EXISTENCE, MULTIPLICITY AND CONCENTRATION OF POSITIVE SOLUTIONS FOR A CLASS OF QUASILINEAR PROBLEMS

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Abstract. Using variational methods we establish existence and multiplicity of positive solutions for the following class of quasilinear problems

\[-\Delta_p u + \lambda V(x)|u|^{p-2}u = \mu|u|^{p-2}u + |u|^{p^*-2}u \text{ in } \mathbb{R}^N\]

where \(\Delta_p u\) is the \(p\)-Laplacian operator, \(2 \leq p < N, p^* = pN/(N-p)\), \(\lambda, \mu \in (0, \infty)\) and \(V: \mathbb{R}^N \to \mathbb{R}\) is a continuous function verifying some hypothesis.

1. Introduction

We are concerned with the existence of positive solutions for the following class of quasilinear elliptic problems

\[
(P_{\lambda, \mu}) \quad \left\{ \begin{array}{l}
-\Delta_p u + \lambda V(x)|u|^{p-2}u = \mu|u|^{p-2}u + |u|^{p^*-2}u \quad \text{in } \mathbb{R}^N, \\
u > 0 \quad \text{in } \mathbb{R}^N, \\
u \in W^{1,p}(\mathbb{R}^N)
\end{array} \right.
\]

where \(\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u), 2 \leq p < N, p^* = pN/(N-p), \lambda, \mu \in (0, \infty)\) and \(V: \mathbb{R}^N \to \mathbb{R}\) is a continuous function. Nonlinear equations involving the
$p$-Laplacian $\Delta_p$ have been studied extensively in the last years, see for example [1]–[4], [14]–[16] and the references cited in these works. In this paper we study the problems $(P_{\lambda,\mu})$ with $V$ verifying the following hypotheses:

$(H_1)$ $V \geq 0, \Omega = \text{int}, V^{-1}(0)$ is a nonempty bounded set with smooth boundary.

$(H_2)$ There exists $M_0 > 0$ such that $\mathcal{L}\{x \in \mathbb{R}^N : V(x) \leq M_0\} < \infty$ where $\mathcal{L}$ denotes the Lebesgue measure in $\mathbb{R}^N$.

Such hypotheses were firstly posed to the potentials of a class of Schrödinger equations by Bartsch and Wang in the paper [5]. See also [6], [11] and [12]. Motivated by [6] and [11], we are here interested in the following problems related to $(P_{\lambda,\mu})$:

- Existence of least energy solutions for large $\lambda$.
- The concentration behaviour of the solutions as $\lambda \to \infty$.
- Multiplicity of solutions involving the Lusternick–Schnirelmann category of $\Omega$.

Here by a least energy solution we understand a positive solution with minima energy over all nontrivial solutions of $(P_{\lambda,\mu})$. By concentration behaviors we describe tendencies of solutions $u_\lambda$ of $(P_{\lambda,\mu})$ as $\lambda \to \infty$. Precisely, letting $(D_\mu)$ denote the limit problem

\[
\begin{cases}
-\Delta_p u = \mu |u|^{p-2}u + |u|^{p^*-2}u & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{in } \partial \Omega.
\end{cases}
\]

we say that the solutions $(u_n)$ of $(P_{\lambda_n,\mu})$ will be concentrate at a solution $u$ of $(D_\mu)$ if a subsequence converges strongly to $u$ in $W^{1,p}(\mathbb{R}^N)$ as $\lambda_n \to \infty$.

We say that a sequence $(u_n)$ of solutions of $(P_{\lambda_n,\mu})$ concentrates at a solution $u$ of $(D_\mu)$ if along a subsequence it converges to $u$ strongly in $W^{1,p}(\mathbb{R}^N)$ as $\lambda_n \to \infty$.

The paper is organized as follows. In Section 2 we shall fix some notations and give several technical results. Section 3 is devoted to prove the existence of positive solution for $(P_{\lambda,\mu})$, the main result reads as follows:

**Theorem A.** Assume $(H_1)$ and $(H_2)$ hold and $N \geq p^2$. Then, for every $0 < \mu < \mu_1$, there exists $\lambda_\mu > 0$ such that $(P_{\lambda,\mu})$ has at least energy solution $u_\lambda$ for each $\lambda \geq \lambda(\mu)$.

Here by $\mu_1$ we denote the first eigenvalue of $(-\Delta_p, W^{1,p}_0(\Omega))$. In Section 4 we shall study the concentrate behavior of the solutions found in the Theorem A, and the main result is:
Theorem B. Every sequence of solutions \((u_n)\) of \((P_{\lambda, \mu})\) such that \(\mu \in (0, \mu_1), \lambda_n \to \infty\) and \(I_{\lambda_n, \mu}(u_n) \to c < 1/NS^N/p\) as \(n \to \infty\) concentrates at a solution of \((D_{\mu})\).

In the above theorem, \(S\) is the best Sobolev constant of the imbedding \(D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)\), given by

\[
S = \inf_{u \in W^{1,p}(\mathbb{R}^N)} \frac{\|\nabla u\|^p}{\|u\|^{p^*}},
\]

and \(I_{\lambda, \mu}\) is functional related to \((P_{\lambda, \mu})\) given by

\[
I_{\lambda, \mu}(u) = \int_{\mathbb{R}^N} \left( \frac{1}{p} |\nabla u|^p + \frac{1}{p} (\lambda V(x) - \mu) |u|^p - \frac{1}{p^*} |u|^{p^*} \right) dx.
\]

In Section 5, we conclude the paper by showing a result of multiplicity which is related to the Lusternik–Schineralmann category of \(\Omega\) denoted by \(\text{cat}(\Omega)\). The result is the following:

Theorem C. Assume (H1) and (H2) hold and that \(N \geq p^2\). Then there exist \(0 < \mu^* < \mu_1\) and for each \(0 < \mu < \mu^*\) two numbers \(\Lambda(\mu) > 0\) and \(0 < c(\mu) < 1/NS^{(N/p)}\) such that, if \(\lambda \geq \Lambda(\mu)\), then \((P_{\lambda, \mu})\) has at least \(\text{cat}(\Omega)\) solutions with energy \(I_{\lambda, \mu} \leq c(\mu)\).

Our methods to the problems are variational. The solutions are obtained from critical points of \(I_{\lambda, \mu}\) on its Nehari manifold. Since the problem is posed on \(\mathbb{R}^N\) and the imbedding of \(W^{1,p}(\mathbb{R}^N)\) into \(L^{p^*}(\mathbb{R}^N)\) is not compact, we analyze the Palais–Smale sequences with the aid of the parameter \(\lambda\). We adapt an argument similar to that of Brézis and Nirenberg [10] to deal with the critical nonlinearity. By letting \(\mu\) small and \(\lambda\) large we connect the multiplicity of solutions with the topology of \(\Omega\); the idea here may go back to the work of Benci and Cerami [7] (see also, e.g. [6], [11] and [20]). In addition, since the \(p\)-Laplacian operator \(\Delta_p\) is nonlinear, it is clear that the arguments for general \(p \geq 2\) are more subtle than that for \(p = 2\).

2. Notations and technical results

From now on we always assume that (H1)–(H2) hold and that \(N \geq p^2\). We denote by \(\| \cdot \|_q\) and \(\| \cdot \|_{1,p}\) the usual norms in the Banach spaces \(L^q(\mathbb{R}^N)\) for \(q \in [1, \infty]\) and \(W^{1,p}(\mathbb{R}^N)\) respectively, and by \(\mu_1\) the first eigenvalue of the following problem

\[
\begin{cases}
-\Delta_p u = \eta |u|^{p-2} u & \text{in } \Omega, \\
u = 0 & \text{in } \partial \Omega.
\end{cases}
\]

Let

\[
E = \left\{ u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^p dx < \infty \right\}
\]
be the Banach space endowed with the norm
\[
\|u\| = \left(\|u\|_{L_p}^p + \int_{\mathbb{R}^N} V(x)|u|^p \, dx\right)^{1/p}
\]
which is equivalent to each of the norms
\[
\|u\|_\lambda = \left(\|u\|_{L_p}^p + \lambda \int_{\mathbb{R}^N} V(x)|u|^p \, dx\right)^{1/p}
\]
for \(\lambda > 0\).

**Lemma 2.1.** Let \(\lambda_n \geq 1\) and \(u_n \in E\) be such that \(\lambda_n \to \infty\) and \(\|u_n\|_{\lambda_n}^p < K\) for some positive constant \(K\). Then there exists \(u \in W_0^{1,p}(\Omega)\) such that, up to a subsequence, \(u_n \rightharpoonup u\) weakly in \(E\) and \(u_n \to u\) in \(L^p(\mathbb{R}^N)\).

**Proof.** Since \(\|u_n\|^p \leq \|u_n\|_{\lambda_n}^p < K\) we may assume that \(u_n \rightharpoonup u\) weakly in \(E\) and \(u_n \to u\) in \(L^p(\mathbb{R}^N)\). Let \(C_m = \{x : |x| \leq m, \, V(x) \geq 1/m\}, \, m \in \mathbb{N}\). Then
\[
\int_{C_m} |u_n|^p \leq m \int_{C_m} V(x)|u_n|^p \leq \frac{mK}{\lambda_n} \to 0 \text{ as } n \to \infty
\]
for every \(m\). This implies that \(u(x) = 0\) for a.e. \(x \in \mathbb{R}^N \setminus \Omega\). Hence, since \(\partial \Omega\) is smooth, \(u \in W_0^{1,p}(\Omega)\).

We now show that \(u_n \to u\) in \(L^p(\mathbb{R}^N)\). Let \(F = \{x \in \mathbb{R}^N : V(x) \leq M_0\}\) with \(M_0\) as in \((H_2)\). Then
\[
\int_{F^c} |u_n|^p \leq \frac{1}{\lambda_n M_0} \int_{F^c} \lambda_n V(x)|u_n|^p \leq \frac{K}{\lambda_n M_0} \to 0 \text{ as } n \to \infty.
\]
Setting \(B_R = \{x \in \mathbb{R}^N : |x| \leq R\}\), and choosing \(r \in (1, N/(N - p))\), \(r' = r/(r - 1)\), we have
\[
\int_{B_R \cap F} |u_n - u|^p \leq |u_n - u|_{L^p(F)}^{r'} \mathcal{L}(B_R \cap F)^{1/r'} \leq c\|u_n - u\|^p \mathcal{L}(B_R \cap F)^{1/r'} \to 0
\]
as \(R \to \infty\) due to \((H_2)\). Finally, since \(u_n \rightharpoonup u\) in \(L^p_{\text{loc}}(\mathbb{R}^N)\),
\[
\int_{B_R} |u_n - u|^p \, dx \text{ as } n \to \infty
\]
from where follows \(u_n \to u\) in \(L^p(\mathbb{R}^N)\). \(\square\)

Hereafter we denote by \(L_\lambda : W^{1,p}(\mathbb{R}^N) \to (W^{1,p}(\mathbb{R}^N))'\) the operator given by
\[
\langle L_\lambda u, v \rangle = \int_{\mathbb{R}^N} (|\nabla u|^{p-2} \nabla u \nabla v + \lambda V(x)|u|^{p-2}uv) \, dx
\]
and the number
\[
\gamma_\lambda = \inf \left\{ \int_{\mathbb{R}^N} (|\nabla u|^{p} + \lambda V(x)|u|^{p}) \, dx : u \in E, \, |u|_p = 1 \right\}
\]
It is easy to check that \(\gamma_\lambda\) is a nondecreasing function in \(\lambda\).
Lemma 2.2. For each $\mu \in (0, \mu_1)$ there is $\lambda(\mu) > 0$ such that
$$\gamma \lambda \geq \frac{(\mu + \mu_1)}{2} \quad \text{for all } \lambda \geq \lambda(\mu).$$
Consequently, there exists $\alpha_\mu > 0$ such that
$$\alpha_\mu \|u\|_\lambda^p \leq \int_{\mathbb{R}^N} (|\nabla u|^p + (\lambda V(x) - \mu)|u|^p) \, dx \quad \text{for all } u \in E \text{ and } \lambda \geq \lambda(\mu).$$

Proof. Assume by contradiction that there exists a sequence $\lambda_n \to \infty$ such that
$$\gamma \lambda_n < \frac{\mu + \mu_1}{2} \quad \text{for all } n \in \mathbb{N}$$
and
$$\gamma \lambda_n \to \tau \leq \frac{\mu + \mu_1}{2} \quad \text{as } n \to \infty.$$
Let $u_n \in E$ be such that $|u_n|_p = 1$ and $\langle L_{\lambda_n} u_n, u_n \rangle = \tau + o_n(1)$. Since
$$\|u_n\|_{\lambda_n}^p = \int_{\mathbb{R}^N} (|\nabla u_n|^p + (1 + \lambda_n V(x))|u_n|^p) \, dx$$
we have
$$\|u_n\|_{\lambda_n}^p \leq 2(1 + \mu_1)$$
for all $n$ large. By Lemma 2.1 there is $u \in W^{1,p}_0(\Omega)$ such that
$$u_n \rightharpoonup u \quad \text{weakly in } E \text{ and } u_n \to u \quad \text{in } L^p(\mathbb{R}^N).$$
Therefore
$$|u|_p = 1 \quad \text{and } \liminf_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^p \, dx \geq \int_{\mathbb{R}^N} |\nabla u|^p \, dx$$
so
$$\int_{\Omega} (|\nabla u|^p - \tau|u|^p) \, dx \leq \liminf_{n \to \infty} \int_{\mathbb{R}^N} (|\nabla u_n|^p - \tau|u_n|^p) \, dx$$
which implies
$$\int_{\Omega} (|\nabla u|^p - \tau|u|^p) \, dx \leq \liminf_{n \to \infty} (\langle L_{\lambda_n} u_n, u_n \rangle - \tau) = 0$$
and thus
$$\int_{\Omega} |\nabla u|^p \, dx \leq \tau \int_{\Omega} |u|^p \, dx = \tau < \mu_1$$
obtaining this way a contradiction. \qed

Consider the functional
$$I_{\lambda, \mu}(u) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u|^p + \lambda V(x)|u|^p - \mu|u|^p) \, dx - \frac{1}{p^*} \int_{\mathbb{R}^N} |u|^{p^*} \, dx$$
that is,
$$I_{\lambda, \mu}(u) = \frac{1}{p} (\langle L\lambda u, u \rangle - \mu|u|_p^p) - \frac{1}{p^*} |u|_{p^*}^{p^*}.$$
Then \( I_{\lambda,\mu} \in C^1(E, \mathbb{R}) \) and critical points of \( I_{\lambda,\mu} \) are solutions of
\[
-\Delta_p u + \lambda V(x) |u|^{p-2} u = \mu |u|^{p-2} u + |u|^{p^* - 2} u, \quad u \in W^{1,p}(\mathbb{R}^N).
\]

Recall that a sequence \( (u_n) \subseteq E \) is called a \((PS)_c\) sequence for \( I_{\lambda,\mu} \), if
\( I_{\lambda,\mu}(u_n) \rightarrow c \) and \( I'_{\lambda,\mu}(u_n) \rightarrow 0 \) as \( n \rightarrow \infty \). \( I_{\lambda,\mu} \) is said to satisfy the \((PS)_c\) condition if any \((PS)_c\) sequence contains a convergent subsequence.

**Lemma 2.3.** If \( \mu \in (0, \mu_1) \) and \( \lambda \geq \lambda(\mu) \), the functional \( I_{\lambda,\mu} \) satisfies the \((PS)_c\) condition for all \( c < 1/NS^{(N/p)} \).

**Proof.** By definition,
\[
(2.1) \quad I_{\lambda,\mu}(u_n) - \frac{1}{\lambda} I'_{\lambda,\mu}(u_n) u_n = \frac{1}{N} \langle (L\lambda u_n, u_n) - \mu |u_n|^{p^*} \rangle
\]
and
\[
(2.2) \quad I_{\lambda,\mu}(u_n) - \frac{1}{\lambda} I'_{\lambda,\mu}(u_n) u_n = \frac{1}{N} |u_n|^{p^*}.
\]

Using Lemma 2.2 and (2.1), we get that \( u_n \) is a bounded sequence in \( E \).

To prove that \( (u_n) \) has a strongly convergent subsequence in \( E \), we assume that \( \lambda(\mu) \) verifies the following inequality \( \lambda(\mu) \geq \mu/M_0 \), thus
\[
(2.3) \quad \lambda M_0 - \mu \geq 0 \quad \text{for all } \lambda \in [\lambda(\mu), \infty).
\]

Since \( (u_n) \) is a bounded in \( E \), we may assume without loss of generality that
\[
\begin{align*}
&u_n \rightharpoonup u \quad \text{in } E, \\
&u_n \rightarrow u \quad \text{in } L^p_{\text{loc}}(\mathbb{R}^N), \\
&u_n(x) \rightarrow u(x) \quad \text{a.e. in } x \in \mathbb{R}^N.
\end{align*}
\]

Moreover, using the same arguments developed in Garcia Azorero and Peral Alonso [14], Gueda and Veron [16] and Alves [1], we have
\[
|\nabla u_n|^{p-2} \frac{\partial u_n}{\partial x_i} \rightarrow |\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \quad \text{in } L^p(\mathbb{R}^N), \quad i = 1, \ldots, N.
\]

The above informations imply that \( u \) is a weak solution of
\[
-\Delta_p u + \lambda V(x) |u|^{p-2} u = \mu |u|^{p-2} u + |u|^{p^* - 2} u \quad \text{in } \mathbb{R}^N.
\]

Let \( w_n = u_n - u \). By the Brézis and Lieb Lemma [9], we have
\[
(2.4) \quad |V^{1/p} u_n|_{p^*} = |V^{1/p} u|_{p^*} + |V^{1/p} w_n|_{p^*} + o_n(1),
\]
\[
(2.5) \quad |u_n|_{p^*} = |u|_{p^*} + |w_n|_{p^*} + o_n(1).
\]

Moreover, using a lemma proved by Alves in [2], we also have
\[
(2.6) \quad \int_{\mathbb{R}^N} ||\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u - |\nabla w_n|^{p-2} \nabla w_n|^{p/(p-1)} \, dx = o_n(1).
\]
From (2.4)–(2.6) together $I'_\lambda(u_n) \to 0$ follow

\begin{equation}
(2.7) \quad (\langle L_\lambda w_n, w_n \rangle - \mu |w_n|^p_p) - |w_n|_{p^*}^{p^*} = o_n(1).
\end{equation}

By the last equality, up to a subsequence, we can assume that

$$
\lim_{n \to \infty} (\langle L_\lambda w_n, w_n \rangle - \mu |w_n|^p_p) = l \quad \text{and} \quad \lim_{n \to \infty} |w_n|_{p^*}^{p^*} = l \leq Nc < S^{N/p}.
$$

As in the proof of Lemma 2.1 one shows that

$$
\int_F |w_n|^p_p \, dx \to 0 \quad \text{as} \quad n \to \infty
$$

where $F = \{ x \in \mathbb{R}^N : V(x) \leq M_0 \}$. Using the inequality (2.3)

$$
S |w_n|_{p^*}^{p^*} \leq |\nabla w_n|^p_p \leq |\nabla w_n|^p_p + \int_F (\lambda V(x) - \mu) |w_n|^p_p \, dx
$$

hence

$$
S |w_n|_{p^*}^{p^*} \leq (\langle L_\lambda w_n, w_n \rangle - \mu |w_n|^p_p) + \mu \int_F |w_n|^p_p \, dx,
$$

or equivalently

$$
S |w_n|_{p^*}^{p^*} \leq (\langle L_\lambda w_n, w_n \rangle - \mu |w_n|^p_p) + o_n(1).
$$

Passing to the limit in the last inequality, we obtain $S^{(p/p^*)} \leq l$. Since $l < S^{(N/p)}$ it follows $l = 0$, hence $w_n \to 0$ in $E$. \hfill \Box

## 3. Existence of positive solutions

The main objective of this section is to prove the Theorem A. We begin recalling the definition of the Nehari manifold $M_{\lambda, \mu}$ related to the functional $I_{\lambda, \mu}$ given by

$$
M_{\lambda, \mu} = \{ u \in E \setminus \{ 0 \} : I'_{\lambda, \mu}(u) = 0 \}.
$$

Note that by well known arguments, we have that following equality

$$
c_{\lambda, \mu} = \inf_{u \in M_{\lambda, \mu}} I_{\lambda, \mu}(u) = \frac{1}{N} \inf_{v \in \mathcal{V}} \langle (L_0 u, u) - \mu |u|^p_p \rangle^{N/p}
$$

where $\mathcal{V} = \{ v \in E : |v|^p_p = 1 \}$.

Using arguments explored by Benci and Cerami [7], we have the following result:

**Proposition 3.1.** Let $u \in M_{\lambda, \mu}$ be a critical point of $I_{\lambda, \mu}$ with $I_{\lambda, \mu}(u) < 2c_{\lambda, \mu}$. Then $u$ does not change sign, hence, we can assume that it is a positive function of $(P_{\lambda, \mu})$.

Below, for every domain $D \subset \mathbb{R}^N$, we consider the functional

$$
I_{\mu, D}(u) = \frac{1}{p} \int_D (|\nabla u|^p - \mu |u|^p_p) \, dx - \frac{1}{p^*} \int_D |u|^{p^*} \, dx = \frac{1}{p} \langle (L_0 u, u) - \mu |u|^p_p \rangle - \frac{1}{p^*} |u|_{p^*}^{p^*}.
$$
on \( W^{1,p}_0(D) \). Its Nehari manifold is
\[
\mathcal{M}_{\mu, D} = \{ u \in W^{1,p}_0(D) \setminus \{0\} : \langle L_0 u, u \rangle - \mu |u|^p_p = |u|^p_p \}
\]
and
\[
c(\mu, D) = \inf_{u \in \mathcal{M}_{\mu, D}} I_{\mu, D}(u) = \frac{1}{N} \inf_{v \in \mathcal{V}_D} (\langle L_0 u, u \rangle - \mu |u|^p_p)^{N/p}
\]
where \( \mathcal{V}_D = \{ v \in W^{1,p}_0(D) : |v|^p_p = 1 \} \).

**Lemma 3.2.** If \( \mu \in (0, \mu_1) \), and \( \lambda \geq \lambda(\mu) \) then
\[
\frac{1}{N} (\alpha_\mu S)^{N/p} \leq c(\mu, \Omega) < \frac{1}{N} S^{N/p}.
\]

**Proof.** By Lemma 2.2,
\[
\alpha_\mu \|v\|^p_{W^{1,p}} \leq c(\mu, \Omega).
\]
Using the definitions of the numbers \( S, c_{\lambda, \mu} \) and \( c(\mu, \Omega) \), we have the following inequalities
\[
\frac{1}{N} (\alpha_\mu S)^{N/p} \leq c(\mu, \Omega).
\]
From the results showed by Guedda and Veron in [16], we know that
\[
c(\mu, \Omega) < \frac{1}{N} S^{N/p} \quad \text{for all} \ \mu \in (0, \mu_1)
\]
and \( c(\mu, \Omega) \) is achieved at some \( u_0 > 0 \) with \( u_0 \in W^{1,p}_0(\Omega) \cap C(\Omega) \). Therefore \( c_{\lambda, \mu} < c(\mu, \Omega) \), because otherwise would be also achieved at \( u_0 \) which vanish outside \( \Omega \). From Harnack’s inequality (see Trudinger [19]) follows that \( u_0 \equiv 0 \) in \( \mathbb{R}^N \), contradicting the fact that \( u_0 \) is positive on \( \Omega \).

**Proof of Theorem A.** Let \( (u^n) \) be a minimizing sequence for \( I_{\lambda, \mu} \) on \( \mathcal{M}_{\lambda, \mu} \). Then by Ekeland’s variational principle (see Ekeland [13]), we may assume that it is a (PS) sequence. It follows from Proposition 3.1 and Lemmas 2.3 and 2.4 that a subsequence converges to a least energy solution \( u_{\lambda} \) of \((P_{\lambda, \mu})\). \( \square \)

4. Concentration of the solutions

Now we prove Theorem B. We need two technical results. The first one is the following (cf. Alves, Carrião and Medeiros [3])

**Lemma 4.1.** Let \( F \in C^2(\mathbb{R}, \mathbb{R}^+) \) a convex and even function such \( F(0) = 0 \) and \( f(s) = F'(s) \geq 0 \) for all \( s \in [0, \infty) \). Then, for all \( \phi, \varphi \geq 0 \) we have
\[
|F(\phi - \varphi) - F(\phi) - F(\varphi)| \leq 2(f(\phi)\varphi + f(\varphi)\phi).
\]

**Proof.** Indeed, we have two cases to be considered. If \( \varphi \leq \phi \), by convexity we have
\[
\frac{F(\varphi) - F(0)}{\varphi - 0} \leq f(\phi),
\]
that is, $F(\phi) \leq f(\phi)\phi$. On the other hand, since $f' = F'' \geq 0$ we have that $f$ is nondecreasing and consequently

$$|F(\phi - \varphi) - F(\phi)| \leq \varphi \int_0^1 f(\phi - t\varphi) \, dt \leq \varphi f(\phi).$$

Therefore,

(4.1) \quad |F(\phi - \varphi) - F(\phi) - F(\varphi)| \leq 2\varphi f(\phi).

If $\phi \leq \varphi$, we repeat the above argument to find

(4.2) \quad |F(\phi - \varphi) - F(\phi) - F(\varphi)| \leq 2\phi f(\varphi).

From (4.1)–(4.2) the lemma follows. \hfill \Box

The second one reads as

**Proposition 4.2.** Let $u_n$ be a sequence of solutions related to $(P_{\lambda_n,\mu})$ with $\lambda_n \to \infty$. Then, if $w_n = u_n - u$ where $u$ is the weak limit of $u_n$ in $E$, we have

$$\langle L\lambda_n u_n, u_n \rangle = \langle L_0 u, u \rangle + \langle L\lambda_n w_n, w_n \rangle + o_n(1).$$

**Proof.** Using Lemma 4.1 with $F(u) = |u|^p$ ($p \geq 2$), $\phi = u_n$ and $\varphi = u$, we get

(4.3) \quad |u_n|^p + |u|^p - 2p\Theta_n \leq |w_n|^p \leq |u_n|^p + |u|^p + 2p\Theta_n

where $\Theta_n = |u_n|^{p-2}u_n u + |u|^{p-2}uw_n$. Repeating the same arguments explored in the proof of Lemma 2.1, we observe that $u \in W_0^{1,p}(\Omega)$, thus

$$\int_{\mathbb{R}^N} V(x)\Theta_n \, dx = 0$$

and, by (4.3),

$$\int_{\mathbb{R}^N} V(x)|u_n|^p \, dx = \int_{\mathbb{R}^N} V(x)|w_n|^p \, dx.$$ 

The last equality and Brézis and Lieb’s Lemma imply

$$\langle L\lambda_n u_n, u_n \rangle = \langle L_0 u, u \rangle + \langle L\lambda_n w_n, w_n \rangle + o_n(1).$$ \hfill \Box

**Proof of Theorem B.** Let $(u_n)$ be a sequence of solutions of $(P_{\lambda_n,\mu})$, $\mu \in (0, \mu_1), \lambda_n \to \infty$ such that

$$NI_{\lambda_n,\mu}(u_n) = \langle L\lambda_n u_n, u_n \rangle - \mu|u_n|^p \to Nc < S^{N/p}.$$ 

Then, it follows from Lemmas 2.1 and 2.2 that there exists a $u \in W_0^{1,p}(\Omega)$ such that along a subsequence $u_n \rightharpoonup u$ weakly in $E$ and

(4.4) \quad u_n \to u \quad \text{in } L^p(\mathbb{R}^N).
Since \( u_n \) is a solution of \((P_{\lambda_n, \mu})\), we have, for all \( v \in E \), the following equality:
\[
\int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u_n \nabla v + \lambda_n V(x)|u_n|^{p-2} u_n v - \mu|u_n|^{p-2} u_n v = \int_{\mathbb{R}^N} |u_n|^{p^*-2} u_n v.
\]
Using the Concentration-Compactness Principle by Lions \([17]\), and similar arguments found in \([14]\) and \([1]\), we have that
\[
g\text{arguments found in [14] and [1], we have that } u_n \rightarrow u \text{ in } L^{p^*}_{\text{loc}}(\Omega)
\]
which implies
\[
u \rightarrow u \text{ in } W^{1,p}_{\text{loc}}(\Omega).
\]
If \( v \in W^1_0(\Omega) \) then \( \int_{\mathbb{R}^N} V(x)|u_n|^{p-2} u_n v \, dx = 0 \) for all \( n \in \mathbb{N} \). So, letting \( n \rightarrow \infty \) in the above equality yields
\[
\int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u \nabla v - \mu|u_n|^{p-2} uv = \int_{\mathbb{R}^N} |u|^{p^*-2} uv \text{ for all } v \in W^{1,p}_0(\Omega).
\]
This implies that \( u \) is a solution of \((D_\mu)\). Setting \( w_n = u_n - u \), by Proposition 4.2 and Brézis and Lieb's Lemma
\[
\left\langle (L_{\lambda_n} w_n, w_n) - \mu|w_n|^p \right\rangle - |w_n|^{p^*}_{p^*} = o_n(1).
\]
We claim that \( |w_n|_{p^*} \rightarrow 0 \). Assume by contradiction that \( |w_n|^{p^*}_{p^*} \rightarrow t > 0 \). Then, since
\[
S|w_n|^{p^*}_{p^*} \leq |\nabla w_n|^p \leq \left( (L_{\lambda_n} w_n, w_n) - \mu|w_n|^p \right) + o_n(1)
\]
we have
\[
S|w_n|^{p^*}_{p^*} \leq |w_n|^{p^*}_{p^*} + o_n(1).
\]
Using the fact that \( |u_n|^{p^*}_{p^*} \geq |w_n|^{p^*}_{p^*} + o_n(1) \), we get
\[
S^{N/p} \leq \lim_{n \rightarrow \infty} |u_n|^{p^*}_{p^*} = Nc < S^{N/p},
\]
which is a contradiction. Therefore, \( |w_n|_{p^*} \rightarrow 0 \) and \( (L_{\lambda_n} w_n, w_n) - \mu|w_n|^p \rightarrow 0 \) which, jointly with (4.4), implies \( (L_{\lambda_n} w_n, w_n) \rightarrow 0 \) consequently,
\[
(4.5) \quad \int_{\mathbb{R}^N} (|\nabla w_n|^p + \lambda_n V|w_n|^p) \rightarrow 0.
\]
Now the combination of (4.4) and (4.5) shows that \( u_n \rightarrow u \) in \( E \) finishing the proof. \( \square \)

**Corollary 4.3.** For each \( \mu \in (0, \mu_1) \), \( \lim_{\lambda \rightarrow \infty} c_{\lambda, \mu} = c(\mu, \Omega) \).

**Proof.** By Lemma 3.2, \( c_{\lambda, \mu} \rightarrow c \leq c(\mu, \Omega) < (1/N)S^{N/p} \) and, by Theorem A, \( c_{\lambda, \mu} \) is achieved for \( \lambda \geq \lambda(\mu) \). So Theorem B implies that \( c \) is achieved by \( I_\mu, \Omega \) on \( M_{\mu, \Omega} \). Hence, \( c \geq c(\mu, \Omega) \). \( \square \)
5. Multiplicity of solutions involving $\text{cat} (\Omega)$

In this section we prove Theorem C which establishes the existence of multiple solutions related with category of set $\Omega$.

Following the arguments of Benci and Cerami [7], since $\Omega$ is a bounded smooth domain of $\mathbb{R}^N$, we may fix $r > 0$ small enough such that

$$\Omega^+_2 = \{ x \in \mathbb{R}^N : \text{dist}(x, \Omega) < 2r \} \quad \text{and} \quad \Omega^-_r = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > r \}$$

are homotopically equivalent to $\Omega$. Moreover, we may assume that $B_r = \{ x \in \mathbb{R}^N : |x| < r \} \subset \Omega$. We write $c(\mu, \Omega) < c(\mu, r) < \frac{1}{N} S^{N/p}$ for $0 < \mu < \mu_1$.

By Talenti [18], we know that the numbers $c(0, G)$ with $G \subset \mathbb{R}^N$ are independent of $G$, in the sense that $c(0, G) = (1/N) S^{N/p}$. Moreover, in Alves and Ding [4, Lemma 2.4] it was proved that

$$\lim_{\mu \to 0} c(\mu, G) = \frac{1}{N} S^{N/p}. \quad (5.1)$$

For $0 \neq u \in L^{p^*}(\Omega)$ we consider its center of mass

$$\beta(u) = \frac{\int_{\Omega} |u|^{p^*} x \, dx}{\int_{\Omega} |u|^{p^*} \, dx}.$$

Using the same arguments explored by Alves and Ding in [4, Lemma 3.3], we have the following result

**Lemma 5.1.** There exists a $\mu^* = \mu^*(r) \in (0, \mu_1)$ such that, for $0 < \mu < \mu^*$,

(a) $c(\mu, r) < 2c(\mu, \Omega)$,
(b) $\beta(u) \in \Omega^+_2$ for every $u \in M_{\mu, \Omega}$ with $I_{\mu, \Omega}(u) \leq c(\mu, r)$.

As in Bartsch and Wang [6], we choose $R > 0$ with $\Omega \subset B_R$ and set

$$\xi(t) = \begin{cases} 
1 & \text{for } 0 \leq t \leq R, \\
R/t & \text{for } R \leq t.
\end{cases}$$

Define

$$\beta_0(u) = \frac{\int_{\Omega} |u|^{p^*} \xi(|x|) x \, dx}{\int_{\Omega} |u|^{p^*} \, dx} \quad \text{for } u \in L^{p^*}(\mathbb{R}^N) \setminus \{0\}.$$ 

**Lemma 5.2.** There exist $\mu^* = \mu^*(r) \in (0, \mu_1)$ and for each $0 < \mu < \mu^*$ a number $\Lambda(\mu) \geq \lambda(\mu)$ with the following properties:

(a) $c(\mu, r) < 2\lambda_{\mu}$ for all $\lambda \geq \Lambda(\mu)$, and
(b) $\beta_0(u) \in \Omega^+_2$ for all $\lambda \geq \Lambda(\mu)$ and all $u \in M_{\lambda, \mu}$ with $I_{\lambda, \mu} \leq c(\mu, r)$. 


Proof. Assertion (a) follows immediately from Lemma 5.1 and Corollary 4.3. We now prove (b). Assume, by contradiction, that for \( \mu \) arbitrarily small there is a sequence \( (u_n) \) such that \( u_n \in \mathcal{M}_{\lambda_n, \mu}, \lambda_n \to \infty, I_{\lambda_n, \mu}(u_n) \to c \leq c(\mu, r) \) and \( \beta_0(u_n) \notin \Omega_{2r}^+ \). Then, by Lemma 2.1, there is \( u \in W_0^{1, p}(\Omega) \) such that \( u_n \rightharpoonup u \) weakly in \( E \) and \( u_n \to u \) in \( L^p(\mathbb{R}^N) \). We distinguish two cases:

**Case 1.** \( |u|^p_{p^*} \leq \langle L_0 u, u \rangle - \mu |u|^p_p \).

Let \( w_n = u_n - u \). Since \( V(x) = 0 \) for \( x \in \Omega \), as before, we have

\[
\langle L_{\lambda_n} u_n, u_n \rangle - \mu |u_n|^p_p = \langle L_0 u, u \rangle - \mu |u|^p_p + \langle L_{\lambda_n} u_n, w_n \rangle - \mu |w_n|^p_p + o_n(1).
\]

Using the fact that \( u_n \in \mathcal{M}_{\lambda_n, \mu} \),

\[
\langle L_{\lambda_n} u_n, w_n \rangle - \mu |w_n|^p_p \leq |w_n|^{p^*}_{p^*} + o_n(1).
\]

We claim that \( |w_n|^p_{p^*} \to 0 \). Assume by contradiction that \( |w_n|^p_{p^*} \to l > 0 \). Then, since

\[
S|w_n|^p_p \leq |\nabla w_n|^p_p \leq \langle L_{\lambda_n} w_n, w_n \rangle - \mu |w_n|^p_p + o_n(1)
\]

that is,

\[
S|w_n|^p_{p^*} \leq |w_n|^{p^*}_{p^*} + o_n(1).
\]

Recalling that \( |u_n|^p_{p^*} \geq |w_n|^p_{p^*} \), follows that

\[
S^{N/p} \leq \lim_{n \to \infty} |u_n|^{p^*}_{p^*} = Nc < S^{N/p},
\]

which a contradiction. Consequently, \( u_n \to u \) in \( L^p(\mathbb{R}^N) \) and, therefore, \( \beta_0(u_n) \to \beta(u) \). But, since \( I_{\mu, \Omega}(u) \leq c(\mu, r) \), it follows from Lemma 5.1 that \( \beta(u) \in \Omega_{2r}^+ \). This contradicts our assumptions that \( \beta_0(u_n) \notin \Omega_{2r}^+ \).

**Case 2.** \( |u|^p_{p^*} > \langle L_0 u, u \rangle - \mu |u|^p_p \).

In this case \( tu \in \mathcal{M}_{\mu, \Omega} \) for some \( t \in (0, 1) \) and, therefore,

\[
c(\mu, \Omega) \leq I_{\mu, \Omega}(tu) = \frac{t^p}{N} \langle L_0 u, u \rangle - \mu |u|^p_p \leq \lim_{n \to \infty} I_{\lambda_n, \mu}(u_n) \leq c(\mu, r).
\]

Since, by (5.1),

\[
\lim_{\mu \to 0} c(\mu, \Omega) = \lim_{\mu \to 0} c(\mu, r) = \frac{1}{N} S^{N/p},
\]

we have that for each \( \epsilon > 0 \),

\[
\lim_{n \to \infty} I_{\lambda_n, \mu}(u_n) - I_{\mu, \Omega}(tu) \leq \frac{\epsilon}{2N} \text{ for all } \mu \in (0, \mu^*)
\]

Consequently, there is a \( n(\mu) \) large enough such that

\[
||u_n(\mu)||_{p^*} - |tu|_{p^*}^* < \epsilon
\]

which implies

\[
|\beta_0(u_n(\mu)) - \beta(tu)| < r.
\]
From Lemma 5.1, \( \beta(tu) \in \Omega^+ \), consequently by the last inequality \( \beta_0(u_0(\mu)) \in \Omega^+ \), which is a contradiction. \( \square \)

We will apply the following result of [11] to prove Theorem C.

**Proposition 5.3.** Let \( I : M \to \mathbb{R} \) be an even \( C^1 \)–functional on a complete symmetric \( C^{1,1} \)-submanifold \( M \subset X \setminus 0 \) of some Banach space \( X \). Assume that \( I \) is bounded below and satisfies the Palais–Smale condition \((PS)_c\) for all \( c \leq b \). Further, assume that there are maps

\[
i : Z \to I^{\leq b} \quad \text{and} \quad \beta_0 : I^{\leq b} \to W
\]

where \( I^{\leq b} = \{ u \in M : I(u) \leq b \} \), whose compositions \( \beta_0 \circ i \) is a homotopy equivalence, and that \( \beta_0(u) = \beta_0(-u) \) for all \( u \in M \cap I^{\leq b} \). Then \( I \) has at least \( \text{cat}(Z) \) pairs \( \{ u, -u \} \) of critical points with \( I(u) = I(-u) \leq b \).

**Proof of Theorem C.** We are going to apply Proposition 5.3. Take \( X = E, Z = \Omega^+ \) and \( W = \Omega^+_r \). For \( 0 < \mu \leq \mu^* \) and \( \lambda \geq \Lambda(\mu) \) we consider \( I = I_{\lambda, \mu}, M = M_{\lambda, \mu} \) and \( b = c(\mu, r) \). As mentioned before, \( b < (1/N)^{N/p} \), hence by Lemma 5.2 shows that it is well defined from \( I_{\lambda, \mu}^c \) into \( M_{\lambda, \mu} \). By definition \( \beta_0(u) = \beta_0(-u) \). Let \( u_r \in W^{1,p}_0(B_r) \subset E \) be a minimizer of \( I_{\mu, B_r} \) on \( M_{\mu, B_r} \) with \( u_r > 0 \), radially symmetric. We define the map \( i \) by setting \( i(x) = u_r(\cdot - x) \). Since \( i(x) \equiv 0 \) in \( \mathbb{R}^N \setminus \Omega \) for every \( x \in \Omega^+ \), we have \( i(x) \in M_{\lambda, \mu} \) and \( I_{\lambda, \mu}(i(x)) = I_{\mu, B_r}(u_r) = c(\mu, r) \). The radial symmetry implies that \( \beta_0(i(x)) = x \) for every \( x \in \Omega^+ \). Now it follows from Proposition 5.3 that \( (\beta_{\lambda, \mu}) \) has at least \( \text{cat}(\Omega) \) solutions, finishing the proof. \( \square \)

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