omega |D^\alpha_x u(x)|^p r^{p(\mu-s+|\alpha|)}\,dx\bigg)^{1/p}.$$

For weak solutions to problem (1.1) we have

Lemma 2.1. ([18], [19]). *Assume that* $v(0)\in L_2(\Omega)$, $f\in L_\infty(\mathbb{R}_+,L_{6/5}(\Omega))$, $\int_{\Omega^t} f_\eta\,dx\,dt'\in L_\infty(\mathbb{R}_+)$, *where* $f_\eta=f\cdot\eta$, $\eta=(-x_2,x_1,0)$. *Assume that* $T_*>0$ *is given. Then there exist constants*

$$d_1^2=\frac{c}{\nu_1}\bigg[(\sup_t|f(t)|_{6/5,\Omega})^2+\bigg(\sup_t\int_{\Omega^t} f_\eta\,dx\,dt'\bigg)^2+\bigg(\int_\Omega v_\eta(0)\,dx\bigg)^2\bigg]+|v(0)|^2_{2,\Omega},$$
$$d_2^2(T)=(3+e^{\nu_1T_*})d_1^2,$$

where $\nu=\nu_1+\nu_2$, $\nu_i>0$, $i=1,2$, *independent of* k *such that*

$$|v(t)|_{2,\Omega}\le d_1\quad\text{for } t>0, \tag{2.2}$$

$$|v(t)|^2_{2,\Omega}+\nu_2\int_{kT_*}^t\|v(t')\|^2_{1,\Omega}\,dt'\le d_2^2\quad\text{for } t\in[kT_*,(k+1)T_*], \tag{2.3}$$

where $k\in\mathbb{N}$.

Now we show that axially symmetric weak solution to problem (1.1) behaves as three-dimensional in a neighbourhood of the axis of symmetry. Let $\widetilde\Omega$ be an intersection of Ω with the plane $\varphi=\text{const}$. In domain $\widetilde\Omega$ we introduce the weighted spaces

$$\|u\|_{W^s_{p,\alpha}(\widetilde\Omega)}=\bigg(\sum_{|\alpha|\le s}\int_{\widetilde\Omega}|D^\alpha_{x'}u|^p r^{p\alpha}\,dr\,dz\bigg)^{1/p},$$

where $x'=(r,z)$, $s\in\mathbb{N}\cup\{0\}$, $p\in[1,\infty]$, $\alpha\in\mathbb{R}$.

From [8] the following theorems of imbedding

$$\|u\|_{L_{q,\beta}(\widetilde\Omega)}\le\varepsilon^{1-\varkappa}\|D^l_{x'}u\|_{L_{p,\alpha}(\widetilde\Omega)}+c\varepsilon^{-\varkappa}\|u\|_{L_{p,\alpha}(\widetilde\Omega)}, \tag{2.4}$$

hold for $1<p<q<\infty$, $\alpha\ge\beta-1/q$, where

$$\varkappa=\bigg(\frac1p-\frac1q\bigg)\frac2l+\frac1l(\alpha-\beta)<1. \tag{2.5}$$

In view of (2.4) the following interpolation inequality takes place

$$\|u\|_{L_{q,\beta}(\widetilde\Omega)}\le c\|D^l_{x'}u\|^\varkappa_{L_{p,\alpha}(\widetilde\Omega)}\|u\|^{1-\varkappa}_{L_{p,\alpha}(\widetilde\Omega)}+c\|u\|_{L_{p,\alpha}(\widetilde\Omega)}. \tag{2.6}$$

Lemma 2.2. *Assume that* v *is the axially symmetric weak solution to problem* (1.1) *from Lemma* 2.1. *Then the following estimate*

$$\|v\|_{L_\sigma(0,T;L_p(\Omega))} \le c\|v\|_{V_2^0(\Omega^T)}, \tag{2.7}$$

holds for

$$\frac{2}{\sigma} + \frac{3}{p} = \frac{3}{2}. \tag{2.8}$$

Remark 2.3. Estimate (2.7), (2.8) is exactly the same as for arbitrary weak solution to problem (1.1). Hence near the axis of symmetry axially symmetric weak solution behaves as arbitrary weak solution.

Considering a neighbourhood in a positive distance from the axis of symmetry ($r \ge R_0 > 0$), inequality (2.7) takes the form

$$\|v\|_{L_4(\Omega^T)} \le c\|v\|_{V_2^0(\Omega^T)},$$

so the estimate for two-dimensional regular solution holds.

Proof of Lemma 2.2. For the axially symmetric weak solution inequality (2.3) takes the form

$$\|v\|_{V_2^0(\Omega^T)} = (2\pi)^{1/2}\left[\sup_{t\le T}|v(t)|_{L_{2,1/2}(\widetilde\Omega)} + \left(\nu_2\int_0^t \|v(t')\|^2_{W^1_{2,1/2}(\widetilde\Omega)}\,dt'\right)^{1/2}\right] \le d_2.$$

To show (2.7) we look for (2.6) in the form

$$\|v\|_{L_{p,1/p}(\widetilde\Omega)} \le c\|\nabla v\|^{\varkappa}_{L_{2,1/2}(\widetilde\Omega)}\|v\|^{1-\varkappa}_{L_{2,1/2}(\widetilde\Omega)} + c\|v\|_{L_{2,1/2}(\widetilde\Omega)}, \tag{2.9}$$

which holds because (2.5) implies

$$\varkappa = \frac{3}{2} - \frac{3}{p} < 1 \quad \text{for } 2 < p < 6.$$

From (2.9) we get

$$\begin{aligned}\left(\int_0^T \|v(t)\|^{\frac{2}{\varkappa}}_{L_{p,1/p}(\widetilde\Omega)}\,dt\right)^{\varkappa/2} &\le c\sup_t\|v\|^{1-\varkappa}_{L_{2,1/2}(\widetilde\Omega)}\left(\int_0^T\|\nabla v\|^2_{L_{2,1/2}(\widetilde\Omega)}\,dt\right)^{\varkappa/2} \\ &\quad + cT^{\varkappa/2}\sup_t\|v\|_{L_{2,1/2}(\widetilde\Omega)} \le c(T)\|v\|_{V_2^0(\Omega^T)}.\end{aligned} \tag{2.10}$$

The norm on the l.h.s. of (2.10) equals

$$c\left(\int_0^T\|v(t)\|^{2/\varkappa}_{L_p(\Omega)}dt\right)^{\varkappa/2}.$$

Expressing the integral in the form $\|v\|_{L_\sigma(0,T;L_p(\Omega))}$ we have

$$\frac{2}{\sigma}+\frac{2}{p}=\frac{2}{2/\varkappa}+\frac{2}{p}=\varkappa+\frac{2}{p}=\frac{3}{2}-\frac{3}{p}+\frac{2}{p}=\frac{3}{2}-\frac{1}{p}.$$

Hence (2.8) holds. This ends the proof. □

3. Estimates near the axis of symmetry

In this section we find some inequalities for solutions of problem (1.1) in a neighbourhood of the axis of symmetry. The inequalities are necessary to find an estimate for solutions to (1.1) and to prove the existence of solutions.

To obtain estimates in this section we follow the considerations from [18]. First we examine (1.4). Multiplying (1.4) by by $\zeta^{(1)}$ we obtain

$$\begin{aligned}
&\chi^{(1)}_{,t}+v\cdot\nabla\chi^{(1)}-\frac{v_r}{r}\chi^{(1)}-\nu\Delta\chi^{(1)}+\nu\frac{\chi^{(1)}}{r^2}\\
&\qquad=\chi v\cdot\nabla\zeta^{(1)}-\nu(2\nabla\zeta^{(1)}\nabla\chi+\chi\Delta\zeta^{(1)})+\frac{2}{r}wu^{(1)}+F^{(1)}_\varphi \quad &&\text{in } \Omega^T_\varepsilon,\\
&\chi^{(1)}=0 &&\text{on } S^T_\varepsilon,\\
&\chi^{(1)}|_{t=0}=\chi^{(1)}(0) &&\text{in } \Omega_\varepsilon.
\end{aligned}\tag{3.1}$$

Lemma 3.1. *Assume that $v\in L_\infty(0,T;L_2(\Omega))\cap L_2(0,T;W^1_3(\Omega))$, $w\in L_\infty(0,T;H^1_0(\Omega_{\varepsilon(r_0+\delta_1)}))$, $u^{(1)}\in L_2(0,T;L_{4,-3/4-\varepsilon_0}(\Omega_\varepsilon))$, where $\varepsilon_0>0$ and $\varepsilon>0$ are arbitrary small. Assume that $F^{(1)}_\varphi\in L_2(0,T;L_{2,-1}(\Omega))$, $\chi^{(1)}(0)\in L_{2,-1}(\Omega)$. Let*

$$Y_1^2(t)=\int\limits_0^t\left|\frac{F^{(1)}_\varphi(t')}{r}\right|^2_{2,\Omega}dt'+\left|\frac{\chi^{(1)}(0)}{r}\right|^2_{2,\Omega}.$$

Then

$$\begin{aligned}
&\left|\frac{\chi^{(1)}(t)}{r}\right|^2_{2,\Omega_\varepsilon}+\nu\int\limits_0^t\left|\nabla\frac{\chi^{(1)}(t')}{r}\right|^2_{2,\Omega_\varepsilon}dt'\\
&\qquad\le\frac{c}{r_0^2}|\nabla\zeta^{(1)}|^2_{\infty,\Omega}\sup_t|v|^2_{2,\Omega}\int\limits_0^t|||v(t')|||^2_{1,3,\Omega}\,dt'\\
&\qquad\quad+c\sup_t\|w\|^2_{1,0,\Omega_{\varepsilon(r_0+\delta_1)}}\int\limits_0^t|u^{(1)}(t')|^2_{4,-3/4-\varepsilon_0,\Omega_\varepsilon}\,dt'+cY_1^2(t),
\end{aligned}\tag{3.2}$$

where $t\le T$.

PROOF. Multiplying $(3.1)_1$ by $\chi^{(1)}/r^2$ and integrating over Ω_ε implies

$$(3.3)\quad \frac{1}{2}\frac{d}{dt}\bigg|\frac{\chi^{(1)}}{r}\bigg|^2_{2,\Omega_\varepsilon}+\nu\bigg|\nabla\frac{\chi^{(1)}}{r}\bigg|^2_{2,\Omega_\varepsilon}\le\bigg|\int\limits_{\Omega_\varepsilon}v\chi\nabla\zeta^{(1)}\frac{\chi^{(1)}}{r^2}\,dx\bigg|$$
$$-\nu\int\limits_{\Omega_\varepsilon}(2\nabla\zeta^{(1)}\nabla\chi+\chi\Delta\zeta^{(1)})\frac{\chi^{(1)}}{r^2}\,dx+2\int\limits_{\Omega_\varepsilon}\frac{w}{r}u^{(1)}\frac{\chi^{(1)}}{r^2}\,dx+\int\limits_{\Omega_\varepsilon}\frac{F_\varphi^{(1)}}{r}\frac{\chi^{(1)}}{r}\,dx.$$

Now we examine the particular terms on the r.h.s. of (3.3). We estimate the first term by

$$\varepsilon_1\bigg|\frac{\chi^{(1)}}{r}\bigg|^2_{2,\Omega_\varepsilon}+c\bigg(\frac{1}{\varepsilon_1}\bigg)\bigg|\frac{v\chi\nabla\zeta^{(1)}}{r}\bigg|^2_{6/5,\Omega_\varepsilon},$$

where the second integral is bounded by

$$c|v|^2_{2,\Omega'_{(1)}}\bigg|\frac{\chi}{r}\nabla\zeta^{(1)}\bigg|^2_{3,\Omega_\varepsilon}.$$

Integrating by parts the second term on the r.h.s. of (3.3) takes the form

$$\nu\int\limits_{\Omega_\varepsilon}\frac{|\nabla\zeta^{(1)}|^2}{r^2}\chi^2\,dx-2\nu\int\limits_{\Omega_\varepsilon}\frac{\nabla\zeta^{(1)}\zeta^{(1)}\nabla r\chi^2}{r^3}\,dx,$$

where the second integral is estimated by

$$\varepsilon_2\bigg|\frac{\chi^{(1)}}{r}\bigg|^2_{2,\Omega_\varepsilon}+c\bigg(\frac{1}{\varepsilon_2}\bigg)\bigg|\nabla\zeta^{(1)}\frac{\chi}{r^2}\bigg|^2_{2,\Omega_\varepsilon}.$$

The third term on the r.h.s. of (3.3) is bounded, by the Hölder and Young inequalities, by

$$\varepsilon_3\bigg|\frac{\chi^{(1)}}{r^{2-\varepsilon_0}}\bigg|^2_{2,\Omega_\varepsilon}+c\bigg(\frac{1}{\varepsilon_3}\bigg)\int\limits_{\Omega}\bigg|\frac{w}{r}\bigg|^2\bigg|\frac{u^{(1)}}{r^{\varepsilon_0}}\bigg|^2\,dx,$$

where $\varepsilon_0>0$ is an arbitrary small number. Further, we estimate the second integral by

$$c|w|^2_{4,-1/4,\Omega_{\varepsilon(r_0+\delta_1)}}|u^{(1)}|^2_{4,-3/4-\varepsilon_0,\Omega_\varepsilon}\le c\|w\|^2_{1,0,\Omega_{\varepsilon(r_0+\delta_1)}}|u^{(1)}|^2_{4,-3/4-\varepsilon_0,\Omega_\varepsilon}$$

Finally the last term on the r.h.s. of (3.3) is estimated by

$$\varepsilon_4\bigg|\frac{\chi^{(1)}}{r}\bigg|^2_{2,\Omega_\varepsilon}+c\bigg(\frac{1}{\varepsilon_4}\bigg)\bigg|\frac{F_\varphi^{(1)}}{r}\bigg|^2_{2,\Omega_\varepsilon}.$$

Utilizing the above estimates in (3.3) and assuming that $\varepsilon_1 - \varepsilon_4$ are sufficiently small we obtain

$$\begin{aligned}(3.4)\quad &\frac{d}{dt}\bigg|\frac{\chi^{(1)}}{r}\bigg|^2_{2,\Omega_\varepsilon} + \nu\bigg|\nabla\frac{\chi^{(1)}}{r}\bigg|^2_{2,\Omega_\varepsilon}\\ &\qquad\le c|\nabla\zeta^{(1)}|^2_{\infty,\Omega}\bigg(|v|^2_{2,\Omega'_{(1)}}\bigg|\frac{\chi}{r}\bigg|^2_{3,\Omega'_{(1)}} + \bigg|\frac{\chi}{r^2}\bigg|^2_{2,\Omega'_{(1)}}\bigg)\\ &\qquad\quad + c\|w\|^2_{1,0,\Omega_{\varepsilon(r_0+\delta_1)}}|u^{(1)}|^2_{4,-3/4-\varepsilon_0,\Omega_\varepsilon} + c\bigg|\frac{F_\varphi^{(1)}}{r}\bigg|^2_{2,\Omega_\varepsilon}.\end{aligned}$$

Integrating (3.4) with respect to time and using that

$$\bigg|\frac{\chi}{r}\bigg|^2_{2,\Omega'_{(1)}} + \bigg|\frac{\chi}{r}\bigg|^2_{3,\Omega'_{(1)}} \le \frac{c}{r_0^2}|||v|||^2_{1,3,\Omega_{r_0+\delta_1}},$$

we obtain (3.2). □

To estimate the second factor of the second term on the r.h.s. of (3.2) we localize problem (1.9) by multiplying it by $\zeta^{(1)}$. Hence we have

$$\begin{aligned}&u^{(1)}_{,t} + v\cdot\nabla u^{(1)} + \frac{v_r}{r}u^{(1)} - vu\nabla\zeta^{(1)} + v_{,z}\cdot\nabla w\zeta^{(1)} + \frac{v_{r,z}}{r}w\zeta^{(1)}\\ &\quad - \nu\Delta u^{(1)} + \nu(2\nabla u\nabla\zeta^{(1)} + u\Delta\zeta^{(1)}) + \nu\frac{u^{(1)}}{r^2} = f^{(1)}_{\varphi,z} && \text{in } \Omega^T_\varepsilon,\\ (3.5)\quad &u^{(1)}_{,r} = 0 && \text{on } S^T_{1\varepsilon},\\ &u^{(1)} = 0 && \text{on } S^T_{2\varepsilon},\\ &u^{(1)}|_{t=0} = u^{(1)}(0) && \text{in } \Omega.\end{aligned}$$

Let

$$\begin{aligned}&X_1(t) = |f^{(1)}_\varphi|_{2,-\delta,\Omega^t} + |u^{(1)}(0)|_{2,-\delta,\Omega},\\ (3.6)\quad &b_1(t) = |v|_{\infty,2,\Omega^t},\\ &b_2(t) = |v_{,x}|_{q,2,\Omega^t} + |v_{,x}|_{3,2,\Omega^t} + |v|_{3,\infty,\Omega^t},\end{aligned}$$

where $0<\delta\le 1-3/q$. Then for solutions of (3.5) we have

Lemma 3.2. *Assume that*

$$\begin{gathered}v \in L_2(0,T;W^1_q(\Omega)) \cap L_2(0,T;W^1_3(\Omega)) \cap L_\infty(0,T;L_3(\Omega)),\\ w^{(1)} \in L_\infty(0,T;H^1_0(\Omega)),\\ f^{(1)}_\varphi \in L_{2,-\delta}(\Omega^T),\quad u^{(1)}(0)\in L_{2,-\delta}(\Omega),\quad 0<\delta<1-3/q.\end{gathered}$$

Then solutions of (3.5) *satisfy the inequality*

$$\begin{aligned}(3.7)\quad &|u^{(1)}(t)|^2_{2,-\delta,\Omega_\varepsilon} + |u^{(1)}_{,x}|^2_{2,-\delta,\Omega^t_\varepsilon} + |u^{(1)}|^2_{2,-(1+\delta),\Omega^t_\varepsilon}\\ &\le c\exp(cb_1^2(t))\bigg[b_2^2\sup_t\|w^{(1)}\|^2_{1,0,\Omega_\varepsilon} + \frac{d_2^2}{r_0^{2\delta}}|\nabla\zeta^{(1)}|^2_{\infty,\Omega}(b_2^2(t)+1) + X_1^2(t)\bigg].\end{aligned}$$

Proof. Multiplying $(3.5)_1$ by $u^{(1)}r^{-2\delta}$, $\delta > 0$, and integrating over Ω_ε yields

$$
\begin{aligned}
(3.8)\quad & \frac{1}{2}\frac{d}{dt}|u^{(1)}|^2_{2,-\delta,\Omega_\varepsilon} + \nu\left(1 - \frac{\varepsilon_*}{2}\right)|u^{(1)}_{,x}|^2_{2,-\delta,\Omega_\varepsilon} + \nu\left(1 - \frac{2\delta^2}{\varepsilon_*}\right)|u^{(1)}|^2_{2,-(1+\delta),\Omega_\varepsilon} \\
&= -\int_{\Omega_\varepsilon} v\cdot\nabla u^{(1)}u^{(1)}r^{-2\delta}\,dx - \int_{\Omega_\varepsilon}\frac{v_r}{r}|u^{(1)}|^2r^{-2\delta}\,dx \\
&\quad + \int_{\Omega_\varepsilon} vu\nabla\zeta^{(1)}u^{(1)}r^{-2\delta}\,dx - \int_{\Omega_\varepsilon}\left(v_{,z}\cdot\nabla w + \frac{v_{r,z}}{r}w\right)\zeta^{(1)}u^{(1)}r^{-2\delta}\,dx \\
&\quad - \nu\int_{\Omega_\varepsilon}(2\nabla u\nabla\zeta^{(1)} + u\Delta\zeta^{(1)})u^{(1)}r^{-2\delta}\,dx + \int_{\Omega_\varepsilon} f^{(1)}_{\varphi,z}u^{(1)}r^{-2\delta}\,dx.
\end{aligned}
$$

Now we have to examine the particular terms on the r.h.s. of (3.8). We examine the first term in the way

$$
-\frac{1}{2}\int_{\Omega_\varepsilon} v\cdot\nabla|u^{(1)}|^2r^{-2\delta}\,dx = -\delta\int_{\Omega_\varepsilon} v\cdot\nabla r|u^{(1)}|^2r^{-2\delta-1}\,dx
$$

and the last expression is bounded by

$$
\varepsilon_1|u^{(1)}|^2_{2,-(1+\delta),\Omega_\varepsilon} + c(1/\varepsilon_1)|v|^2_{\infty,\Omega_{r_0+\delta_1}}|u^{(1)}|^2_{2,-\delta,\Omega_\varepsilon} \equiv I_1.
$$

The second term on the r.h.s. of (3.8) is also bounded by I_1.

The third term on the r.h.s. of (3.8) is restricted by

$$
\varepsilon_2\left|\frac{u^{(1)}}{r^\delta}\right|^2_{6,\Omega_\varepsilon} + c\left(\frac{1}{\varepsilon_2}\right)\frac{1}{r_0^{2\delta}}|\nabla\zeta^{(1)}|^2_{\infty,\Omega}|v|^2_{3,\Omega}|u|^2_{2,\Omega'_{(1)}}.
$$

We bound the fourth term on the r.h.s. of (3.8) by

$$
\begin{aligned}
\varepsilon_3|u^{(1)}|^2_{2q/(q-2),-2\delta,\Omega_\varepsilon} &+ c\left(\frac{1}{\varepsilon_3}\right)|v_{,z}|^2_{q,\Omega_{r_0+\delta_1}}\|w^{(1)}\|^2_{1,0,\Omega_\varepsilon} \\
&+ c\left|\int_{\Omega_\varepsilon} v_{,z}w\cdot\nabla\zeta^{(1)}u^{(1)}r^{-2\delta}\,dx\right| \equiv I_2,
\end{aligned}
$$

where $q\in(2,\infty)$. Choosing q sufficiently large, estimate (2.1) implies

$$
|u^{(1)}|_{2q/(q-2),-2\delta,\Omega_\varepsilon} \le c\|u^{(1)}\|_{1,-\delta,\Omega_\varepsilon},
$$

with $\delta \le 1 - 3/q$.

Finally the last integral in I_2 is bounded by

$$
\varepsilon_4\left|\frac{u^{(1)}}{r^\delta}\right|^2_{6,\Omega_\varepsilon} + c\left(\frac{1}{\varepsilon_4}\right)\frac{1}{r_0^{2\delta}}|\nabla\zeta^{(1)}|^2_{\infty,\Omega}|v_{,z}|^2_{3,\Omega}|w|^2_{2,\Omega'_{(1)}}.
$$

The fifth term on the r.h.s. of (3.8) takes the form

$$
\nu\int_{\Omega}[u^2|\nabla\zeta^{(1)}|^2r^{-2\delta} - 2\delta u^2\zeta^{(1)}\nabla\zeta^{(1)}\cdot\nabla r r^{-2\delta-1}]\,dx,
$$

where the second integral is bounded by

$$\frac{c}{r_0^{2\delta+1}}|\nabla\zeta^{(1)}|^2_{\infty,\Omega}|u|^2_{2,\Omega'_{(1)}}.$$

Integrating by parts and utilizing $(3.5)_3$, the last term on the r.h.s. of (3.8) takes the form

$$-\int_{\Omega_\varepsilon} f_\varphi^{(1)}u_{,z}^{(1)}r^{-2\delta}\,dx$$

and it is restricted, by the Hölder and Young inequalities, by

$$\varepsilon_5|u_{,z}^{(1)}|^2_{2,-\delta,\Omega_\varepsilon}+c\left(\frac{1}{\varepsilon_5}\right)|f_\varphi^{(1)}|^2_{2,-\delta,\Omega_\varepsilon}.$$

Utilizing the above estimates in (3.8) and assuming that $\varepsilon_1-\varepsilon_5$ are sufficiently small yields

$$\begin{aligned}
\frac12\frac{d}{dt}|u^{(1)}|^2_{2,-\delta,\Omega_\varepsilon}&+\left[\nu\left(1-\frac{\varepsilon_*}{2}\right)-\overline{\varepsilon}\right]|u_{,x}^{(1)}|^2_{2,-\delta,\Omega_\varepsilon}\\
&+\left[\nu\left(1-\frac{2\delta^2}{\varepsilon_*}\right)-\overline{\varepsilon}\right]|u^{(1)}|^2_{2,-(1+\delta),\Omega_\varepsilon}\\
&\le c\left(\frac{1}{\overline{\varepsilon}}\right)[|v|^2_{\infty,\Omega_{r_0+\delta_1}}|u^{(1)}|^2_{2,-\delta,\Omega_\varepsilon}+|v_{,z}|^2_{q,\Omega_{r_0+\delta_1}}\|w^{(1)}\|^2_{1,0,\Omega_\varepsilon}\\
&+\frac{1}{r_0^{2\delta}}|\nabla\zeta^{(1)}|^2_{\infty,\Omega}(|v_{,z}|^2_{3,\Omega}|w|^2_{2,\Omega'_{(1)}}+|v|^2_{3,\Omega}|u|^2_{2,\Omega'_{(1)}}+|u|^2_{2,\Omega'_{(1)}})\\
&+|f_\varphi^{(1)}|^2_{2,-\delta,\Omega_\varepsilon}],
\end{aligned}\tag{3.9}$$

where $\overline{\varepsilon}\ge\sum_{i=1}^5\varepsilon_i$.

Taking $\varepsilon_*<2$, $\delta<1$ and choosing $\overline{\varepsilon}$ sufficiently small we obtain that the coefficients near the second and third norm on the l.h.s. of (3.9) are positive. Then integrating (3.9) with respect to time yields

$$\begin{aligned}
|u^{(1)}(t)|^2_{2,-\delta,\Omega_\varepsilon}&+|u_{,x}^{(1)}|^2_{2,-\delta,\Omega_\varepsilon^t}+|u^{(1)}|^2_{2,-(1+\delta),\Omega_\varepsilon^t}\\
&\le ce^{c\|v\|^2_{\infty,2,\Omega^t_{r_0+\delta_1}}}\Big[\sup_t\|w^{(1)}\|^2_{1,0,\Omega_\varepsilon}\|v_{,z}\|^2_{q,2,\Omega^t_{r_0+\delta_1}}\\
&+\frac{1}{r_0^{2\delta}}|\nabla\zeta^{(1)}|^2_{\infty,\Omega}\Big(\sup_t|w|^2_{2,\Omega'_{(1)}}\|v_{,z}\|^2_{3,2,\Omega^t}+\|v\|^2_{3,\infty,\Omega^t}|u|^2_{2,\Omega'^t_{(1)}}\\
&+|u|^2_{2,\Omega'^t_{(1)}}\Big)+|f_\varphi^{(1)}|^2_{2,-\delta,\Omega_\varepsilon^t}+|u^{(1)}(0)|^2_{2,-\delta,\Omega_\varepsilon}\Big].
\end{aligned}$$

Utilizing that v is the weak solution to problem (1.1) we obtain (3.7). □

To estimate $\sup_t \|w^{(1)}\|_{1,0,\Omega_\varepsilon}$ in (3.7) we localize problem (1.5). Multiplying (1.5) by $\zeta^{(1)}$ we get

$$
\begin{aligned}
& w^{(1)}_{,t} + v\cdot\nabla w^{(1)} - vw\cdot\nabla\zeta^{(1)} + \frac{v_r}{r}w^{(1)} - \nu\Delta w^{(1)} \\
& \qquad + \nu(2\nabla w\nabla\zeta^{(1)} + w\Delta\zeta^{(1)}) + \nu\frac{w^{(1)}}{r^2} = f^{(1)}_\varphi \qquad \text{in } \Omega^T, \\
& w^{(1)}_{,r} = 0 \qquad \text{on } S_1^T, \\
& w^{(1)}_{,z} = 0 \qquad \text{on } S_2^T, \\
& w^{(1)}|_{t=0} = w^{(1)}(0) \qquad \text{in } \Omega.
\end{aligned} \tag{3.10}
$$

Let us introduce the quantities

$$
\begin{aligned}
X_2(t) &= \|w^{(1)}(0)\|_{1,0,\Omega} + |w^{(1)}(0)|_{2,\Omega} + |f^{(1)}_\varphi|_{2,\Omega^t}, \\
b_3(t) &= |v|_{5,\Omega^t}.
\end{aligned} \tag{3.11}
$$

Employing proofs of Lemmas 6.3.2–6.3.4 from [18] we obtain

Lemma 3.3. *Assume that v is the weak solution to problem* (1.1). *Assume that $v\in L_2(0,T;L_\infty(\Omega))\cap L_5(\Omega^T)$, $f^{(1)}_\varphi\in L_2(\Omega^T)$, $w^{(1)}(0)\in H^1_0(\Omega)$. Then solutions of* (3.10) *satisfy the following inequality*

$$
\begin{aligned}
& |w^{(1)}(t)|^2_{2,\Omega_\varepsilon} + \|w^{(1)}(t)\|^2_{1,0,\Omega_\varepsilon} + e^{-t}\int_0^t |w^{(1)}_{,x}(t')|^2_{2,\Omega_\varepsilon} e^{t'}\,dt' \\
& \quad \le ce^{cb_1^2(t)}\bigg[e^{-t}\int_0^t |w^{(1)}(t')|^2_{2,\Omega_\varepsilon}e^{t'}\,dt' \\
& \qquad + (|\nabla\zeta^{(1)}|^2_{\infty,\Omega} + |\nabla^2\zeta^{(1)}|^2_{\infty,\Omega})(d_2^2 + b_3^2(t)) + X_2^2(t)\bigg].
\end{aligned} \tag{3.12}
$$

Proof. Multiplying $(3.10)_1$ by $w^{(1)}$ and integrating over Ω_ε implies

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}|w^{(1)}|^2_{2,\Omega_\varepsilon} + \nu|\nabla w^{(1)}|^2_{2,\Omega_\varepsilon} + \nu|w^{(1)}|^2_{2,-1,\Omega_\varepsilon} \\
& \qquad - \int_{\Omega_\varepsilon} vw\nabla\zeta^{(1)}w^{(1)}\,dx + \int_{\Omega_\varepsilon}\frac{v_r}{r}|w^{(1)}|^2\,dx \\
& \qquad + \nu\int_{\Omega_\varepsilon}(2\nabla w\nabla\zeta^{(1)} + w\Delta\zeta^{(1)})w^{(1)}\,dx = \int_{\Omega_\varepsilon} f^{(1)}_\varphi w^{(1)}\,dx.
\end{aligned} \tag{3.13}
$$

The fourth term on the l.h.s. of (3.13) can be treated in the way

$$
\bigg|\int_{\Omega_\varepsilon} vw\nabla\zeta^{(1)}w^{(1)}\,dx\bigg| \le c|w|^2_{10/3,\Omega'_{(1)}}|w^{(1)}|^2_{2,\Omega_\varepsilon} + c|\nabla\zeta^{(1)}|^2_{\infty,\Omega}|v|^2_{5,\Omega'_{(1)}}.
$$

We estimate the fifth term on the l.h.s. of (3.13) by

$$\varepsilon_1|w^{(1)}|^2_{2,-1,\Omega_\varepsilon}+c(1/\varepsilon_1)|v_r|^2_{\infty,\Omega}|w^{(1)}|^2_{2,\Omega_\varepsilon}.$$

The last term on the l.h.s. of (3.13) equals $-\nu\int_{\Omega_\varepsilon}w^2|\nabla\zeta^{(1)}|^2\,dx$ so it is bounded by $c|\nabla\zeta^{(1)}|^2_{\infty,\Omega}|w|^2_{2,\Omega'_{(1)}}$. Finally the r.h.s. is estimated by

$$\varepsilon_2|w^{(1)}|^2_{2,-1,\Omega_\varepsilon}+c(1/\varepsilon_2)|f_\varphi|^2_{2,1,\Omega_\varepsilon}.$$

In view of above considerations (3.13) takes the form

$$\begin{aligned}(3.14)\quad &\frac{1}{2}\frac{d}{dt}|w^{(1)}|^2_{2,\Omega_\varepsilon}+\nu|\nabla w^{(1)}|^2_{2,\Omega_\varepsilon}+(\nu-\varepsilon_*)|w^{(1)}|^2_{2,-1,\Omega_\varepsilon}\\ &\quad\le c(|\nabla\zeta^{(1)}|^2_{\infty,\Omega}|v|^2_{5,\Omega'_{(1)}}+|w|^2_{10/3,\Omega}|w^{(1)}|^2_{2,\Omega_\varepsilon})\\ &\qquad+c\bigg(\frac{1}{\varepsilon_*}\bigg)(|v_r|^2_{\infty,\Omega}|w^{(1)}|^2_{2,\Omega_\varepsilon}+|f_\varphi|^2_{2,1,\Omega_\varepsilon})+c|\nabla\zeta^{(1)}|^2_{\infty,\Omega}|w|^2_{2,\Omega'_{(1)}},\end{aligned}$$

where $\varepsilon_*=\varepsilon_1+\varepsilon_2$. Multiplying $(3.10)_1$ by $w^{(1)}_{,t}$ and integrating over Ω_ε yields

$$\begin{aligned}&|w^{(1)}_{,t}|^2_{2,\Omega_\varepsilon}+\int\limits_{\Omega_\varepsilon}v\cdot\nabla w^{(1)}w^{(1)}_{,t}\,dx-\int\limits_{\Omega_\varepsilon}vw\nabla\zeta^{(1)}w^{(1)}_{,t}\,dx\\ &\quad+\int\limits_{\Omega_\varepsilon}\frac{v_r}{r}w^{(1)}w^{(1)}_{,t}\,dx-\nu\int\limits_{\Omega_\varepsilon}\Delta w^{(1)}w^{(1)}_{,t}\,dx+\nu\int\limits_{\Omega_\varepsilon}\frac{w^{(1)}w^{(1)}_{,t}}{r^2}\,dx\\ &\quad+\nu\int\limits_{\Omega_\varepsilon}(2\nabla w\nabla\zeta^{(1)}+w\Delta\zeta^{(1)})w^{(1)}_{,t}\,dx=\int\limits_{\Omega_\varepsilon}f^{(1)}_\varphi w^{(1)}_{,t}\,dx.\end{aligned}$$

Continuing, we have

$$\begin{aligned}(3.15)\qquad &\frac{1}{2}|w^{(1)}_{,t}|^2_{2,\Omega_\varepsilon}+\frac{\nu}{2}\frac{d}{dt}(|\nabla w^{(1)}|^2_{2,\Omega_\varepsilon}+|w^{(1)}|^2_{2,-1,\Omega_\varepsilon})\\ &\quad\le c|v|^2_{\infty,\Omega_\varepsilon}\|w^{(1)}\|^2_{1,0,\Omega_\varepsilon}+c\int\limits_{\Omega_\varepsilon}v^2w^2|\nabla\zeta^{(1)}|^2\,dx\\ &\qquad+c\int\limits_{\Omega_\varepsilon}(|\nabla w|^2|\nabla\zeta^{(1)}|^2+|w|^2|\Delta\zeta^{(1)}|^2)\,dx+c|f^{(1)}_\varphi|^2_{2,\Omega_\varepsilon}.\end{aligned}$$

Adding (3.14) and (3.15) yields

$$\begin{aligned}(3.16)\quad &\frac{d}{dt}|w^{(1)}|^2_{2,\Omega_\varepsilon}+\nu\frac{d}{dt}\|w^{(1)}\|^2_{1,0,\Omega_\varepsilon}+\nu\|w^{(1)}\|^2_{1,0,\Omega_\varepsilon}+\nu|w^{(1)}_{,x}|^2_{2,\Omega_\varepsilon}+|w^{(1)}_{,t}|^2_{2,\Omega_\varepsilon}\\ &\quad\le c|v|^2_{\infty,\Omega}\|w^{(1)}\|^2_{1,0,\Omega_\varepsilon}+c(|w|^2_{10/3,\Omega}+|v|^2_{\infty,\Omega})|w^{(1)}|^2_{2,\Omega_\varepsilon}\\ &\qquad+c|\nabla\zeta^{(1)}|^2_{\infty,\Omega}|v|^2_{5,\Omega'_{(1)}}+c|\nabla^2\zeta^{(1)}|^2_{\infty,\Omega}|w|^2_{2,\Omega'_{(1)}}\\ &\qquad+c|\nabla\zeta^{(1)}|^2_{\infty,\Omega}|\nabla w|^2_{2,\Omega'_{(1)}}+c|f^{(1)}_\varphi|^2_{2,\Omega_\varepsilon}.\end{aligned}$$

Multiplying (3.16) by $e^{t-c\int_0^t|v(t')|^2_{\infty,\Omega}dt'}$ and using that $|w|_{10/3,\Omega}\le c|v|_{\infty,\Omega}$ gives

$$
\begin{aligned}
(3.17)\quad &\frac{d}{dt}(|w^{(1)}|^2_{2,\Omega_\varepsilon}e^{t-c\int_0^t|v(t')|^2_{\infty,\Omega}\,dt'})+\nu\frac{d}{dt}(\|w^{(1)}\|^2_{1,0,\Omega_\varepsilon}e^{t-c\int_0^t|v(t')|^2_{\infty,\Omega}\,dt'})\\
&\quad+\nu|w^{(1)}_{,x}|^2_{2,\Omega_\varepsilon}e^{t-c\int_0^t|v(t')|^2_{\infty,\Omega}dt'}\\
&\le|w^{(1)}|^2_{2,\Omega_\varepsilon}e^{t-c\int_0^t|v(t')|^2_{\infty,\Omega}\,dt'}+c(|\nabla\zeta^{(1)}|^2_{\infty,\Omega}|v|^2_{5,\Omega'_{(1)}}\\
&\quad+|\nabla\zeta^{(1)}|^2_{\infty,\Omega}|\nabla w|^2_{2,\Omega'_{(1)}}+|\nabla^2\zeta^{(1)}|^2_{\infty,\Omega}|w|^2_{2,\Omega'_{(1)}})e^{t-c\int_0^t|v(t')|^2_{\infty,\Omega}\,dt'}\\
&\quad+c|f^{(1)}_\varphi|^2_{2,\Omega_\varepsilon}e^{t-c\int_0^t|v(t')|^2_{\infty,\Omega}\,dt'}.
\end{aligned}
$$

Integrating (3.17) with respect to time yields

$$
\begin{aligned}
&|w^{(1)}(t)|^2_{2,\Omega_\varepsilon}e^{t-c\int_0^t|v(t')|^2_{\infty,\Omega}\,dt'}+\nu\|w^{(1)}(t)\|^2_{1,0,\Omega_\varepsilon}e^{t-c\int_0^t|v(t')|^2_{\infty,\Omega}\,dt'}\\
&\qquad+\nu\int_0^t|w^{(1)}_{,x}(t')|^2_{2,\Omega_\varepsilon}e^{t'-c\int_0^{t'}|v(t'')|^2_{\infty,\Omega}\,dt''}\,dt'\\
&\quad\le c\int_0^t|w^{(1)}(t')|^2_{2,\Omega_\varepsilon}e^{t'-c\int_0^{t'}|v(t'')|^2_{\infty,\Omega}dt''}\,dt'+c(|\nabla\zeta^{(1)}|^2_{\infty,\Omega}\\
&\qquad+|\nabla^2\zeta^{(1)}|^2_{\infty,\Omega})\int_0^t(|v|^2_{5,\Omega'_{(1)}}+\|w\|^2_{1,\Omega'_{(1)}})e^{t'-c\int_0^{t'}|v(t'')|^2_{\infty,\Omega}dt''}\,dt'\\
&\qquad+c\int_0^t|f_\varphi(t')|^2_{2,\Omega_\varepsilon}e^{t'-c\int_0^{t'}|v(t'')|^2_{\infty,\Omega}dt''}\,dt'+|w^{(1)}(0)|^2_{2,\Omega_\varepsilon}+\nu\|w^{(1)}(0)\|^2_{1,0,\Omega_\varepsilon}.
\end{aligned}
$$

Simplifying we obtain

$$
\begin{aligned}
(3.18)\quad &|w^{(1)}(t)|^2_{2,\Omega_\varepsilon}+\nu\|w^{(1)}(t)\|^2_{1,0,\Omega_\varepsilon}+\nu e^{-t}\int_0^t|w^{(1)}_{,x}(t')|^2_{2,\Omega_\varepsilon}e^{t'}\,dt'\\
&\le ce^{c\|v\|^2_{\infty,2,\Omega^t}}\bigg[e^{-t}\int_0^t|w^{(1)}(t')|^2_{2,\Omega_\varepsilon}e^{t'}\,dt'\\
&\quad+(|\nabla\zeta^{(1)}|^2_{\infty,\Omega}+|\nabla^2\zeta^{(1)}|^2_{\infty,\Omega})(|v|_{5,\Omega'^t_{(1)}}+|w_{,x}|^2_{2,\Omega'^t_{(1)}}+|w|^2_{2,\Omega'^t_{(1)}})\\
&\quad+|f_\varphi|^2_{2,\Omega^t_\varepsilon}+|w^{(1)}(0)|^2_{2,\Omega_\varepsilon}+\|w^{(1)}(0)\|^2_{1,0,\Omega_\varepsilon}\bigg].
\end{aligned}
$$

Utilizing $(3.6)_2$ and (3.11) in (3.18) implies (3.12). □

Let

$$
(3.19)\qquad b_4(t)=\|v\|_{3,2,\Omega^T}.
$$

To estimate the first term on the r.h.s. of (3.12) we need

Lemma 3.4. *Assume that v is a weak solution to problem* (1.1). *Assume that $v \in L_2(0,T;L_\infty(\Omega)) \cap L_4(0,T;L_3(\Omega))$, $f_\varphi^{(1)} \in L_2(\Omega^T)$, $w^{(1)}(0) \in L_2(\Omega)$. Then solutions of problem* (3.10) *satisfy*

$$
\begin{aligned}
(3.20)\quad & |w^{(1)}(t)|^2_{2,\Omega_\varepsilon} + \nu\|w^{(1)}(t)\|^2_{1,0,\Omega_\varepsilon} \\
&\qquad \le c\exp(cb_1^2(t))[|\nabla\zeta^{(1)}|^2_{\infty,\Omega}(b_4^2(t)+1)d_2^2 + X_2^2(t)].
\end{aligned}
$$

Proof. Multiplying $(3.10)_1$ by $w^{(1)}$ and integrating over Ω_ε implies

$$
\begin{aligned}
(3.21)\quad & \frac{1}{2}\frac{d}{dt}|w^{(1)}|^2_{2,\Omega_\varepsilon} + \nu(|\nabla w^{(1)}|^2_{2,\Omega_\varepsilon} + |w^{(1)}|^2_{2,-1,\Omega_\varepsilon}) \\
&= -\int_{\Omega_\varepsilon} \frac{v_r}{r}|w^{(1)}|^2\,dx + \int_{\Omega_\varepsilon} vw\nabla\zeta^{(1)}w^{(1)}\,dx \\
&\quad - \nu\int_{\Omega_\varepsilon}(2\nabla w\nabla\zeta^{(1)} + w\Delta\zeta^{(1)})w^{(1)}\,dx + \int_{\Omega_\varepsilon} f_\varphi^{(1)}w^{(1)}\,dx.
\end{aligned}
$$

We estimate the first term on the r.h.s. by

$$
\varepsilon_1|w^{(1)}|^2_{2,-1,\Omega_\varepsilon} + c(1/\varepsilon_1)|v_r|^2_{\infty,\Omega}|w^{(1)}|^2_{2,\Omega_\varepsilon}.
$$

The second term on the r.h.s. of (3.21) is bounded by

$$
\varepsilon_2|w^{(1)}|^2_{6,\Omega_\varepsilon} + c(1/\varepsilon_2)|\nabla\zeta^{(1)}|^2_{\infty,\Omega}|v|^2_{3,\Omega'_{(1)}}|w|^2_{2,\Omega'_{(1)}}.
$$

The third term on the r.h.s. of (3.21) takes the form

$$
\nu\int_\Omega w^2|\nabla\zeta^{(1)}|^2\,dx \le c|\nabla\zeta^{(1)}|^2_{\infty,\Omega}|w|^2_{2,\Omega'_{(1)}}.
$$

Finally, the last term on the r.h.s. of (3.21) is bounded by

$$
\varepsilon_3|w^{(1)}|^2_{2,-1,\Omega_\varepsilon} + c(1/\varepsilon_3)|f_\varphi^{(1)}|^2_{2,1,\Omega_\varepsilon}.
$$

Utilizing the above estimates in (3.21) and assuming that $\varepsilon_1 - \varepsilon_3$ are sufficiently small we obtain

$$
\begin{aligned}
(3.22)\quad & \frac{d}{dt}|w^{(1)}|^2_{2,\Omega_\varepsilon} + \nu|\nabla w^{(1)}|^2_{2,\Omega_\varepsilon} + \nu|w^{(1)}|^2_{2,-1,\Omega_\varepsilon} \le c|v_r|^2_{\infty,\Omega}|w^{(1)}|^2_{2,\Omega_\varepsilon} \\
&\quad + c|\nabla\zeta^{(1)}|^2_{\infty,\Omega}(|v|^2_{3,\Omega'_{(1)}}|w|^2_{2,\Omega'_{(1)}} + |w|^2_{2,\Omega'_{(1)}}) + c|f_\varphi^{(1)}|^2_{2,1,\Omega_\varepsilon}.
\end{aligned}
$$

Integrating (3.22) with respect to time yields

$$
\begin{aligned}
(3.23)\quad & |w^{(1)}(t)|^2_{2,\Omega_\varepsilon} + \nu\|w^{(1)}\|^2_{1,0,\Omega_\varepsilon} \le c\exp(c|v|^2_{\infty,2,\Omega^t}) \\
&\quad \cdot [|\nabla\zeta^{(1)}|^2_{\infty,\Omega}(|v|^2_{3,2,\Omega^t}+1)|w|^2_{2,\Omega^t} + |f_\varphi^{(1)}|^2_{2,1,\Omega^t} + |w^{(1)}(0)|^2_{2,\Omega}].
\end{aligned}
$$

In view of $(3.6)_2$, $(3.11)_1$, (3.19) and the fact v is the weak solution we obtain (3.20) from (3.23). $\square$

We estimate the first term under the square brackets on the r.h.s. of (3.12) by

$$e^{-t}\sup_t |w^{(1)}(t)|^2_{2,\Omega_\varepsilon}\int\limits_0^t e^{t'}\,dt' \le \sup_{t'\le t}|w^{(1)}(t')|^2_{2,\Omega_\varepsilon} \le c\exp(cb_1^2(t))[|\nabla\zeta^{(1)}|^2_{\infty,\Omega}(1+b_4^2(t))d_2^2+X_2^2(t)], \tag{3.24}$$

where (3.20) was employed. Utilizing (3.24) in (3.12) yields

$$|w^{(1)}(t)|^2_{2,\Omega_\varepsilon}+\|w^{(1)}(t)\|^2_{1,0,\Omega_\varepsilon} \le ce^{cb_1^2}[|\nabla\zeta^{(1)}|^2_{\infty,\Omega}+|\nabla^2\zeta^{(1)}|^2_{\infty,\Omega})\cdot(d_2^2+b_3^2+b_4^2d_2^2)+X_2^2]. \tag{3.25}$$

Repeating the considerations leading to (3.25) for the function $w^{(3)}=w\zeta^{(3)}$ we obtain

$$|w^{(3)}(t)|^2_{2,\Omega_\varepsilon}+\|w^{(3)}(t)\|^2_{1,0,\Omega_\varepsilon} \le ce^{cb_1^2}[(|\nabla\zeta^{(3)}|^2_{\infty,\Omega}+|\nabla^2\zeta^{(3)}|^2_{\infty,\Omega})(d_2^2+b_3^2+b_4^2d_2^2)+X_3^2], \tag{3.26}$$

where $X_3(t)=\|w^{(3)}(0)\|_{1,0,\Omega}+|w^{(3)}(0)|_{2,\Omega}+|f_\varphi^{(3)}|_{2,\Omega^t}$. From (3.25) and (3.26) we have

$$\sum_{i=1,3}(|w^{(i)}(t)|^2_{2,\Omega_\varepsilon}+\|w^{(i)}(t)\|^2_{1,0,\Omega_\varepsilon}) \le ce^{cb_1^2}\Big[\sum_{i=1,3}(|\nabla\zeta^{(i)}|^2_{\infty,\Omega}+|\nabla^2\zeta^{(i)}|^2_{\infty,\Omega})(d_2^2+b_3^2+b_4^2d_2^2)+X_2^2+X_3^2\Big]. \tag{3.27}$$

Using that

$$\sup_t\|w(t)\|_{1,0,\Omega_{\varepsilon(r_0+\delta_1)}}\le\sup_t\|w^{(3)}(t)\|_{1,0,\Omega_\varepsilon}$$

and applying (3.27) in (3.2) we get

$$\begin{aligned}\Big|\frac{\chi^{(1)}(t)}{r}\Big|^2_{2,\Omega_\varepsilon}+\nu\int\limits_0^t\Big|\nabla\frac{\chi^{(1)}(t')}{r}\Big|^2_{2,\Omega_\varepsilon}dt' &\le \frac{c}{r_0^2}|\nabla\zeta^{(1)}|^2_{\infty,\Omega}b_5^4(t)\\ &\quad+ce^{cb_1^2}[\sum_{i=1,3}(|\nabla\zeta^{(i)}|^2_{\infty,\Omega}+|\nabla^2\zeta^{(i)}|^2_{\infty,\Omega})(d_2^2+b_3^2+b_4^2d_2^2)+X_2^2+X_3^2]\\ &\quad\cdot\int\limits_0^t|u^{(1)}(t')|^2_{4,-3/4-\varepsilon_0,\Omega_\varepsilon}dt'+cY_1^2(t),\end{aligned} \tag{3.28}$$

where $b_5(t)=\|v\|_{L_\infty(0,t;L_2(\Omega))}+\|v\|_{L_2(0,T;W_3^1(\Omega))}$.

Finally, we have to estimate the expression

$$I=\int\limits_0^t|u^{(1)}(t')|^2_{4,-3/4-\varepsilon_0,\Omega_\varepsilon}dt'.$$

Using imbedding (2.1) in the form $|u|_{4,-(\delta+1/4),\Omega} \le c\|u\|_{1,-\delta,\Omega}$, we obtain for $\delta = 1/2+\varepsilon_0$, $\varepsilon_0 > 0$ arbitrary small,

$$|u^{(1)}|_{4,-(3/4+\varepsilon_0),\Omega_\varepsilon} \le c\|u^{(1)}\|_{1,-(1/2+\varepsilon_0),\Omega_\varepsilon}.$$

Hence

$$I \le c\int_0^t \|u^{(1)}(t')\|^2_{1,-(1/2+\varepsilon_0),\Omega_\varepsilon}\,dt' \equiv I_1,$$

which is estimated by (3.7) with $\delta = 1/2+\varepsilon_0$. Then $1/2+\varepsilon_0 = 1-3/q$, so $q = 6/(1-3\varepsilon_0)$.

Now, in view of (3.7) and (3.20) we have

$$I_1 \le c\exp(cb_1^2)[|\nabla\zeta^{(1)}|^2_{\infty,\Omega}d_2^2(1+b_2^2+b_2^2b_4^2)+b_2^2X_2^2+X_1^2].$$

Utilizing the above estimates in (3.28) yields

$$\begin{aligned}(3.29)\quad &\left|\frac{\chi^{(1)}(t)}{r}\right|^2_{2,\Omega_\varepsilon}+\nu\int_0^t\left|\nabla\frac{\chi^{(1)}(t')}{r}\right|^2_{2,\Omega_\varepsilon}dt' \le \frac{c}{r_0^2}|\nabla\zeta^{(1)}|^2_{\infty,\Omega}b_5^4\\ &+c\exp(cb_1^2)\left[\sum_{i=1,3}(|\nabla\zeta^{(i)}|^2_{\infty,\Omega}+|\nabla^2\zeta^{(i)}|^2_{\infty,\Omega})(d_2^2+b_3^2+b_4^2d_2^2)+X_2^2+X_3^2\right]\\ &\qquad\cdot[|\nabla\zeta^{(1)}|^2_{\infty,\Omega}d_2^2(1+b_2^2+b_2^2b_4^2)+b_2^2X_2^2+X_1^2]+cY_1^2.\end{aligned}$$

Now we shall obtain estimates for the vorticity vector. First we localize problem (1.7),

$$\begin{aligned}(3.30)\quad &\alpha^{(1)}_{r,t}+v\cdot\nabla\alpha_r^{(1)}-v\cdot\nabla\alpha_r\nabla\zeta^{(1)}-(\alpha_r^{(1)}v_{r,r}+\alpha_z^{(1)}v_{r,z})\\ &\quad-\nu\Delta\alpha_r^{(1)}+\nu\frac{\alpha_r^{(1)}}{r^2}+\nu(2\nabla\alpha_r\nabla\zeta^{(1)}+\alpha_r\Delta\zeta^{(1)}) = F_r^{(1)} &&\text{in } \Omega^T,\\ &\alpha_r^{(1)} = 0 &&\text{on } S^T,\\ &\alpha_r^{(1)}|_{t=0} = \alpha_r^{(1)}(0) &&\text{in } \Omega.\end{aligned}$$

Next, localizing (1.8) we have

$$\begin{aligned}(3.31)\quad &\alpha^{(1)}_{z,t}+v\cdot\nabla\alpha_z^{(1)}-v\alpha_z\nabla\zeta^{(1)}-(\alpha_r^{(1)}v_{z,r}+\alpha_z^{(1)}v_{z,z})\\ &\quad-\nu\Delta\alpha_z^{(1)}+\nu(2\nabla\alpha_z\nabla\zeta^{(1)}+\alpha_z\Delta\zeta^{(1)}) = F_z^{(1)} &&\text{in } \Omega^T,\\ &\alpha_z^{(1)} = 0 &&\text{on } S_1^T,\\ &\alpha_{z,z}^{(1)} = 0 &&\text{on } S_2^T,\\ &\alpha_z^{(1)}|_{t=0} = \alpha_z^{(1)}(0) &&\text{in } \Omega.\end{aligned}$$

For solutions of problems (3.30) and (3.31) we obtain

Lemma 3.5. *Assume that* $\alpha'=(\alpha_r,\alpha_z)$, $F'=(F_r,F_z)$, $v\in L_2(0,T;L_\infty(\Omega))$, $v_{,x}\in L_2(0,T;L_3(\Omega))$, $F^{(1)'}\in L_2(\Omega^T)$, $\alpha^{(1)'}(0)\in L_2(\Omega)$. *Then*

$$
\begin{aligned}
(3.32)\quad & |\alpha^{(1)'}(t)|^2_{2,\Omega_\varepsilon}+\nu|\alpha^{(1)'}_{,x}|^2_{2,\Omega^t_\varepsilon}+\nu|\alpha^{(1)}_r|^2_{2,-1,\Omega^t_\varepsilon}\\
&\le c\exp(c|v_{,x}|^2_{3,2,\Omega^t}+c|v|^2_{\infty,2,\Omega^t})[d_1^2|\nabla\zeta^{(1)}|^2_{\infty,\Omega}|\alpha'|^2_{3,2,\Omega'^t_{(1)}}\\
&\quad+|\nabla\zeta^{(1)}|^2_{\infty,\Omega}|\alpha'|^2_{2,\Omega'^t_{(1)}}+|F^{(1)'}|^2_{2,\Omega^t_\varepsilon}+|\alpha^{(1)'}(0)|^2_{2,\Omega_\varepsilon}].
\end{aligned}
$$

Proof. Multiplying (3.30) by $\alpha^{(1)}_r$ and integrating over Ω_ε gives

$$
\begin{aligned}
(3.33)\quad & \frac12\frac{d}{dt}|\alpha^{(1)}_r|^2_{2,\Omega_\varepsilon}+\nu|\alpha^{(1)}_{r,x}|^2_{2,\Omega_\varepsilon}+\nu|\alpha^{(1)}_r|^2_{2,-1,\Omega_\varepsilon}\\
&=\int_{\Omega_\varepsilon} v\alpha_r\nabla\zeta^{(1)}\alpha^{(1)}_r\,dx+\int_{\Omega_\varepsilon}(\alpha^{(1)}_r v_{r,r}+\alpha^{(1)}_z v_{r,z})\alpha^{(1)}_r\,dx\\
&\quad-\nu\int_{\Omega_\varepsilon}(2\nabla\alpha_r\nabla\zeta^{(1)}+\alpha_r\Delta\zeta^{(1)})\alpha^{(1)}_r\,dx+\int_{\Omega_\varepsilon}F^{(1)}_r\alpha^{(1)}_r\,dx.
\end{aligned}
$$

We estimate the first term on the r.h.s. of (3.33) by

$$\varepsilon_1|\alpha^{(1)}_r|^2_{6,\Omega_\varepsilon}+c(1/\varepsilon_1)|\nabla\zeta^{(1)}|^2_{\infty,\Omega}|v|^2_{2,\Omega}|\alpha_r|^2_{3,\Omega'_{(1)}},$$

and the second by

$$\varepsilon_2|\alpha^{(1)}_r|^2_{6,\Omega_\varepsilon}+c(1/\varepsilon_2)|v_{,x}|^2_{3,\Omega}|\alpha^{(1)'}|^2_{2,\Omega_\varepsilon}.$$

The third term on the r.h.s. of (3.33) equals

$$\nu\int_{\Omega_\varepsilon}\alpha_r^2|\nabla\zeta^{(1)}|^2\,dx.$$

Finally, we restrict the last term on the r.h.s. of (3.33) by

$$\varepsilon_3|\alpha^{(1)}_r|^2_{2,\Omega_\varepsilon}+c(1/\varepsilon_3)|F^{(1)}_r|^2_{2,\Omega_\varepsilon}.$$

Utilizing the above estimates in (3.33), assuming that $\varepsilon_1-\varepsilon_3$ are sufficiently small and applying the Poincaré inequality we obtain

$$
\begin{aligned}
(3.34)\quad & \frac{d}{dt}|\alpha^{(1)}_r|^2_{2,\Omega_\varepsilon}+\nu\|\alpha^{(1)}_r\|^2_{1,0,\Omega_\varepsilon}\le c(|\nabla\zeta^{(1)}|^2_{\infty,\Omega}|v|^2_{2,\Omega}|\alpha_r|^2_{3,\Omega'_{(1)}}\\
&\quad+|v_{,x}|^2_{3,\Omega}|\alpha^{(1)'}|^2_{2,\Omega_\varepsilon}+|\nabla\zeta^{(1)}|^2_{\infty,\Omega}|\alpha_r|^2_{2,\Omega'_{(1)}}+|F^{(1)}_r|^2_{2,\Omega_\varepsilon}).
\end{aligned}
$$

Next we examine problem (3.31). Multiplying $(3.31)_1$ by $\alpha_z^{(1)}$ and integrating over Ω_ε yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&|\alpha_z^{(1)}|^2_{2,\Omega_\varepsilon} + \nu|\alpha_{z,x}^{(1)}|^2_{2,\Omega_\varepsilon}\\ &= \int\limits_{\Omega_\varepsilon} v\alpha_z\nabla\zeta^{(1)}\alpha_z^{(1)}\,dx + \int\limits_{\Omega_\varepsilon}(\alpha_r^{(1)}v_{z,r} + \alpha_z^{(1)}v_{z,z})\alpha_z^{(1)}\,dx\\ &\quad - \nu\int\limits_{\Omega_\varepsilon}(2\nabla\alpha_z\nabla\zeta^{(1)} + \alpha_z\Delta\zeta^{(1)})\alpha_z^{(1)}\,dx + \int\limits_{\Omega_\varepsilon} F_z^{(1)}\alpha_z^{(1)}\,dx.\end{aligned}$$

Repeating the similar considerations leading to (3.34), with difference

$$\int_{\Omega_\varepsilon}(\alpha_z^{(1)})^2 v_{z,z}\,dx = -2\int_{\Omega_\varepsilon}\alpha_z^{(1)}\alpha_{z,z}^{(1)}v_z\,dx \equiv I,$$

where

$$|I| \le \varepsilon_1|\alpha_{z,z}^{(1)}|^2_{2,\Omega_\varepsilon} + c(1/\varepsilon_1)|v_z|^2_{\infty,\Omega}|\alpha_z^{(1)}|^2_{2,\Omega_\varepsilon}$$

we obtain

$$\begin{aligned}\frac{d}{dt}|\alpha_z^{(1)}|^2_{2,\Omega_\varepsilon} + \nu|\alpha_{z,x}^{(1)}|^2_{2,\Omega_\varepsilon} &\le c(|\nabla\zeta^{(1)}|^2_{\infty,\Omega}|v|^2_{2,\Omega}|\alpha_z|^2_{3,\Omega'_{(1)}}\\ &+ \varepsilon_1|\alpha_r^{(1)}|^2_{6,\Omega_\varepsilon} + c(1/\varepsilon_1)(|v_{z,r}|^2_{3,\Omega} + |v_z|^2_{\infty,\Omega})|\alpha_z^{(1)}|^2_{2,\Omega_\varepsilon}\\ &+ |\nabla\zeta^{(1)}|^2_{\infty,\Omega}|\alpha_z|^2_{2,\Omega_\varepsilon} + |F_z^{(1)}|^2_{2,\Omega_\varepsilon}).\end{aligned}\tag{3.35}$$

Summing up inequalities (3.34) and (3.35) and assuming that ε is sufficiently small yields

$$\begin{aligned}\frac{d}{dt}&|\alpha^{(1)'}|^2_{2,\Omega_\varepsilon} + \nu|\alpha_{,x}^{(1)'}|^2_{2,\Omega_\varepsilon} + \nu|\alpha_r^{(1)}|^2_{2,-1,\Omega_\varepsilon}\\ &\le c(|\nabla\zeta^{(1)}|^2_{\infty,\Omega}|v|^2_{2,\Omega}|\alpha'|^2_{3,\Omega'_{(1)}} + (|v_{,x}|^2_{3,\Omega} + |v|^2_{\infty,\Omega})|\alpha^{(1)'}|^2_{2,\Omega_\varepsilon}\\ &\quad + |\nabla\zeta^{(1)}|^2_{\infty,\Omega}|\alpha'|^2_{2,\Omega'_{(1)}} + |F^{(1)'}|^2_{2,\Omega_\varepsilon}).\end{aligned}\tag{3.36}$$

Integrating (3.36) with respect to time implies (3.32). □

Utilizing (3.19), $(3.6)_2$ and assuming

$$X_4(t) = |F^{(1)'}(t)|_{2,\Omega^t} + |\alpha^{(1)'}(0)|_{2,\Omega}$$

we simplify (3.32) to the following form

$$\begin{aligned}|\alpha^{(1)'}(t)|^2_{2,\Omega_\varepsilon} &+ \nu|\alpha_{,x}^{(1)'}|^2_{2,\Omega_\varepsilon^t} + \nu|\alpha_r^{(1)}|^2_{2,-1,\Omega_\varepsilon^t}\\ &\le c\exp(c(b_1^2+b_4^2))[|\nabla\zeta^{(1)}|^2_{\infty,\Omega}b_4^2(1+d_1^2) + X_4^2].\end{aligned}\tag{3.37}$$

Finally, we localize problem (1.6)

$$\begin{aligned}&\operatorname{rot} v^{(1)} = v\times\nabla\zeta^{(1)} + \alpha^{(1)} && \text{in } \Omega_\varepsilon,\\ &\operatorname{div} v^{(1)} = v\cdot\nabla\zeta^{(1)} && \text{in } \Omega_\varepsilon,\\ &\overline{n}\cdot v^{(1)}|_S = 0 && \text{on } S_\varepsilon.\end{aligned}\tag{3.38}$$

For solutions of problem (3.38) we have

$$\|v^{(1)}\|_{V_2^1(\Omega_\varepsilon^t)} \le c(d_2(t) + \|\alpha^{(1)}\|_{V_2^0(\Omega_\varepsilon^t)}). \tag{3.39}$$

Now we have to find an estimate for the r.h.s. of (3.39). Simplifying (3.29) yields

$$\begin{aligned}\left|\frac{\chi^{(1)}(t)}{r}\right|_{2,\Omega_\varepsilon}^2 + \nu\int_0^t \left|\nabla\frac{\chi^{(1)}(t')}{r}\right|_{2,\Omega_\varepsilon}^2 dt' \\ \le \varphi(b,d_1,d_2)\left[\frac{1}{\delta_1^2} + X_1^2 + X_2^2 + X_3^2\right]^2 + cY_1^2,\end{aligned} \tag{3.40}$$

$b = b_1 + b_2 + b_3 + b_4$, and (3.37) takes the form

$$\|\alpha^{(1)'}\|_{V_2^1(\Omega_\varepsilon^t)}^2 \le \varphi(b,d_1)\left[\frac{1}{\delta_1^2} + X_4^2\right]. \tag{3.41}$$

We need an additional estimate for $\chi^{(1)}$. Multiplying $(3.1)_1$ by $\chi^{(1)}$ and integrating over Ω_ε^t we have

$$\begin{aligned}&\frac{1}{2}|\chi^{(1)}|_{2,\Omega_\varepsilon}^2 + \nu|\nabla\chi^{(1)}|_{2,\Omega_\varepsilon^t}^2 + \nu\left|\frac{\chi^{(1)}}{r}\right|_{2,\Omega_\varepsilon^t}^2 \\ &\le \left|\int_{\Omega_\varepsilon^t}\frac{v_r}{r}(\chi^{(1)})^2\,dx\,dt'\right| + \left|\int_{\Omega_\varepsilon^t}\chi v\cdot\nabla\zeta^{(1)}\chi^{(1)}\,dx\,dt'\right| \\ &\quad + \nu\left|\int_{\Omega_\varepsilon^t}(2\nabla\zeta^{(1)}\nabla\chi + \chi\Delta\zeta^{(1)})\chi^{(1)}\,dx\,dt'\right| + \left|\int_{\Omega^t}\frac{2wu^{(1)})}{r}\chi^{(1)}\,dx\,dt'\right| \\ &\quad + \left|\int_{\Omega^t}F_\varphi^{(1)}\chi^{(1)}\,dx\,dt'\right| + \frac{1}{2}|\chi^{(1)}(0)|_{2,\Omega_\varepsilon}^2.\end{aligned} \tag{3.42}$$

We estimate the first term on the r.h.s. of (3.42) by

$$\begin{aligned}\int_0^t dt'\left|\int_{\Omega_\varepsilon}\frac{v_r}{r}(\chi^{(1)})^2\,dx\right| &\le \int_0^t |v_r|_{2,\Omega_\varepsilon}|\chi^{(1)}|_{6,\Omega_\varepsilon}\left|\frac{\chi^{(1)}}{r}\right|_{3,\Omega_\varepsilon} \\ &\le \sup_t |v_r|_{2,\Omega_\varepsilon}\int_0^t |\chi^{(1)}|_{6,\Omega_\varepsilon}\left|\frac{\chi^{(1)}}{r}\right|_{3,\Omega_\varepsilon} dt' \\ &\le \varepsilon_1\int_0^t |\chi^{(1)}|_{6,\Omega_\varepsilon}^2\,dt' + c\left(\frac{1}{\varepsilon_1}\right)d_1^2\int_0^t \left\|\frac{\chi^{(1)}(t')}{r}\right\|_{1,\Omega_\varepsilon}^2 dt'.\end{aligned}$$

The second term on the r.h.s. of (3.42) is bounded by

$$|\nabla\zeta^{(1)}|_{\infty,\Omega}\int\limits_{\Omega_\varepsilon^t}|\chi v\chi^{(1)}|\,dx\,dt'\le|\nabla\zeta^{(1)}|_{\infty,\Omega}\sup_t|v|_{2,\Omega_\varepsilon}\int\limits_0^t|\chi|_{3,\Omega_\varepsilon}|\chi^{(1)}|_{6,\Omega_\varepsilon}\,dt'$$
$$\le\varepsilon_2\int\limits_0^t|\chi^{(1)}(t')|^2_{6,\Omega_\varepsilon}\,dt'+c\Big(\frac{1}{\varepsilon_2}\Big)d_1^2|\nabla\zeta^{(1)}|^2_{\infty,\Omega}\int\limits_0^t|\chi|^2_{3,\Omega_\varepsilon}\,dt'.$$

The third term on the r.h.s. of (3.42) equals

$$\nu\int_{\Omega_\varepsilon^t}|\nabla\zeta^{(1)}|^2\chi^2\,dx\,dt'.$$

Integrating by parts in the fourth term on the r.h.s. of (3.42) yields

$$\bigg|\int_{\Omega_\varepsilon^t}\frac{w}{r}u^{(1)}\chi^{(1)}\,dx\,dt'\bigg|\le\varepsilon_3|\chi^{(1)}|^2_{6,2,\Omega_\varepsilon^t}+c(1/\varepsilon_3)\sup_t\bigg|\frac{w}{r}\bigg|^2_{2,\Omega_\varepsilon}|u^{(1)}|^2_{3,2,\Omega_\varepsilon^t}.$$

Finally, we estimate the fifth term on the r.h.s. of (3.42) by

$$\varepsilon_4|\chi^{(1)}|^2_{2,\Omega_\varepsilon^t}+c(1/\varepsilon_4)|F_\varphi^{(1)}|^2_{2,\Omega_\varepsilon^t}.$$

Utilizing the above estimates in (3.42) and assuming that $\varepsilon_1-\varepsilon_4$ are sufficiently small implies

$$|\chi^{(1)}(t)|^2_{2,\Omega_\varepsilon}+\nu|\nabla\chi^{(1)}|^2_{2,\Omega_\varepsilon^t}+\nu\bigg|\frac{\chi^{(1)}}{r}\bigg|^2_{2,\Omega_\varepsilon^t}\tag{3.43}$$
$$\le cd_1^2\int\limits_0^t\bigg\|\frac{\chi^{(1)}(t')}{r}\bigg\|^2_{1,\Omega_\varepsilon}dt'+c(d_1^2+d_2^2)|\nabla\zeta^{(1)}|^2_{\infty,\Omega}$$
$$+c\sup_t\bigg|\frac{w}{r}\bigg|^2_{2,\Omega_\varepsilon}|u^{(1)}|^2_{3,2,\Omega_\varepsilon^t}+c|F_\varphi^{(1)}|^2_{2,\Omega_\varepsilon^t}+|\chi^{(1)}(0)|^2_{2,\Omega_\varepsilon}.$$

In view of (3.7), (3.26) and (3.40) inequality (3.43) takes the form

$$|\chi^{(1)}(t)|^2_{2,\Omega_\varepsilon}+\nu|\nabla\chi^{(1)}|^2_{2,\Omega_\varepsilon^t}+\nu|\chi^{(1)}/r|^2_{2,\Omega_\varepsilon^t}\le\varphi(b,d_1,d_2)[1/\delta_1^2+X_1^2+X_2^2+X_3^2]+cY_1^2.\tag{3.44}$$

Finally, from (3.41) and (3.44) we obtain

$$\|\alpha^{(1)}\|^2_{V_2^0(\Omega_\varepsilon^t)}\le\varphi(b,d_1,d_2)[1/\delta_1^2+X_1^2+X_2^2+X_3^2+X_4^2]+cY_1^2.\tag{3.45}$$

Since the r.h.s. of (3.45) does not depend on ε we can pass with ε to 0 in (3.45) and also in (3.39). Then by applying the properties of the cutoff functions we obtain from (3.39) and (3.45) the inequality

$$|v^{(1)}|_{10,\Omega^t}+|\nabla v^{(1)}|_{10/3,\Omega^t}\le c\|v^{(1)}\|_{V_2^1(\Omega^t)}\le\varphi(b,d_1,d_2)(1/\delta_1+X_1+X_2+X_3+X_4)+c(d_2+Y_1).\tag{3.46}$$

4. Estimates in a positive distance from the axis of symmetry

In this section we follow the considerations from [19]. First we examine problem (1.4). Multiplying (1.4) by $(\zeta^{(2)})^2$ we obtain

$$
\begin{aligned}
&\chi^{(2)'}_{,t} + v\cdot\nabla\chi^{(2)'} - \frac{v_r}{r}\chi^{(2)'} - \nu\Delta\chi^{(2)'} + \nu\frac{\chi^{(2)'}}{r^2} = v\chi\nabla(\zeta^{(2)})^2 \\
&\qquad - \nu(2\nabla(\zeta^{(2)})^2\nabla\chi + \chi\Delta(\zeta^{(2)})^2) + \frac{2}{r}w^{(2)}w^{(2)}_{,z} + F^{(2)'}_\varphi \qquad \text{in } \Omega^T, \\
&\chi^{(2)'} = 0 \qquad \text{on } S^T, \\
&\chi^{(2)'}|_{t=0} = \chi^{(2)'}(0) \qquad \text{in } \Omega,
\end{aligned}
\tag{4.1}
$$

where $\chi^{(2)'} = \chi(\zeta^{(2)})^2$, $F^{(2)'} = F(\zeta^{(2)})^2$, $w^{(2)} = w\zeta^{(2)}$. Let

$$
Y_2(t) = |F^{(2)}_\varphi|_{2,\Omega^t} + \left|\frac{\chi^{(2)}(0)}{r}\right|_{2,\Omega}, \qquad b_6(t) = \|v\|_{L_2(0,t;W^1_3(\Omega))}.
\tag{4.2}
$$

Lemma 4.1. *Assume that v is the weak solution to problem* (1.1). *Assume that $v\in L_2(0,T;W^1_3(\Omega))$, $F^{(2)}_\varphi\in L_2(\Omega^T)$, $\chi^{(2)}(0)\in L_2(\Omega)$. Then*

$$
\begin{aligned}
&\left|\frac{\chi^{(2)'}(t)}{r}\right|^2_{2,\Omega} + \nu\int_0^t\left\|\frac{\chi^{(2)'}(t')}{r}\right\|^2_{1,\Omega}dt' \\
&\qquad\le c|\nabla\zeta^{(2)}|^2_{\infty,\Omega}(d_2^2+1)b_6^2 + c\left|\frac{w^{(2)}}{r}\right|^4_{4,\Omega^t} + cY_2^2(t).
\end{aligned}
\tag{4.3}
$$

Proof. Multiplying $(4.1)_1$ by $\chi^{(2)'}/r^2$ and integrating over Ω implies

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left|\frac{\chi^{(2)'}}{r}\right|^2_{2,\Omega} + \nu\left\|\frac{\chi^{(2)'}}{r}\right\|^2_{1,\Omega} &\le \int_\Omega\left|\chi v\cdot\nabla(\zeta^{(2)})^2\frac{\chi^{(2)'}}{r^2}\right|dx \\
&\quad - \nu\int_\Omega(2\nabla\chi\nabla(\zeta^{(2)})^2 + \chi\Delta(\zeta^{(2)})^2)\frac{\chi^{(2)'}}{r^2}\,dx \\
&\quad + \int_\Omega\frac{2}{r}w^{(2)}w^{(2)}_{,z}\frac{\chi^{(2)'}}{r^2}\,dx + \int_\Omega\frac{F^{(2)'}_\varphi\chi^{(2)'}}{r^2}\,dx.
\end{aligned}
\tag{4.4}
$$

Now we shall estimate the particular terms on the r.h.s. of (4.4). We restrict the first term by

$$
\varepsilon_1\left|\frac{\chi^{(2)'}}{r}\right|^2_{6,\Omega} + c\left(\frac{1}{\varepsilon_1}\right)|\nabla\zeta^{(2)}|^2_{\infty,\Omega}|v|^2_{2,\widehat\Omega'_{(1)}}|\chi|^2_{3,\widehat\Omega'_{(1)}}.
$$

The second term on the r.h.s. of (4.4) equals

$$\nu\int\limits_\Omega \chi^2\nabla(\zeta^{(2)})^2\nabla\frac{(\zeta^{(2)})^2}{r^2}\,dx = \nu\int\limits_\Omega \frac{\chi^2}{r^2}|\nabla(\zeta^{(2)})^2|^2\,dx - 2\nu\int\limits_\Omega \frac{\nabla(\zeta^{(2)})^2(\zeta^{(2)})^2\nabla r\chi^2}{r^3}\,dx,$$

where the second integral is estimated by

$$\varepsilon_2\bigg|\frac{\chi^{(2)\prime}}{r}\bigg|_{2,\Omega}^2 + c\bigg(\frac{1}{\varepsilon_2}\bigg)\bigg|\frac{\nabla\zeta^{(2)}\chi}{r^2}\bigg|_{2,\Omega}^2.$$

The third term on the r.h.s. of (4.4) equals

$$\int\limits_\Omega\bigg(\frac{|w^{(2)}|^2}{r^2}\bigg)_{,z}\frac{\chi^{(2)\prime}}{r}\,dx = -\int\limits_\Omega\frac{|w^{(2)}|^2}{r^2}\bigg(\frac{\chi^{(2)\prime}}{r}\bigg)_{,z}\,dx = I.$$

Hence,

$$|I|\le\varepsilon_3\bigg|\bigg(\frac{\chi^{(2)\prime}}{r}\bigg)_{,z}\bigg|_{2,\Omega}^2 + c\bigg(\frac{1}{\varepsilon_2}\bigg)\bigg|\frac{w^{(2)}}{r}\bigg|_{4,\Omega}^4.$$

Finally, the last term on the r.h.s. of (4.4) is bounded by

$$\varepsilon_4\bigg|\frac{\chi^{(2)\prime}}{r}\bigg|_{2,\Omega}^2 + c\bigg(\frac{1}{\varepsilon_4}\bigg)\bigg|\frac{F_\varphi^{(2)\prime}}{r}\bigg|_{2,\Omega}^2.$$

Utilizing the above estimates in (4.4) and assuming that $\varepsilon_1-\varepsilon_4$ are sufficiently small we obtain

$$\begin{aligned}(4.5)\quad \frac{d}{dt}\bigg|\frac{\chi^{(2)\prime}}{r}\bigg|_{2,\Omega}^2 + \nu\bigg\|\frac{\chi^{(2)\prime}}{r}\bigg\|_{1,\Omega}^2 &\le c\bigg(|\nabla\zeta^{(2)}|_{\infty,\Omega}^2|v|_{2,\Omega}^2|v_{,x}|_{3,\Omega}^2\\ &+\frac{1}{(r_0-\delta_2)^2}|\nabla\zeta^{(2)}|_{\infty,\Omega}^2|v_{,x}|_{2,\Omega}^2 + \bigg|\frac{w^{(2)}}{r}\bigg|_{4,\Omega}^4 + \frac{1}{(r_0-\delta_2)^2}|F_\varphi^{(2)}|_{2,\Omega}^2\bigg).\end{aligned}$$

Integrating (4.5) with respect to time and using (4.2) we get (4.3). □

To estimate the second term on the r.h.s. of (4.3) we need the following localized version of problem (1.5)

$$(4.6)\quad\begin{aligned} &w_{,t}^{(2)} + v\cdot\nabla w^{(2)} - vw\cdot\nabla\zeta^{(2)} + \frac{v_r}{r}w^{(2)} - \nu\Delta w^{(2)}\\ &\quad+\nu(2\nabla w\nabla\zeta^{(2)} + w\Delta\zeta^{(2)}) + \nu\frac{w^{(2)}}{r^2} = f_\varphi^{(2)} && \text{in } \Omega^T,\\ &w_{,r}^{(2)} = \frac{1}{R}w^{(2)} && \text{on } S_1^T,\\ &w^{(2)}|_{r=r_0-\delta_2} = 0,\\ &w_{,z}^{(2)} = 0 && \text{on } S_2^T,\\ &w^{(2)}|_{t=0} = w^{(2)}(0) && \text{in } \Omega.\end{aligned}$$

Lemma 4.2. *Assume that v is the weak solution to problem* (1.1) *satisfying* (2.2)–(2.3). *Assume that $v\in L_4(\Omega^T)$, $f_\varphi^{(2)}\in L_4(0,T;L_{4/3}(\Omega))$, $w^{(2)}(0)\in L_4(\Omega)$. Then solutions of* (4.6) *satisfy*

$$
\begin{aligned}
(4.7)\quad & |w^{(2)}(t)|_{4,\Omega}^4+\nu|\nabla|w^{(2)}|^2|_{2,\Omega^t}^2+\nu|w^{(2)}|_{4,\Omega^t}^4\\
&\le c(d_1)d_1^4T+\frac{c}{\delta_2^4}(1+d_1^4)|v|_{4,\Omega^t}^4+c\|f_\varphi^{(2)}\|_{4/3,4,\Omega^t}^4+|w^{(2)}(0)|_{4,\Omega}^4.
\end{aligned}
$$

Proof. Multiplying $(4.6)_1$ by $w^{(2)}|w^{(2)}|^2$, integrating over Ω and utilizing boundary conditions implies

$$
\begin{aligned}
(4.8)\quad & \frac14\frac{d}{dt}|w^{(2)}|_{4,\Omega}^2+\frac34\nu|\nabla|w^{(2)}|^2|_{2,\Omega}^2+\nu\left|\frac{w^{(2)}}{\sqrt r}\right|_{4,\Omega}^4\\
&\le\int_\Omega\left|\frac{v_r}{r}\right||w^{(2)}|^4\,dx+\int_\Omega|wv\nabla\zeta^{(2)}||w^{(2)}|^3\,dx\\
&\quad-\nu\int_\Omega(2\nabla w\nabla\zeta^{(2)}+w\Delta\zeta^{(2)})w^{(2)}|w^{(2)}|^2\,dx\\
&\quad+\int_\Omega|f_\varphi^{(2)}||w^{(2)}|^3\,dx+\frac1R\int_{S_1}|w^{(2)}(R)|^4\,dS_1.
\end{aligned}
$$

We estimate the first term on the r.h.s. of (4.8), by the interpolation inequality (see [6], Section 15), by

$$
\varepsilon_1|\nabla|w^{(2)}|^2|_{2,\Omega}^2+c(1/\varepsilon_1,d_1)|w^{(2)}|^2|_{1,\Omega}^2,
$$

where estimate (2.2)–(2.3) was utilized and $\varepsilon_1\in(0,1)$. We restrict the second term on the r.h.s. of (4.8) by

$$
|wv\cdot\nabla\zeta^{(2)}|_{4/3,\Omega}|w^{(2)}|_{12,\Omega}^2\le\varepsilon_2|w^{(2)}|_{12,\Omega}^4+c(1/\varepsilon_2)|wv\cdot\nabla\zeta^{(2)}|_{4/3,\Omega}^4,
$$

where $\varepsilon_2\in(0,1)$, the first norm is bounded by $\varepsilon_2c\|\,|w^{(2)}|^2\,\|_{1,\Omega}^2\le\varepsilon_2c(|\nabla|w^{(2)}|^2|^2+|w^{(2)}/\sqrt r|_{4,\Omega}^4)$ and the second by $c(1/\varepsilon_2)|\nabla\zeta^{(2)}|_{\infty,\Omega}^4|v|_{2,\widehat\Omega_{(1)}}^4|w|_{4,\widehat\Omega_{(1)}}^4$.

The third term on the r.h.s. of (4.8) equals

$$
\begin{aligned}
&-\nu\int_\Omega(\nabla w^2\nabla\zeta^{(2)}\zeta^{(2)}+w^2\Delta\zeta^{(2)}\zeta^{(2)})|w^{(2)}|^2\,dx\\
&\qquad=\nu\int_\Omega w^2|\nabla\zeta^{(2)}|^2|w^{(2)}|^2\,dx+\nu\int_\Omega w^2\nabla\zeta^{(2)}\zeta^{(2)}\nabla|w^{(2)}|^2\,dx\equiv I_1+I_2,
\end{aligned}
$$

where

$$|I_1| \le \bigg(\int_\Omega (w^2|\nabla\zeta^{(2)}|^2)^{6/5}\,dx\bigg)^{5/6}\bigg(\int_\Omega |w^{(2)}|^{12}\,dx\bigg)^{2/12}$$
$$\le \varepsilon_3|w^{(2)}|^4_{12,\Omega} + c\bigg(\frac{1}{\varepsilon_3}\bigg)|\nabla\zeta^{(2)}|^4_{\infty,\Omega}|w|^4_{12/5,\widehat{\Omega}'_{(1)}},$$
$$|I_2| \le \varepsilon_4|\nabla|w^{(2)}|^2|^2_{2,\Omega} + c\bigg(\frac{1}{\varepsilon_4}\bigg)|\nabla\zeta^{(2)}|^2_{\infty,\Omega}|w|^4_{4,\widehat{\Omega}'_{(1)}},$$

and $\varepsilon_3, \varepsilon_4 \in (0,1)$.

The fourth term on the r.h.s. of (4.8) is estimated, by the Hölder and Young inequalities, by

$$\varepsilon_5|w^{(2)}|^4_{12,\Omega} + c(1/\varepsilon_5)|f_\varphi^{(2)}|^4_{4/3,\Omega},$$

where $\varepsilon_5 \in (0,1)$.

Finally, the boundary term is bounded by

$$\varepsilon_6|\nabla(w^{(2)})^2|^2_{2,\Omega} + c(1/\varepsilon_6)|\,|w^{(2)}|^2\,|^2_{1,\Omega},$$

where $\varepsilon_6 \in (0,1)$.

Utilizing the above inequalities in (4.8) and assuming that $\varepsilon_1 - \varepsilon_6$ are sufficiently small we obtain

$$\frac{1}{4}\frac{d}{dt}|w^{(2)}|^4_{4,\Omega} + \frac{\nu}{2}|\nabla|w^{(2)}|^2|^2_{2,\Omega} + \nu\bigg|\frac{w^{(2)}}{\sqrt{r}}\bigg|^4_{4,\Omega}$$
$$\le c(d_1)d_1^4 + c|\nabla\zeta^{2)}|^4_{\infty,\Omega}(d_1^4|w|^4_{4,\widehat{\Omega}'_{(1)}} + |w|^4_{\frac{12}{5},\widehat{\Omega}'_{(1)}} + |w|^4_{4,\widehat{\Omega}'_{(1)}}) + c|f_\varphi^{(2)}|^4_{4/3,\Omega}. \tag{4.9}$$

The second term on the r.h.s. of (4.9) is estimated by

$$\frac{c}{\delta_2^4}(1+d_1^4)(|v|^4_{4,\Omega} + |v|^4_{12/5,\Omega}).$$

Utilizing this in (4.9) and integrating the result with respect to time implies (4.7). This concludes the proof. □

Next we consider the elliptic problem

$$\begin{aligned} &v_{r,z} - v_{z,r} = \chi,\\ &v_{r,r} + v_{z,z} = -\frac{v_r}{r},\\ &v_r|_{S_1} = 0, \quad v_z|_{S_2} = 0. \end{aligned} \tag{4.10}$$

Multiplying (4.10) by $(\zeta^{(2)})^2$ yields

$$\begin{aligned} v_{r,z}^{(2)'} - v_{z,r}^{(2)'} &= \chi^{(2)'} - v_z((\zeta^{(2)})^2)_{,r},\\ v_{r,r}^{(2)'} + v_{z,z}^{(2)'} &= -\frac{1}{r}v_r^{(2)'} + v_r((\zeta^{(2)})^2)_{,r},\\ v_r^{(2)'}|_{S_1} = 0, \quad v_z^{(2)'}|_{S_2} &= 0, \quad v_r^{(2)'}|_{r=r_0-\delta_2} = 0, \end{aligned} \tag{4.11}$$

where $v^{(2)'} = v(\zeta^{(2)})^2$, $\chi^{(2)'} = \chi(\zeta^{(2)})^2$. For solutions of (4.11) we have

$$\|v^{(2)'}\|_{L_\infty(0,T;H^1(\Omega))} + \|v^{(2)'}\|_{L_{10/3}(0,T;W^1_{10/3}(\Omega))} + \|v^{(2)'}\|_{L_2(0,T;H^2(\Omega))} \le c(\|\chi^{(2)'}\|_{L_\infty(0,T;L_2(\Omega))} + \|\chi^{(2)'}\|_{L_2(0,T;H^1(\Omega))}) + c(d_1+d_2). \tag{4.12}$$

Let us introduce the notation

$$Y_3(t) = |f_\varphi^{(2)}|_{4/3,4,\Omega^t} + |w^{(2)}(0)|_{4,\Omega}.$$

Then (4.7) takes the form

$$|w^{(2)}|_{4,\Omega} + |\nabla|w^{(2)}|^2|_{2,\Omega^t} + |w^{(2)}|_{4,\Omega^t} \le c(d_1)T + \frac{c}{\delta_2}(1+d_1)|v|_{4,\Omega^t} + cY_3(t). \tag{4.13}$$

Utilizing (4.13) in (4.3) yields

$$\|\chi^{(2)}\|_{V_2^1(\Omega^t)} \le \frac{c}{\delta_2}(1+d_2)b_6 + \frac{c}{\delta_2}(1+d_1)|v|_{4,\Omega^t} + c(d_1)T + c(Y_2(t)+Y_3(t)). \tag{4.14}$$

Employing (4.14) in (4.12) gives

$$\|v^{(2)'}\|_{L_\infty(0,t;H^1(\Omega))} + \|v^{(2)'}\|_{L_{10/3}(0,t;W^1_{10/3}(\Omega))} + \|v^{(2)'}\|_{L_2(0,t;H^2(\Omega))} \le \frac{c}{\delta_2}(1+d_1+d_2)(b_6 + |v|_{4,\Omega^t}) + c(d_1)T + c(d_1+d_2) + c(Y_2(t)+Y_3(t)). \tag{4.15}$$

Next we need

Lemma 4.3. *Assume that $w^{(4)}(0) \in H^1(\Omega)$, $f_\varphi^{(4)} \in L_2(\Omega^T)$, $F_\varphi^{(4)} \in L_2(\Omega^T)$, $\chi^{(4)}(0) \in L_2(\Omega)$. Assume the v is a weak solution to problem* (1.1) *such that $v \in L_2(0,T;W_3^1(\Omega)) \cap L_4(\Omega^T)$. Then solutions to problem* (4.6) *satisfy*

$$\|w^{(2)}\|_{2,\Omega^t} \le \varphi(Z(t))d_2 + c\bigg(\frac{1}{\delta_2}|v|^2_{4,\Omega^t} + d_2 + |f_\varphi^{(2)}|_{2,\Omega^t} + \|w^{(2)}(0)\|_{1,\Omega}\bigg) \equiv Z_1(t), \tag{4.16}$$

where $t \le T$, $Z(t)$ is defined by (4.18) *and φ is a power function because it follows from interpolation inequalities.*

Proof. For solutions of problem (4.6) we obtain

$$\|w^{(2)}\|_{2,\Omega^t} \le c\bigg(|v'\cdot\nabla' w^{(2)}|_{2,\Omega^t} + |v' w \nabla'\zeta^{(2)}|_{2,\Omega^t} + \bigg|\frac{v_r}{r}w^{(2)}\bigg|_{2,\Omega^t} + |2\nabla w\nabla\zeta^{(2)} + w\Delta\zeta^{(2)}|_{2,\Omega^t} + |w^{(2)}|_{2,\Omega^t} + |f_\varphi^{(2)}|_{2,\Omega^t} + \|w^{(2)}\|_{1/2,S_1^t} + \|w^{(2)}(0)\|_{1,\Omega}\bigg). \tag{4.17}$$

Now we estimate the particular terms from the r.h.s. of (4.17). To estimate the first term we need (4.15) for $v^{(4)'}$, where $v^{(4)} = v\zeta^{(4)}$. Therefore, repeating the considerations leading to (4.13)–(4.15) in the case where $\zeta^{(2)}$ is replaced by $\zeta^{(4)}$ we obtain

$$|w^{(4)}|_{4,\Omega} + |\nabla |w^{(4)}|^2|_{2,\Omega^t} + |w^{(4)}|_{4,\Omega^t} \le c(d_1)T + \frac{c}{\delta_2}(1+d_1)|v|_{4,\Omega^t} + cY_5(t),$$

where

$$Y_5(t) = |f_\varphi^{(4)}|_{4/3,4,\Omega^t} + |w^{(4)}(0)|_{4,\Omega}.$$

Next, instead of (4.14) we have

$$\|\chi^{(4)}\|_{V_2^1(\Omega^t)} \le \frac{c}{\delta_2}(1+d_1+d_2)(b_6 + |v|_{4,\Omega^t}) + c(d_1)T + c(Y_4(t)+Y_5(t)),$$

where

$$Y_4(t) = |F_\varphi^{(4)}|_{2,\Omega^t} + \left|\frac{\chi^{(4)}(0)}{r}\right|_{2,\Omega}.$$

Finally, (4.15) takes the form

$$\begin{aligned}&\|v^{(4)'}\|_{L_\infty(0,t;H^1(\Omega))} + \|v^{(4)'}\|_{L_{10/3}(0,t;W^1_{10/3}(\Omega))} + \|v^{(4)'}\|_{L_2(0,t;H^2(\Omega))}\\ &\quad\le \frac{c}{\delta_2}(1+d_1+d_2)(b_6+|v|_{4,\Omega^t}) + c(d_1)T + c(d_1+d_2) + c(Y_4(t)+Y_5(t)).\end{aligned}$$

The above inequality implies

$$\begin{aligned}(4.18)\quad |v^{(4)'}|_{10,\Omega^t} + |\nabla v^{(4)'}|_{10/3,\Omega^t} &\le \frac{c}{\delta_2}(1+d_1+d_2)(b_6+|v|_{4,\Omega^t})\\ &\quad + c(d_1)T + c(d_1+d_2) + c(Y_4(t)+Y_5(t)) \equiv Z(t),\end{aligned}$$

where $t \le T$.

Applying the Hölder inequality we estimate the first term on the r.h.s. of (4.17) by

$$\begin{aligned}|v'\cdot\nabla' w^{(2)}|_{2,\Omega^t} &\le |v'|_{10,\widehat{\Omega}^t_{r_0-\delta_2}} |\nabla' w^{(2)}|_{5/2,\Omega^t} \le |v^{(4)'}|_{10,\Omega^t}|\nabla' w^{(2)}|_{5/2,\Omega^t}\\ &\le \varepsilon_1\|w^{(2)}\|_{2,\Omega^t} + \varphi(1/\varepsilon_1, |v^{(4)'}|_{10,\Omega^t})|w^{(2)}|_{2,\Omega^t} \equiv I_1,\end{aligned}$$

where in the last inequality some interpolation inequality is used (see [6, Chapter 3, Section 10]) and φ is a positive increasing function.

In view of the energy estimate (2.2)–(2.3) and (4.18) we have

$$I_1 \le \varepsilon_1\|w^{(2)}\|_{2,\Omega^t} + \varphi(1/\varepsilon_1, Z(t))d_2.$$

By the Hölder inequality the second term on the r.h.s. of (4.17) is estimated by

$$\frac{c}{\delta_2}|v'|_{4,\Omega^t}|w|_{4,\Omega^t} \le \frac{c}{\delta_2}|v|^2_{4,\Omega^t}.$$

Applying the Hölder inequality and some interpolation inequality (see [6, Chapter 3, Section 10]) the third term on the r.h.s. of (4.17) is bounded by

$$\begin{aligned}c|v_r|_{10,\widehat{\Omega}^t_{r_0-\delta_2}}|w^{(2)}|_{5/2,\Omega^t} &\le c|v^{(4)'}|_{10,\Omega^t}|w^{(2)}|_{5/2,\Omega^t}\\ &\le \varepsilon_2\|w^{(2)}\|_{2,\Omega^t} + \varphi(1/\varepsilon_2, |v^{(4)'}|_{10,\Omega^t})|w^{(2)}|_{2,\Omega^t} \equiv I_2.\end{aligned}$$

Utilizing the energy estimate (2.2)–(2.3) and (4.18) we obtain

$$I_2 \le \varepsilon_2 \|w^{(2)}\|_{2,\Omega^t} + \varphi(1/\varepsilon_2, Z(t))d_2.$$

In view of (2.2)–(2.3) we estimate the fourth and fifth terms on the r.h.s. of (4.17) by cd_2.

By some interpolation inequality (see [6, Chapter 3, Section 10]) the seventh term on the r.h.s. of (4.17) is bounded by

$$\varepsilon_3 \|w^{(2)}\|_{2,\Omega^t} + c(1/\varepsilon_3)d_2.$$

Utilizing the above estimates in (4.17) and assuming that $\varepsilon_1 - \varepsilon_3$ are sufficiently small we obtain (4.16). □

From (4.16) we have

$$|w^{(2)}|_{10,\Omega^t} + |\nabla w^{(2)}|_{10/3,\Omega^t} \le cZ_1(t), \quad t \le T. \tag{4.19}$$

Inequalities (4.15) and (4.19) imply

$$|v^{(2)}|_{10,\Omega^t} + |\nabla v^{(2)}|_{10/3,\Omega^t} \le cZ_1(t), \quad t \le T. \tag{4.20}$$

5. Local existence

To prove local existence of solutions to problem (1.1) we apply the Leray–Schauder fixed point theorem. Let

$$\begin{aligned} \mathfrak{M}(\Omega^T) = \bigg\{ v : \sum_{i=1}^6 b_i(T) &\le c(\|v\|_{L_2(0,T;L_\infty(\Omega))} + \|v\|_{L_5(\Omega^T)} \\ &+ \|v\|_{L_\infty(0,T;L_3(\Omega))} + \|v\|_{L_2(0,T;W_q^1(\Omega))}) < \infty, \\ &3 \le q = \frac{6}{1-3\varepsilon_0},\ \varepsilon_0 > 0 \text{ arbitrary small number} \bigg\}. \end{aligned}$$

Let

$$\frac{1}{\delta_*} = \frac{1}{\delta_1} + \frac{1}{\delta_2} + X(T),$$

where

$$\begin{aligned} \sum_{i=1}^4 X_i(T) \le c(&|f_\varphi^{(1)}|_{2,-\delta,\Omega^T} + |f_\varphi^{(3)}|_{2,\Omega^T} + |F^{(1)'}|_{2,\Omega^T} \\ &+ |u^{(1)}(0)|_{2,-\delta,\Omega} + \|w^{(3)}(0)\|_{1,0,\Omega} + |\alpha^{(1)'}(0)|_{2,\Omega}) \equiv cX(T), \end{aligned}$$

and $\delta = 1/2 + \varepsilon_0$. Let $Y(T)$ be defined by

$$\begin{aligned} \sum_{i=1}^6 Y_i(T) \le c(&\|F_\varphi^{(1)}\|_{L_2(0,T;L_{2,-1}(\Omega))} + \|\chi^{(1)}(0)\|_{L_{2,-1}(\Omega)} \\ &+ \|F_\varphi^{(4)}\|_{L_2(\Omega^T)} + \|\chi^{(4)}(0)\|_{L_2(\Omega)} + \|f_\varphi^{(4)}\|_{L_4(0,T;L_{4/3}(\Omega))} \\ &+ \|w^{(4)}(0)\|_{L_4(\Omega)} + \|f_\varphi^{(2)}\|_{L_2(\Omega^T)} + \|w^{(2)}(0)\|_{H^1(\Omega)}) \equiv cY(T). \end{aligned}$$

In view of the above notation inequalities (3.46) and (4.20) imply

$$\|v\|_{\mathfrak{N}(\Omega^T)} = |v|_{10,\Omega^T} + |\nabla v|_{10/3,\Omega^T} \le \varphi_*\left(\frac{1}{\delta_*}\|v\|_{\mathfrak{M}(\Omega^T)}, d_1, d_2, Y\right) + c_1 Y, \tag{5.1}$$

where φ_* is an increasing positive function. Therefore (5.1) implies the transformation

$$\Phi_1 \colon \mathfrak{M}(\Omega^T) \to \mathfrak{N}(\Omega^T).$$

Next we introduce the transformation

$$\Phi_2 \colon \mathfrak{N}(\Omega^T) \to W^{2,1}_{5/2}(\Omega^T)$$

defined by the problem

$$\begin{aligned}
&v_{,t} - \operatorname{div}\mathbb{T}(v,p) = -\lambda\widetilde{v}\cdot\nabla\widetilde{v} + f && \text{in } \Omega^T,\\
&\operatorname{div} v = 0 && \text{in } \Omega^T,\\
&v\cdot\overline{n} = 0 && \text{on } S^T,\\
&\overline{n}\cdot\mathbb{T}(v,p)\cdot\overline{\tau}_\alpha = 0, \quad \alpha = 1,2 && \text{on } S^T,\\
&v|_{t=0} = v(0) && \text{in } \Omega,
\end{aligned} \tag{5.2}$$

where $\lambda \in [0,1]$ and $\widetilde{v} \in \mathfrak{N}(\Omega^T)$. Solving (5.2) implies

$$W^{2,1}_{5/2}(\Omega^T) \ni v = \Phi_2(\widetilde{v}, \lambda).$$

Therefore a fixed point of the transformation $\Phi = \Phi_2 \circ \Phi_1$ is a solution of problem (1.1).

Now we check the assumptions of the Leray–Schauder fixed point theorem. We have that

$$\Phi \colon \mathfrak{M}(\Omega^T) \times [0,1] \to W^{2,1}_{5/2}(\Omega^T) \subset \mathfrak{M}(\Omega^T),$$

where the last imbedding is compact and continuous. Therefore, the mapping

$$\Phi \colon \mathfrak{M}(\Omega^T) \times [0,1] \to \mathfrak{M}(\Omega^T)$$

is compact and uniformly continuous. For $\lambda = 0$ problem (5.2) has a unique solution.

Now we shall find an estimate for a fixed point of mapping Φ.

Lemma 5.1. *Assume that v is a weak solution to problem* (1.1) *described by Lemma* 2.1. *Assume that $X < \infty$, $Y < \infty$ and δ_* is so large that there exists a constant A such that*

$$\varphi_0\left(\frac{1}{\delta_*}A, d_1, d_2, Y\right) + c(|f|_{5/2,\Omega^T} + \|v(0)\|_{6/5,5/2,\Omega}) \le A, \tag{5.3}$$

where φ_0 is described below. Then

$$\|v\|_{2,5/2,\Omega^T} \le A. \tag{5.4}$$

Proof. Utilizing (5.1) we obtain for solutions of (5.2) the inequality (see [1])

$$
\begin{aligned}
\|v\|_{2,5/2,\Omega^T} &\le c\bigg[\varphi_*\bigg(\frac{c}{\delta_*}\|v\|_{2,5/2,\Omega^T}, d_1, d_2, Y\bigg) + cY\bigg]^2 \\
&\quad + c(|f|_{5/2,\Omega^T} + \|v(0)\|_{6/5,5/2,\Omega}) \\
&\equiv \varphi_0\bigg(\frac{1}{\delta_*}\|v\|_{2,5/2,\Omega^T}, d_1, d_2, Y\bigg) \\
&\quad + c(|f|_{5/2,\Omega^T} + \|v(0)\|_{6/5,5/2,\Omega}).
\end{aligned}
\tag{5.5}
$$

Hence if (5.3) holds then (5.5) implies (5.4). □

In view of the above considerations we have

Theorem 5.2. *Assume that v is a weak solution determined by Lemma* 2.1. *Assume that $X < \infty$, $Y < \infty$, $f \in L_{5/2}(\Omega^T)$, $v(0) \in W^{6/5}_{5/2}(\Omega)$. Assume that the quantity δ_* is so large that* (5.3) *holds. Then there exists at least one solution to problem* (1.1) *such that $v \in W^{2,1}_{5/2}(\Omega^T)$, $\nabla p \in L_{5/2}(\Omega^T)$ and estimate* (5.4) *is valid.*

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