EXISTENCE AND CONCENTRATION OF NODAL SOLUTIONS TO A CLASS OF QUASILINEAR PROBLEMS

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Abstract. The existence and concentration behavior of nodal solutions are established for the equation $-\varepsilon^p \Delta_p u + V(z)|u|^{p-2}u = f(u)$ in $\Omega$, where $\Omega$ is a domain in $\mathbb{R}^N$, not necessarily bounded, $V$ is a positive Hölder continuous function and $f \in C^1$ is a function having subcritical growth.

1. Introduction

In this paper we are concerned with the existence and concentration of nodal solutions for the problem

\[ (P_\varepsilon) \begin{cases} -\varepsilon^p \Delta_p u + V(z)|u|^{p-2}u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases} \]

where $\varepsilon > 0$, $1 < p < N$, $\Delta_p u$ is the $p$-Laplacian operator defined as

$$\Delta_p u = \sum_{i=1}^N \frac{\partial}{\partial x_i} \left( |\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \right).$$

2000 Mathematics Subject Classification. 35H30, 35A15, 35B40.
Key words and phrases. Quasilinear equation, variational methods, behaviour of solutions.
Research of the first named author supported by CNPq and PADCT 620017/2004-0.
\( \Omega \) is a domain in \( \mathbb{R}^N \) containing the origin, not necessarily bounded, with empty or smooth boundary, \( V: \mathbb{R}^N \to \mathbb{R} \) is a Hölder continuous function satisfying:

\( (V_1) \) The set \( M = \{ z \in \mathbb{R}^N : V(z) = V_0 \} \) is bounded and compactly contained in \( \Omega \), where

\[
V_0 = \inf_{z \in \mathbb{R}^N} V(z) > 0.
\]

\( (V_2) \) There exists an open and bounded domain \( \Lambda \), compactly contained in \( \Omega \), such that

\[
V_0 = \inf_{z \in \Lambda} V(z) < \min_{z \in \partial \Lambda} V(z).
\]

With relation the function \( f \), we assume the following conditions:

\( (f_1) \) \( f \in C^1(\mathbb{R}) \) and \( \lim_{s \to 0} |f(s)|/|s|^{p-1} = 0 \).

\( (f_2) \) There exists \( q \in (p, p^*) \), where \( p^* = Np/(N-p) \) such that

\[
\lim_{|s| \to \infty} \frac{|f(s)|}{|s|^{q-1}} = 0.
\]

\( (f_3) \) There exists \( \theta \in (p, q) \) such that \( 0 < \theta F(s) \leq sf(s) \), for all \( s \neq 0 \), where

\[
F(s) = \int_0^s f(t) \, dt.
\]

\( (f_4) \) The function \( s \mapsto f(s)/|s|^{p-1} \) is an increasing in \( \mathbb{R} \setminus \{0\} \).

For the case \( p = 2 \), equation \( (P_\varepsilon) \) is equal to

\[
-\varepsilon^2 \Delta u + V(z)u = f(u) \quad \text{in} \ \Omega,
\]

and this type of equation arises in some important mathematical models. For example, when \( f(s) = |s|^{q-2}s, \ 2 < q < 2^* = 2N/(N-2) \), these equations are related with the existence of standing waves of the nonlinear Schrödinger equation

\[
\text{(NLS)} \quad i\hbar \frac{\partial \Psi}{\partial t} = -\hbar^2 \Delta \Psi + (V(z) + E)\Psi - f(\Psi) \quad \text{for all} \ z \in \Omega.
\]

A standing wave of \( \text{(NLS)} \) is a solution of the form \( \Psi(z, t) = \exp(-iEt/\hbar)u(z) \), where \( u \) is a solution of \( (P_\nu) \).

Still for the case \( p = 2 \), existence of nodal solutions for general semilinear elliptic equations with superlinear nonlinearity have been established by Bartsch and Wang in [7], [8], by Bartsch, Chang and Wang in [9] and by Bartsch, Weth and Willem in [10]. Moreover, the existence and concentration of positive solutions for \( (P_\nu) \) have been extensively studied in recent years, see for example, Floer and Weinstein [11], Oh [18], [19], Rabinowitz [20], Wang [22], Alves and Souto [1], del Pino and Felmer [13], Alves, do Ó and Souto [4], and the references therein. For the case involving nodal solutions we cite the works of Noussair and Wei [16], [17] and Alves and Soares [5].

For the general case \( p \geq 2 \), some results of existence and concentration for \( (P_\varepsilon) \) were proved by Alves and Figueiredo [2], [3].
In this work, motivated by paper [5] and by some ideas employed in [2] and [3], we prove the existence and concentration of nodal solutions to \((P_\varepsilon)\). To get these nodal solutions we adapt some arguments developed in [10], [13] and in [16], [17], and to prove the concentration of the solutions, we use the same type of ideas found in [5]. However, for our case, we ought to use a different approach of that explored in [5], [13] and [16], [17], since we are working with the \(p\)-Laplacian operator and some estimates for this type of operator can not be obtained using the same type of ideas explored for the case \(p = 2\). For example, results involving convergences in the \(C^2\) sense do not hold for the \(p\)-Laplacian. To overcome these difficulties, we use the same type of arguments developed by the authors in the papers [2], [3] and make a careful analysis of the estimates proved in [5].

Our main result is the following:

**Theorem 1.1.** Suppose that \(f\) and \(V\) satisfy \((f_1)-(f_4)\) and \((V_1)-(V_2)\), respectively. Then, there exists \(\varepsilon_0 > 0\) such that \((P_\varepsilon)\) possesses a nodal solution \(u_\varepsilon \in W^{1,p}_0(\Omega)\), for every \(\varepsilon \in (0, \varepsilon_0)\). Moreover, if \(P^1_\varepsilon \in \Omega\) is a positive global maximum point of \(u_\varepsilon\) and \(P^2_\varepsilon \in \Omega\) is a negative global minimum point of \(u_\varepsilon\), we have that \(P^i_\varepsilon \in \Lambda\) and \(V(P^i_\varepsilon) \to V_0\), for \(i = 1, 2\).

This paper is organized as follows: in Section 2, we work with an auxiliary problem, which is used to show the existence of the nodal solution to \((P_\varepsilon)\). In Section 3, we state some lemmas and propositions used in the proof of main result. In Section 4, we prove Theorem 1.1. In Section 5 we prove the technical lemmas and propositions stated in the Section 3.

### 2. An auxiliary problem

In this section we will work with an auxiliary problem, which is related in some sense with problem \((P_\varepsilon)\).

#### 2.1. Preliminaries and notations

In this section we fix some notations and prove some lemmas which are key points in our arguments.

Hereafter, let us omit the symbol “\(dx\)” in all integrals.

We recall that the weak solutions of \((P_\varepsilon)\) are the critical points of the functional

\[
J_\varepsilon(u) = \frac{1}{p} \int_{\Omega} \varepsilon^p |\nabla u|^p + \frac{1}{p} \int_{\Omega} V(z)|u|^p - \int_{\Omega} F(u)
\]

where \(F(t) = \int_0^t f(s) \, ds\), and \(J_\varepsilon(u)\) is defined for \(u\) in the Banach space

\[
W = \left\{ u \in W^{1,p}_0(\Omega) : \int_{\Omega} V(z)|u|^p < \infty \right\}
\]
envelope with the norm

\[ \|u\| = \left( \int_{\Omega} |\nabla u|^p + \int_{\Omega} V(z)|u|^p \right)^{1/p}. \]

To establish the existence of nodal solution, we will adapt for our case, an argument explored by del Pino and Felmer \[13\] (see also \[3\] and \[5\]), which consists in considering a modified problem. To this end, we need to fix some notations.

Let \( \theta \) be the number given in \( (f_3) \), and \( a, k_1, k_2 > 0 \) be constants satisfying \( k_1, k_2 > (\theta/\theta - p) \), \( (f(a)/a^{p-1}) = (V_0/k_1) \) and \( (f(-a)/a^{p-1}) = -(V_0/k_2) \), where \( V_0 \) appears in \( (V_1) \). Using the above numbers, let us define the functions

\[
\tilde{f}(s) = \begin{cases} 
  f(s) & \text{if } |s| \leq a, \\
  \frac{V_0}{k_1} |s|^{p-2} s & \text{if } s > a, \\
  \frac{V_0}{k_2} |s|^{p-2} s & \text{if } s < -a,
\end{cases}
\]

and the auxiliary problem

\[
(P_{\epsilon}) \left\{ \begin{array}{ll}
-\epsilon^p \Delta_p u + V(z)|u|^{p-2} u = g(z,u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{array} \right.
\]

where \( \chi_{\Lambda} \) is the characteristic function of the set \( \Lambda \). It is easy to check that \( (f_1)-(f_4) \) imply that \( g \) is a Carathéodory function and for \( z \in \Omega \), the function \( s \to g(z,s) \) satisfies the following conditions, uniformly for \( z \in \Omega \):

\[
(g_1) \lim_{|s| \to 0} \frac{g(z,s)}{|s|^{p-1}} = 0, \\
(g_2) \lim_{|s| \to \infty} \frac{g(z,s)}{|s|^{q-1}} = 0, \\
(g_3)_i 0 \leq \theta G(z,s) \leq g(z,s)s, \\
(g_3)_{ii} 0 < pG(z,s) \leq g(z,s)s \leq \frac{1}{k} V(z)|s|^p,
\]

for all \( z \in \Lambda \) and all \( s \neq 0 \), and

\[
(g_3)_{ii} 0 < pG(z,s) \leq g(z,s)s \leq \frac{1}{k} V(z)|s|^p,
\]

for all \( z \not\in \Lambda \) and all \( s \neq 0 \), where \( k = \min\{k_1, k_2\} \) and \( G(z,s) = \int_0^s g(z,t) \, dt \).

The function

\[
g_4 \Rightarrow \frac{g(z,s)}{|s|^{p-1}} \text{ is nondecreasing for each } z \in \Omega \text{ and all } s \neq 0.
\]

**Remark 2.1.** Note that if \( u \) is a nodal solution of \( (P_{\epsilon})_a \) with \( |u(z)| \leq a \) for every \( z \in \Omega \setminus \Lambda \), then \( u \) is also a nodal solution of \( (P_{\epsilon}) \).
2.2. Existence of ground state nodal solution for $(P_\varepsilon)_\alpha$. In this section, we adapt some arguments found in Bartsch, Weth and Willem [10], Alves and Figueiredo [3] and Alves and Soares [5] to establish the existence of ground state nodal solution for problem $(P_\varepsilon)_\alpha$.

Hereafter, let us denote by $I_\varepsilon$ the functional
\[
I_\varepsilon(u) = \frac{1}{p} \int_{\Omega} \varepsilon^p |\nabla u|^p + \frac{1}{p} \int_{\Omega} V(z)|u|^p - \int_{\Omega} G(z,u)
\]
and by $M_\varepsilon$ the set
\[
M_\varepsilon = \{ u \in W : u^\pm \neq 0 \text{ and } I'_\varepsilon(u^\pm)u^\pm = 0 \},
\]
where
\[
u^+(z) = \max\{u(z),0\} \quad \text{and} \quad u^-(z) = \min\{u(z),0\}.
\]

It is easy to check that there exists $\mu^* > 0$ such that
\[
(2.1) \quad \int_\Lambda |u^\pm|^q > \mu^* \quad \text{for all } u \in M_\varepsilon.
\]

Hereafter, we will denote by $c_\varepsilon$ the following real number
\[
c_\varepsilon = \inf_{M_\varepsilon} I_\varepsilon.
\]

**Theorem 2.2.** $c_\varepsilon$ is achieved by some $u_\varepsilon \in M_\varepsilon$. Moreover, $u_\varepsilon$ is a nodal solution of $(P_\varepsilon)_\alpha$.

**Proof.** It is easy to check that $I_\varepsilon$ is bounded from below on $M_\varepsilon$ since, using the definition of $M_\varepsilon$, there exists $C > 0$ such that
\[
(2.2) \quad I_\varepsilon(u) \geq C||u||^p \quad \text{for all } u \in M_\varepsilon.
\]
Thus, there exists a sequence $\{v_n\} \subset M_\varepsilon$ verifying $I_\varepsilon(v_n) \rightarrow c_\varepsilon$ as $n \rightarrow \infty$ which is bounded by (2.2). Since $W$ is reflexive, there exists $v \in W$ such that $v_n \rightharpoonup v$ in $W$. By (2.1) and by the Sobolev imbedding, we have
\[
\int_\Lambda |v^\pm|^q > \mu^*,
\]
showing that $v^\pm \neq 0$. By the weak convergence of the sequence and condition \((g_3)\), it follows that
\[
||v^\pm||^p \leq \int_{\Omega} g(z,v^\pm)v^\pm.
\]
The above inequalities imply that there exist $t^+, t^- \in (0,1]$ such that
\[
||t^+v^\pm||^p = g(z,t^+v^\pm),
\]
which implies that $w = t^+v^+ + t^-v^-$ is an element of $M_\varepsilon$.

Since
\[
I_\varepsilon(t^\pm v^\pm) = \int_{\Omega} \left[ \frac{1}{p} g(z,t^\pm v^\pm) t^\pm v^\pm - G(z,t^\pm v^\pm) \right],
\]

using Fatou’s lemma, it follows that
\[
I_{\varepsilon}(t^{\pm}v^{\pm}) \leq \liminf_{n \to \infty} \int_{\Omega} \left[ \frac{1}{p} g(z, t^{\pm}v^{\pm}_n) t^{\pm}v^{\pm}_n - G(z, t^{\pm}v^{\pm}_n) \right].
\]
For each \( n \), conditions (f4) and (g4) imply that the function
\[
t \mapsto \int_{\Omega} \left[ \frac{1}{p} g(z, tv^{\pm}_n) tv^{\pm}_n - G(z, tv^{\pm}_n) \right]
\]
is increasing and thus
\[
I_{\varepsilon}(t^{\pm}v^{\pm}) \leq \liminf_{n \to \infty} \int_{\Omega} \left[ \frac{1}{p} g(z, v^{\pm}_n) v^{\pm}_n - G(z, v^{\pm}_n) \right] = \liminf_{n \to \infty} I_{\varepsilon}(v^{\pm}_n).
\]
From the last inequality, it follows that
\[
I_{\varepsilon}(w) = I_{\varepsilon}(t^{+}v^{+}) + I_{\varepsilon}(t^{-}v^{-}) \leq \liminf_{n \to \infty} I_{\varepsilon}(v^{n}) = c_{\varepsilon}.
\]
Since \( w \in \mathcal{M}_{\varepsilon} \), the above inequality implies that \( I_{\varepsilon}(w) = c_{\varepsilon} \), and thus \( c_{\varepsilon} \) is achieved. Now, from [3, Proposition 3.1] (see also [14]) the functional \( I_{\varepsilon} \) satisfies the Palais–Smale at all \( c \in \mathbb{R} \) and hence, we can repeat the same arguments found in [10, Proposition 3.1] to conclude that \( c_{\varepsilon} \) is a critical level, that is, there exists \( u_{\varepsilon} \in \mathcal{M}_{\varepsilon} \) such that
\[
I_{\varepsilon}(u_{\varepsilon}) = c_{\varepsilon} \quad \text{and} \quad I'_{\varepsilon}(u_{\varepsilon}) = 0.
\]

**Remark 2.3.** The nodal solution found in Theorem 2.2 satisfies the following properties:

(a) If \( \Omega \) is bounded, the function \( u_{\varepsilon} \) belongs to \( C(\overline{\Omega}) \) (see [12]).

(b) If \( \Omega \) is unbounded, the function \( u_{\varepsilon} \) is a continuous function and verifies
\[
\lim_{|z| \to \infty} u_{\varepsilon}(z) = 0 \quad \text{(see [15])}.
\]

3. **Statement of lemmas and propositions**

Hereafter, let us denote by \( w \in W^{1,p}(\mathbb{R}^N) \) a least energy solution of the problem
\[
-\Delta_p w + V_0 |w|^{p-2}w = f(w).
\]
Consequently, \( w \) satisfies
\[
c_{V_0} = J_{V_0}(w) = \inf_{v \in W^{1,p}(\mathbb{R}^N), \tau \geq 0} J_{V_0}(\tau v),
\]
where \( J_{V_0} \) is defined as
\[
J_{V_0}(v) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla v|^p + V_0 |v|^p) dx - \int_{\mathbb{R}^N} F(v) dx,
\]
for all \( v \in W^{1,p}(\mathbb{R}^N) \) (see [1]).
Let us denote by $u_\varepsilon$ the nodal solution of $(P_\varepsilon)_n$ obtained in Theorem 2.2. Adapting the same type of argument explored in [5, Lemma 2.1], it is easy to check that

\[(3.1) \limsup_{\varepsilon \to 0} \varepsilon^{-N} I_\varepsilon(u_\varepsilon) \leq 2cV_0.\]

The next lemma shows a situation where we have convergence on compact sets.

**Lemma 3.1.** Let $\{x_n\} \subset \overline{\Lambda}$ and let $\{\varepsilon_n\}$ be a sequence of positive numbers such that $\varepsilon_n \to 0$ as $n \to \infty$. If we set $v_n(x) = u_{\varepsilon_n}(\varepsilon_n x + x_n)$, then $\{v_n\}$ converges uniformly on compact subsets of $\mathbb{R}^N$.

**Proof.** See Section 5. \qed

The next result establishes a first information about the localization of maximum and minimum points of $u_\varepsilon$.

**Lemma 3.2.** Let $\{\varepsilon_n\} \subset (0, \infty)$ be a sequence with $\varepsilon_n \to 0$ and $P^1_n$, $P^2_n$ be, respectively, a maximum and minimum point of the function $u_{\varepsilon_n}$. Then, there exist $\delta^* > 0$ and a subsequence, still denoted by $\{u_{\varepsilon_n}\}$, such that $\{P^i_n\}$, $i = 1, 2$, are convergent sequences, $u_{\varepsilon_n}(P^1_n) \geq \delta^*$ and $u_{\varepsilon_n}(P^2_n) \leq -\delta^*$.

**Proof.** See Section 5. \qed

The next result is a key point to prove the existence of nodal solution to the original problem. A version of this result for $p = 2$ can be found in [5].

**Proposition 3.3.** If $P^1_\varepsilon$ is a maximum point of $u_\varepsilon^+$ and $P^2_\varepsilon$ is a minimum point of $u_\varepsilon^-$, then

\[\left| \frac{P^1_\varepsilon - P^2_\varepsilon}{\varepsilon} \right| \to \infty \quad \text{as } \varepsilon \to 0.\]

**Proof.** See Section 5. \qed

In order to use the Remark 2.1, the lemmas below are important in our arguments to prove that the family $\{u_\varepsilon\}$ satisfies the estimate $|u_\varepsilon(z)| \leq a$, for $z \in \Omega \setminus \Lambda$ when $\varepsilon$ is sufficiently small.

**Lemma 3.4.** Let $\{\varepsilon_n\}$ be a sequence of positive number with $\varepsilon_n \to 0$ as $n \to \infty$ and let $\{z^i_n\} \subset \overline{\Lambda}$, $i = 1, 2$, be a sequence such that $u_{\varepsilon_n}(z^1_n) \geq b > 0$ and $u_{\varepsilon_n}(z^2_n) \leq -b < 0$. Then,

\[\lim_{n \to \infty} V(z^i_n) = V_0 \quad \text{for } i = 1, 2.\]

**Proof.** See Section 5. \qed

**Lemma 3.5.** If $m^+_\varepsilon = \max_{\partial \Lambda} u_\varepsilon$ and $m^-_\varepsilon = \min_{\partial \Lambda} u_\varepsilon$, then $\lim_{\varepsilon \to 0} m^\pm_\varepsilon = 0$.

**Proof.** This Lemma follows directly from Lemma 3.4. \qed
4. Proof of Theorem 1.1

By Theorem 2.2, we have that problem \((P_\varepsilon)\) has a nodal solution \(u_\varepsilon\), for all \(\varepsilon \in (0, \varpi)\). By Lemma 3.5, \(m^+_\varepsilon < a\), for all \(\varepsilon \in (0, \varpi)\) and then \((u_\varepsilon - a)^+(z) = 0\) for a neighbourhood of \(\partial \Lambda\). Hence \((u_\varepsilon - a)^+ \in W^{1,p}_0(\Omega \setminus \Lambda)\) and the function \(\Psi\) given by \(\Psi(z) = 0\), if \(z \in \Lambda\) and \(\Psi(z) = (u_\varepsilon - a)^+(z)\), if \(z \in \Omega \setminus \Lambda\) belongs to \(W^{1,p}_0(\Omega)\). Using \(\Psi\) as a test function, we have

\[
\int_{\Omega \setminus \Lambda} \varepsilon^p|\nabla \Psi(z)|^p + \int_{\Omega \setminus \Lambda} \left[ V(z) |u_\varepsilon|^{p-2} - \frac{g(z, u_\varepsilon)}{u_\varepsilon} \right] |\Psi(z)|^2 + \int_{\Omega \setminus \Lambda} \left[ V(z) |u_\varepsilon|^{p-2} - \frac{g(z, u_\varepsilon)}{u_\varepsilon} \right] t_0 \Psi(z) = 0.
\]

The last equality implies

\[\Psi(z) = 0, \quad \text{a.e. in } z \in \Omega \setminus \Lambda.\]

Hence \(u_\varepsilon(z) \leq a\), for \(z \in \Omega \setminus \Lambda\). Since we can assume \(m^-_\varepsilon \geq -a\) for \(\varepsilon \in (0, \varpi)\), working with the function \((u_\varepsilon + a)^-\), it is possible to prove that \(u_\varepsilon(z) \geq -a\) for \(z \in \Omega \setminus \Lambda\). This fact implies that \(|u_\varepsilon(z)| \leq a\) for \(z \in \Omega \setminus \Lambda\), and the existence of a nodal solution follows from Remark 2.1. The concentration of the nodal solutions follows from Lemmas 3.2, 3.4 and 3.5 \(\square\)

5. Proofs of lemmas and propositions

In this section, we will prove the lemmas and propositions established in Section 3.

PROOF OF LEMMA 3.1. First of all, note that the sequence \(\{v_n\}\) satisfies the following problem

\[(P_n) \quad \begin{cases} -\Delta_p v_n + V(x_n + \varepsilon_n x)|v_n|^{p-2}v_n = g(x_n + \varepsilon_n x, v_n) & \text{in } \hat{\Omega}_{\varepsilon_n}, \\ v_n = 0 & \text{on } \partial \hat{\Omega}_{\varepsilon_n}, \end{cases}\]

where \(\hat{\Omega}_{\varepsilon_n} = \varepsilon_n^{-1}\{\Omega - x_n\}\). From (3.1), it follows that \(\{v_n\}\) is bounded in \(W^{1,p}(\mathbb{R}^N)\) and adapting some arguments explored in [3, Lemma 4.1], we have that the sequences \(v^+_n\) and \(v^-_n\) are bounded in \(L^\infty(\mathbb{R}^N)\). Thus, there exists a subsequence, still denoted by \(\{v_n\}\), such that for each compact subset \(K\) of \(\mathbb{R}^N\), there exists a constant \(C_K > 0\) with \(|\nabla v_n|_{\infty,K} \leq C_K\), for \(n\) sufficiently large. Therefore \(\{v_n\}\) converges uniformly on compact subsets of \(\mathbb{R}^N\). \(\square\)

PROOF OF LEMMA 3.2. Firstly, we will show that there exist \(\delta^+ > 0\) and \(n_0 \in \mathbb{N}\) such that

\[u_{\varepsilon_n}(P^1_n) \geq \delta^+ \quad \text{and} \quad u_{\varepsilon_n}(P^2_n) \leq -\delta^+ \quad \text{for } n \geq n_0.\]
Assume, by contradiction, that there exists a subsequence, still denoted by \( \{ u_{\varepsilon_n} \} \), such that

\[
u_{\varepsilon_n}(P^1_n) = \| u_{\varepsilon_n}^+ \|_{L^\infty(\mathbb{R}^N)} \to 0.
\]

Defining \( h_n(x) = u_{\varepsilon_n}(\varepsilon_n x) \), we have that \( \| h_n^+ \|_{L^\infty(\mathbb{R}^N)} \to 0 \). For a fixed \( r = V_0/2 \), it follows by (f) that there exists \( n_0 \in \mathbb{N} \) such that

\[
f(\| h_n^+ \|_{L^\infty(\mathbb{R}^N)}) \leq r \quad \text{for} \quad n \geq n_0.
\]

Hence,

\[
\int_{\mathbb{R}^N} |\nabla h_n^+|^p + \int_{\mathbb{R}^N} V_0 |h_n^+|^p \leq \int_{\mathbb{R}^N} f(\| h_n^+ \|_{L^\infty(\mathbb{R}^N)}) |h_n^+|^p \leq r \int_{\mathbb{R}^N} |h_n^+|^p,
\]

thus \( \| h_n^+ \|_{W^{1,p}(\mathbb{R}^N)} = 0 \) for \( n \geq n_0 \), which is impossible, because \( h_n^+ \neq 0 \), for all \( n \in \mathbb{N} \). Then, there exists \( \delta^* > 0 \) and \( n_1 \in \mathbb{N} \) such that \( \| u_{\varepsilon_n}^+ \|_{L^\infty(\mathbb{R}^N)} \geq \delta^* \), for all \( n \geq n_1 \). Repeating these arguments, we find \( n_2 \in \mathbb{N} \) such that \( \| u_{\varepsilon_n}^+ \|_{L^\infty(\mathbb{R}^N)} \geq \delta^* \), for all \( n \geq n_2 \). Choosing \( n_0 = \max\{n_1, n_2\} \), we have that

\[
u_{\varepsilon_n}(P^1_n) \geq \delta^* \quad \text{and} \quad \nu_{\varepsilon_n}(P^2_n) \leq -\delta^*, \quad \text{for all} \quad n \geq n_0.
\]

On the other hand, the function \( h_n \) satisfies the following problem

\[
(P_n^1) \quad \begin{cases} -\Delta_p h_n + V(\varepsilon_n x)|h_n|^{p-2}h_n = g(\varepsilon_n x, h_n) & \text{in} \; \Omega_{\varepsilon_n}, \\ h_n = 0 & \text{on} \; \partial\Omega_{\varepsilon_n}
\end{cases}
\]

where \( \Omega_{\varepsilon_n} = \Omega/\varepsilon_n \). Since \( V_0 = \inf_{z \in \mathbb{R}^N} V(z) \) and \( \limsup_{\varepsilon_n \to 0} (\varepsilon_n^{-N} I_{\varepsilon_n}(u_{\varepsilon_n})) \leq 2c_{V_0} \), defining

\[
J_n^\varepsilon(h_n) = \frac{1}{p} \int_{\Omega_{\varepsilon_n}} \left( |\nabla h_n|^p + V(\varepsilon_n x)|h_n|^p \right) dx - \int_{\Omega_{\varepsilon_n}} G(\varepsilon_n x, h_n) dx
\]

we have that \( J_n^\varepsilon(h_n) = \varepsilon_n^{-N} I_{\varepsilon_n}(u_{\varepsilon_n}) \) and consequently

\[
J_n^\varepsilon(h_n^+) \to c_{V_0} \quad \text{with} \quad h_n^+ \in \mathcal{N}_{\varepsilon_n},
\]

where \( \mathcal{N}_{\varepsilon_n} = \{ u \in W_n \setminus \{ 0 \} : J_n^\varepsilon(u) = 0 \} \).

Using [2, Lemma 4.3] and [3, Proposition 3.3], there exists a sequence \( \{ \tilde{y}_n \} \subset \mathbb{R}^N \), such that \( h_n(x) = h_n^+(x + \tilde{y}_n) \) converges strongly to a function \( h \in W^{1,p}(\mathbb{R}^N) \setminus \{0\} \). Moreover, the sequence \( \{ y_n \} \) given by \( y_n = \varepsilon_n \tilde{y}_n \), has a subsequence, still denoted by \( y_n \), such that \( y_n \to y \in M \). Now, to adapt some arguments explored in [2, Lemma 4.5], let us consider, for each \( n \in \mathbb{N} \) and \( L > 0 \), the functions

\[
v_{L,n}(x) = \begin{cases} \tilde{h}_n(x) & \text{if} \; \tilde{h}_n(x) \leq L, \\ L & \text{if} \; \tilde{h}_n(x) \geq L,
\end{cases}
\]

and

\[
z_{L,n} = \eta^p v_{L,n}^{(p-1)} \tilde{h}_n \quad \text{and} \quad w_{L,n} = \tilde{h}_n v_{L,n}^{\beta-1}
\]
with $\beta > 1$ to be picked up later. Remarkng that the function $\hat{h}_n(x) = h_n(x + \tilde{y}_n)$ is a critical point of the functional
\[
T(v) = \frac{1}{p} \int_{\Omega_{\varepsilon_n}} (|\nabla v|^p + V(\varepsilon_n x + y_n)|v|^p) \, dx - \int_{\Omega_{\varepsilon_n}} G(\varepsilon_n x + y_n, v) \, dx
\]
where $\hat{\Omega}_{\varepsilon_n} = \varepsilon_n^{-1} \{ \Omega - y_n \}$. Taking $z_{L,n}$ as a test function, it is possible to prove that, for a fixed $\gamma \in (0, \delta)$, there exists $R > 0$ such that
\[
\|\hat{h}_n\|_{L^\infty(|x| \geq R)} < \gamma \quad \text{for all } n \in \mathbb{N}.
\]
Consequently,
\[
u_n^+ (\varepsilon_n x + y_n) < \gamma, \quad \text{for all } n \in \mathbb{N} \text{ and } |x| \geq R.
\]
If $P_n^1$ denotes a maximum point of $u_{\varepsilon_n}$, we have that
\[
u_n (P_n^1) \geq \delta^* \quad \text{for all } n \in \mathbb{N}.
\]
So, if $P_n^1 = \varepsilon_n Q_n + y_n$ it follows from (5.1) and (5.2) that $|Q_n| \leq R$. Using the fact that $\{y_n\}$ converges to $y^1 \in M$, we can conclude that $\{P_n^1\}$ also converges to $y^1 \in M$ and therefore
\[
\lim_{n \to \infty} V(P_n^1) = V_0.
\]
Using similar arguments we can prove that the sequence $\{P_n^2\}$ also converges to some $y^2 \in M$. \hfill \Box

**Proof of Proposition 3.3.** Assume by contradiction that there exists a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ verifying
\[
\left| \frac{P_n^1 - P_n^2}{\varepsilon_n} \right| \to \beta < \infty,
\]
where $P_n^1 = P_{\varepsilon_n}$ and $P_n^2 = P_{\varepsilon_n}^2$.

Defining $v_n(x) = u_{\varepsilon_n}(P_n^1 + \varepsilon_n x)$, it is easy to check that $v_n$ is a solution of the problem
\[
(P_n') \quad -\Delta_p v_n + V(P_n^1 + \varepsilon_n x)|v_n|^{p-2}v_n = g(P_n^1 + \varepsilon_n x, v_n) \quad \text{in } \Omega_{\varepsilon_n}^1, \quad v_n = 0 \partial \Omega_{\varepsilon_n}^1
\]
where $\Omega_{\varepsilon_n}^1 = \varepsilon_n^{-1} \{ \Omega - P_n^1 \}$, that the function $v_n$ is a critical point of the functional
\[
J(v) = \frac{1}{p} \int_{\Omega_{\varepsilon_n}^1} (|\nabla v|^p + V(\varepsilon_n x + P_n^1)|v|^p) \, dx - \int_{\Omega_{\varepsilon_n}^1} G(\varepsilon_n x + P_n^1, v) \, dx
\]
and that $J(v_n) = \varepsilon_n^{-N} I_{\varepsilon_n}(u_{\varepsilon_n})$. Using the fact that $J'(v_n)(v_n) = 0$, from (3.1), we have that $\{v_n\}$ is a bounded sequence in $W^{1,p}(\mathbb{R}^N)$. Using the same type of arguments explored in the proof of Lemma 3.1, there exists a function $v \in W^{1,p}(\mathbb{R}^N)$ such that $\{v_n\}$ converges uniformly on compact subsets of $\mathbb{R}^N$. For each fixed compact subset $K \subset \mathbb{R}^N$, there exists $C_K > 0$ such that $|\nabla v_n|_{\infty,K} \leq C_K$, for all $x \in K$. Moreover, the sequence of functions $\chi_n(y) = \chi(\varepsilon_n y + P_n^1)$ can
be assumed to converge weakly on compacts subsets in any $L^r(\mathbb{R}^N)$ to a function $0 \leq \chi \leq 1$ and the function $v$ satisfies the problem

\[(\text{PL}) \quad -\Delta_p v + V(\mathcal{P})|v|^{p-2}v = \mathcal{g}(x, v) \quad \text{in } \mathbb{R}^N,\]

where

\[\mathcal{g}(z, s) = \chi(z)\mathcal{f}(s) + (1 - \chi(z))\tilde{\mathcal{f}}(s) \quad \text{and} \quad \mathcal{P} = \lim_{n \to \infty} P^1_n.\]

Suppose that $\beta = 0$. By Lemma 3.2 and the Mean Value Inequality, we have

\[(5.3) \quad 2\delta \leq |v_n(0) - v_n(Z_z)| \leq |\nabla v_n(Q_n)||Z_n|,\]

where $Z_n = (P^1_n - P^2_n)/\varepsilon_n$ and $Q_n = t_n Z_n$, for some $t_n \in [0, 1]$. Using the hypothesis that $\beta = 0$, we have that the sequence $\{Q_n\} \subset \mathcal{B}_1(0)$. Hence, there exists $C > 0$ such that $|\nabla v_n(Q_n)| \leq C$, for all $n \in \mathbb{N}$. Using the last inequality in (5.3), we obtain the inequality

\[(5.4) \quad 2\delta \leq |v_n(0) - v_n(Z_z)| \leq C|Z_n|,\]

leading to a contradiction with the fact that $\lim_{n \to \infty} Z_n = 0$. Therefore, $\beta > 0$ and $\mathcal{P} = \lim_{n \to \infty} Z_n \neq 0$. To conclude the proof, taking a subsequence if necessary, $v_n$ converges on compacts to a function $v \in W^{1,p}(\mathbb{R}^N)$. As $v_n(0) = u_{\varepsilon_n}(P^1_n) \geq \delta^*$ and $v_n(Z_z) = u_{\varepsilon_n}(P^2_n) \leq -\delta^* < 0$, it follows that $v(0) > 0$ and $v(P) < 0$, i.e. $v$ is a nodal function of (PL). Associated to (PL) we have the functional $\mathcal{J}: W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ defined by

\[\mathcal{J}(u) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u|^p + V(\mathcal{P})|u|^p) \, dx - \int_{\mathbb{R}^N} \mathcal{g}(z, u) \, dx, \quad u \in W^{1,p}(\mathbb{R}^N)\]

where $\mathcal{g}(z, s) = \int_0^s \mathcal{g}(z, t) \, dt$ and $\mathcal{P} = \lim_{n \to \infty} P^1_n$. Then, $v$ is a critical point of $\mathcal{J}$. Thus, $\mathcal{J}'(v) \varphi = 0$ for every $\varphi \in W^{1,p}(\mathbb{R}^N)$. In particular, $\mathcal{J}'(v)v^\pm = 0$, which implies $v \in M_{V(\mathcal{P})}$, where

\[M_{V(\mathcal{P})} = \left\{ u \in W^{1,p}_0(\Omega) : u^\pm \neq 0, \right. \]

\[\left. \int_\Omega (|\nabla u|^p + V(\mathcal{P})|u|^p) \, dx = \int_\Omega \mathcal{g}(z, u^\pm)u^\pm \, dx \right\}.\]

Since $J(v_n) = \varepsilon_n^{-N} I_{\varepsilon_n}(v_{\varepsilon_n})$, it follows that

\[J((v_n)^+), J((v_n)^-) \to c V_0 \quad \text{as} \quad J'((v_n)^+)((v_n)^+) = J'((v_n)^-)((v_n)^-) = 0,\]

thus, applying the same type of arguments found in [3], we can conclude that $\{(v_n)^+\}$ and $\{(v_n)^-\}$ are convergent sequences in $W^{1,p}(\mathbb{R}^N)$. Considering the sequences $\{t_n^+\} \subset \mathbb{R}$, $w_n^+(z) = t_n^+(v_n)^+(z)$ and $w_n^-(z) = t_n^-(v_n)^-(z)$ such that $J_{V_0}(w_n^+)w_n^+ = 0$ where

\[J_{V_0}(u) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u|^p + V_0|u|^p) \, dx - \int_{\mathbb{R}^N} F(u) \, dx, \quad u \in W^{1,p}(\mathbb{R}^N)\]
it is possible to show that the sequences \( \{t_n^i\} \) are also convergent in \( \mathbb{R} \). Using this information, we have that \( w_n^i \) converges to \( w^i \in W^{1,p}(\mathbb{R}^N) \) \( \setminus \{0\} \) with

\[
J'_V(w^i) = 0 \quad \text{and} \quad J_V(w^i) \geq c_V.
\]

Thus

\[
\varepsilon_n^{-N} I_{\varepsilon_n}(u_{\varepsilon_n}) \geq J'_V(t_n^i(v_n^i)^+) + J_V(t_n^i(v_n^i)^-)
\]

and so

\[
\varepsilon_n^{-N} I_{\varepsilon_n}(u_{\varepsilon_n}) \geq J'_V(w_n^i + v_n^i).
\]

Taking the limit as \( n \to \infty \) in the last inequality, we get

\[
2c_V \geq J'_V(w^i) + J_V(w^2) \geq 2c_V.
\]

The last inequality implies that \( J'_V(w^i) = c_V \). By Theorem 4.3 in [23], \( J'_V(w^i) = 0 \) for \( i = 1, 2 \), and by Maximum Principle implies \( w^1 > 0 \) and \( w^2 < 0 \) in \( \mathbb{R}^N \) (see [21]), consequently \( w^+ > 0 \) and \( w^- < 0 \) in \( \mathbb{R}^N \), obtaining this way an absurd. □

Proof of Lemma 3.4. In the next, we will argue by contradiction. Assume, passing to a subsequence if necessary, that \( z_n^i \to \pi_i \in \Pi, i = 1, 2 \) with \( V(\pi_1) > V_0 \) and \( V(\pi_2) \geq V_0 \). As in the above lemmas, we consider the sequences \( v_n^i(z) = u_{\varepsilon_n}(z_n^i + \varepsilon_n z), i = 1, 2 \) and study their behaviors as \( n \) goes to infinity. For \( i = 1, 2 \), the function \( v_n^i \) satisfies the problem

\[
(P_n)_i \quad \begin{cases} -\Delta_p v_n^i + V(\varepsilon_n z + z_n^i)|v_n^i|^{p-2}v_n^i = g(\varepsilon_n z + z_n^i) & \text{in } \Omega_n^i, \\ v_n^i = 0 & \text{on } \partial \Omega_n^i, \end{cases}
\]

where \( \Omega_n^i = \varepsilon^{-N}(\Omega - z_n^i) \). Again, the sequence \( v_n^i \) is bounded in \( W^{1,p}(\mathbb{R}^N) \), thus it can be assumed to converge uniformly on compacts subsets of \( \mathbb{R}^N \) to a function \( v^i \in W^{1,p}(\mathbb{R}^N) \). Now, the sequence of functions \( \chi_n^i(z) \equiv \chi(\varepsilon_n z + z_n^i) \) can be assumed to converge weakly in any \( L^p(\mathbb{R}^N) \) on compacts to a function \( 0 \leq \chi^i \leq 1 \). Therefore, \( v^i \) satisfies the limiting problem

\[
(PL)_i \quad -\Delta_p v^i + V(\pi)|v^i|^{p-2}v^i = \tilde{g}_i(z, v^i) \quad \text{in } \mathbb{R}^N
\]

where \( \tilde{g}_i(z, s) = \chi^i(z)f(s) + (1 - \chi^i(z))\tilde{f}(s) \) and \( \pi_i = \lim_{n \to \infty} z_n^i \). We claim that \( v^i \) does not change sign, for \( i = 1, 2 \). More precisely, \( v^1 \geq 0 \) and \( v^2 \leq 0 \). In fact, suppose, by contradiction, that \( v^1 \) changes sign. Let \( r > 0 \) be such that \( v^1 \) changes sign on the closed ball \( B[0,r] \) and that there exist \( Q^+_1, Q^-_1 \in B[0,r] \) such that

\[
(v_n^i)^+(Q^+_1) = \max_{z \in \mathbb{R}^N} (v_n^i)^+(z) \quad \text{and} \quad (v_n^i)^-(Q^-_1) = \min_{z \in \mathbb{R}^N} (v_n^i)^-(z).
\]

Since \( v_n^i \) converges uniformly to \( v^i \) on compacts subsets of \( \mathbb{R}^N \), there exists \( n_0 \) such that

\[
(v_n^i)^+(Q^+_1) \geq C > 0 \quad \text{and} \quad (v_n^i)^-(Q^-_1) \leq -C < 0, \quad \text{for all } n \geq n_0
\]
for some positive constant \( C \). Thus, for
\[
(v^+_n)^i(Q^+_1,n) = v^i_1(Q^+_1,n) = u_\varepsilon \zeta_n Q^+_1,n + z^+_1, \\
(v^-_n)^i(Q^-_1,n) = v^i_1(Q^-_1,n) = u_\varepsilon \zeta_n Q^-_1,n + z^-_1,
\]
considering
\[
\tilde{P}^+_n = \varepsilon_n Q^+_1,n + z^+_1 \quad \text{and} \quad \tilde{P}^-_n = \varepsilon_n Q^-_1,n + z^-_1
\]
it follows that \( \tilde{P}^+_n \) and \( \tilde{P}^-_n \) are maximum and minimum points of \( u_\varepsilon \zeta_n \) in \( \mathbb{R}^N \), respectively, with
\[
u_\varepsilon(\tilde{P}^+_n) \geq C \quad \text{and} \quad u_\varepsilon(\tilde{P}^-_n) \leq -C.
\]
Applying the same arguments employed in Proposition 3.1, we have
\[
\left| \frac{\tilde{P}^+_n - \tilde{P}^-_n}{\varepsilon_n} \right| \to \infty.
\]
But this is impossible because
\[
\left| \frac{\tilde{P}^+_n - \tilde{P}^-_n}{\varepsilon_n} \right| = |Q^+_1,n - Q^-_1,n| \leq 2r.
\]
Hence, since \( v^+_1(0) > 0 \) and \( v^+_n \to v^+ \) uniformly on compacts subsets of \( \mathbb{R}^N \), it follows that \( v^1 \geq 0 \). Moreover, using the Maximum Principle it follows that \( v^1 > 0 \) in \( \mathbb{R}^N \) (see [21]). A similar argument implies that \( v^2 < 0 \) in \( \mathbb{R}^N \).

Associated to the limiting problem \( (PL) \), we have the functional
\[
\mathcal{J}_i : H^1(\mathbb{R}^N) \to \mathbb{R}
\]
defined by
\[
\mathcal{J}_i(u) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u|^p + V(z_i)|u|^p) \, dx - \int_{\mathbb{R}^N} \mathcal{G}_i(z,u) \, dx, \quad u \in W^{1,p}(\mathbb{R}^N)
\]
where \( \mathcal{G}_i(z,s) = \int_0^s \mathcal{G}_i(z,t) \, dt \). Then, \( v^1 \) is clearly a critical point of \( \mathcal{J}_i \). Also associated to problem \( (P_n) \), we have the functional
\[
J^i_n(u) = \frac{1}{p} \int_{\Omega_n} (|\nabla u|^p + \varepsilon_n |u|^p) \, dx - \int_{\Omega_n} \mathcal{G}_i(\varepsilon_n z + z^+_1) \, dx,
\]
for \( u \in W^{1,p}_0(\Omega_n) \), which satisfies the following equality involving \( u_\varepsilon \zeta_n \) and \( v_n \)
\[
(5.5) \quad \varepsilon_n^{-N} J^i_n(u_\zeta_n) = \varepsilon_n^{-N} (J^i_n(u^+_\zeta_n) + J^i_n(u^-_\zeta_n)) = J^i_n((v^+_1)^+) + J^i_n((v^-_2)^-),
\]
so, repeating the idea explored in the proof of Proposition 3.1, the sequences \( \{(v^+_1)^+\} \), \( \{(v^-_2)^-\} \) are convergent in \( W^{1,p}(\mathbb{R}^N) \).

Considering again sequences \( \{t^i_n\} \subset \mathbb{R} \), \( w^+_n(z) = t^i_n(v^+_1)^+(z) \) and \( w^-_n(z) = t^i_n(v^-_2)^-(z) \) verifying \( \tilde{J}^i_n,w^+_n(w^-_n) = 0 \) where
\[
\tilde{J}^i_n,u(z) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u|^p + \varepsilon_n |z + z^+_1|^p) \, dx - \int_{\mathbb{R}^N} F(u) \, dx,
\]
for \( u \in W^{1,p}(\mathbb{R}^N) \), it is possible to prove that the sequences \( \{t_n^i\} \) are also convergent in \((0, \infty)\). Using this information, we have that \( w_n^i \) converges to \( w^i \in W^{1,p}(\mathbb{R}^N) \setminus \{0\} \) with

\[
J'_{V(z^i)}(w^i) = 0 \quad \text{and} \quad J_{V(z^i)}(w^i) \geq c_{V(z^i)}.
\]

From (5.5),

\[
\varepsilon_n^{-N} J_{\varepsilon_n}(u_{\varepsilon_n}) \geq \tilde{J}_n^1(t_n^1(v_n^1)^+) + \tilde{J}_n^2(t_n^2(v_n^2)^-)
\]

consequently

\[
\varepsilon_n^{-N} J_{\varepsilon_n}(u_{\varepsilon_n}) \geq \tilde{J}_{n,1}(w_n^1) + \tilde{J}_{n,2}(w_n^2).
\]

Taking the limit as \( n \to \infty \) in the last inequality, we get

\[
2c_{V_0} \geq J_{V(z_1)}(w^1) + J_{V(z_2)}(w^1) \geq c_{V(z_1)} + c_{V(z_2)}.
\]

Since by hypothesis, let us assume \( V(z_1) > V(0) \) and \( V(z_2) \geq V(0) \), it follows the inequality

\[
c_{V(z_1)} + c_{V(z_2)} > 2c_{V(0)}
\]

which contradicts (5.6).

\[ \Box \]

**Acknowledgments.** The authors are grateful to the referee for his/her remarks and to Thomas Bartsch for useful suggestions concerning the themes treated in this paper.

**References**


Manuscript received April 20, 2006

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