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ON LIFESPAN OF SOLUTIONS TO THE EINSTEIN EQUATIONS

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Abstract. We investigate the issue of existence of maximal solutions to the vacuum Einstein solutions for asymptotically flat spacetime. Solutions are established globally in time outside a domain of influence of a suitable large compact set, where singularities can appear. Our approach shows existence of metric coefficients which obey the following behavior: $g_{\alpha\beta} = \eta_{\alpha\beta} + O(r^{-\delta})$ for a small fixed $\delta > 0$ at infinity (where $\eta_{\alpha\beta}$ is the Minkowski metric). The system is studied in the harmonic (wavelike) gauge.

1. Introduction

The analysis of the issue of existence of solutions to the Einstein equations is the subject of this paper. We want to examine an area of the spacetime, where we are able to control a metric describing the sought pseudo-Riemannian manifold. We consider

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \quad \alpha, \beta = 0, 1, 2, 3 \tag{1.1}$$

with the signature $(- + + +)$, where the summing convention is used. Points in the spacetime are denoted by $x = (x^0, x^1, x^2, x^3)$.

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We investigate the Cauchy problem for an initial submanifold which is required to be asymptotically flat. The solutions will be searched by the following initial problem

$$G_{\mu\nu} = T_{\mu\nu}, \tag{1.2}$$

$$g_{\mu\nu}|_{x^0=0} = g^0_{\mu\nu}, \quad \frac{\partial g_{\mu\nu}}{\partial x^0}|_{x^0=0} = g^1_{\mu\nu}, \tag{1.3}$$

where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ — the energy-momentum tensor describing influence of external forces. As examples of tensor $T_{\mu\nu}$ we may consider models of a collisionless gas given by the Vlasov equation [1], [3], [12], [15], or the relativistic Maxwell system taking into account influence of the electro-magnetic field [2], or others [13], [14].

Conditions (1.2)–(1.3) can be replaced by assumptions on the curvature tensor of the initial submanifold and the relations between them can be described via the Gauss–Codazzi equations [5].

The geometric structure of equations implies the Bianchi identity

$$G^{\mu\nu}_{;\nu} = 0, \quad T^{\mu\nu}_{;\nu} = 0, \tag{1.4}$$

where ";" denotes the covariant differentiation. It follows that we may look at system (1.2)–(1.3) as a set of ten equations with constraints (1.4). And it is related to the fact that the same geometry can be described by different metrics.

From the analytical point of view the geometrical invariance of the system causes serious difficulties. It is not obvious which type of coordinates are the best (or the most suitable) to investigate the issue of existence. In the only result [5] about the global in time existence and stability of solutions for the vacuum Einstein equations, the authors consider the so-called traceless coordinates. This form of the metric leads to a good structure of nonlinear terms which is related to the null condition property from the theory of the nonlinear wave equations [4], [10]. This approach enables to control solutions for all times under suitable smallness of initial data.

In our paper we want to consider a more general question about the existence of solutions. We search for maximal solutions to system (1.2)–(1.3) for a suitable large class of initial data, however the analysis will concentrate outside of the cone of influence of possible singularities (see Figure 1.1). As an answer we will obtain information about the asymptotic behavior of metric coefficients and about a domain $\mathcal{U}$, where the solution to (1.2)–(1.3) will be well defined. The result obtained here will be an improvement of paper [6], where similar problem for the vacuum equations has been studied.

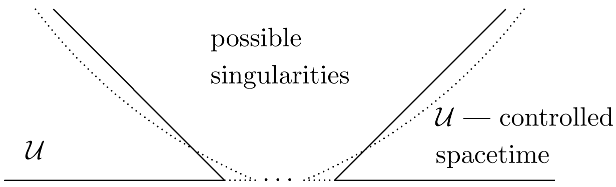

Figure 1.1

The initial data are defined on an initial submanifold Σ_0 as follows

$$g_{\mu\nu}|_{x^0=0} = g^0_{\mu\nu} \quad \text{on } \Sigma_0,$$
$$\left.\frac{\partial g_{\mu\nu}}{\partial x^0}\right|_{x^0=0} = g^1_{\mu\nu} \quad \text{on } \Sigma_0,$$

where $\Sigma_0 = \mathbb{R}^3 \setminus B(0, R)$.

Requirements of the asymptotic flatness can be described by the following relations

$$g^0_{\mu\nu} - \eta_{\mu\nu} \to 0 \quad \text{as } r \to \infty,$$

where $\eta_{\mu\nu}$ is the Minkowski metric

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

and $r = |\overline{x}| = ((x^1)^2 + (x^2)^2 + (x^3)^2)^{1/2}$.

Looking for some reasonable types of behavior of solutions at infinity we consider an important example, it is the Schwarzschild metric given by the following formula (in the spherical coordinates)

$$ds^2 = -(1 - 2M/r)(dx^0)^2 + (1 - 2M/r)^{-1}(dr)^2 + r^2(d\Omega)^2.$$

The above example describes a universe where all mass is localized, and influence of it implies the following spatial asymptotic behavior of metric coefficients

$$g_{\mu\nu} = \eta_{\mu\nu} + O(r^{-1}). \tag{1.5}$$

Thus, more rigoristic (faster) restrictions on vanishing at infinity would have no good physical interpretation.

To concentrate our investigations on dependence from initial data we assume that energy-momentum tensor $T_{\mu\nu}$ is also localized in the spacetime (i.e. we require to the support of $T_{\mu\nu}$ be in the cone of influence of possible singularities, see Figure 1.1), hence we reduce problem (1.2)–(1.3) to the case

$$T_{\mu\nu} \equiv 0. \tag{1.6}$$

We want to consider initial data which fulfill at most relation (1.5). However, we are interested not only in globally regular data, we admit singularities located only in a ball $B(0,R)$. On submanifold Σ_0 metric coefficients $g^0_{\mu\nu}$, $g^1_{\mu\nu}$ are required to be sufficiently regular. As we will see the asymptotic structure allows to examine our issue with no restrictions on largeness of initial data.

To reach our aims we examine system (1.1) in the harmonic (wavelike) coordinates. It follows that the Einstein equations are of hyperbolic type, more precisely, we obtain a set of nonlinear wave equations on metric $g_{\alpha\beta}$ [2], [3], [7]. As we know this gauge is well defined, however for global analysis we meet difficulties with stability of solutions. Nevertheless, to answer on our question this approach will simplify the structure of equations and enables to concentrate our attention only on the analytical difficulties.

We will search for metrics satisfying the asymptotic behavior

$$g_{\mu\nu} = \eta_{\mu\nu} + O(r^{-\delta}) \quad \text{in } \mathcal{U}, \tag{1.7}$$

with initial data fulfilling

$$\begin{aligned} g^0_{\mu\nu} &= \eta_{\mu\nu} + O(r^{-\delta}) && \text{on } \Sigma_0, \\ g^1_{\mu\nu} &= O(r^{-1-\delta}) && \text{on } \Sigma_0 \end{aligned} \tag{1.8}$$

for a small fixed number $\delta > 0$ with suitable regularity which will be stated precisely in the next section of the paper. Conditions (1.8) will be improved, however to define them we will need some analytical notations (see Theorem 2.1). Additionally we assume that (1.8) generates the metric in the harmonic gauge, (so conditions (2.7) will be fulfilled).

The main result of our paper is the following.

THEOREM 1.1. *Let $g^0_{\mu\nu}, g^1_{\mu\nu}$ be defined on the initial submanifold Σ_0 and be sufficiently smooth, moreover let*

$$g^0_{\mu\nu}|_{\Sigma_0},\ g^1_{\mu\nu}|_{\Sigma_0} \text{ satisfy condition (1.8).}$$

Then there exists the maximal solution defined on domain $\mathcal{U}$, see Figure 1.1 such that metric coefficients $g_{\mu\nu}$ fulfill (1.7) and set $\mathcal{U}$ satisfies the following inclusion

$$\mathcal{U} \supset ((0,\infty)\times\mathbb{R}^3) \setminus \{x : (1+\varepsilon)x^0 \le r + M\} \tag{1.9}$$

for any $\varepsilon > 0$, provided M sufficiently large.

The result of Theorem 1.1 characterizes the spacetime generated by the initial data satisfying asymptotic behavior (1.8). The method enables to prove inclusion (1.9), which says that outside of the domain of influence of "large data" our spacetime is, by relation (1.7), a perturbation of the Minkowski manifold. The magnitude of constant M and conditions (1.8) will reduce any large initial data to the case of small solutions.

A similar result for the studied problem has been proved in [6] with the same asymptotic behavior as in (1.7), however here we get a better characterization of the lifespan of solutions, see (1.9). The difference follows from another analytical approach which is applied in proofs. Also we refer to [11], where perturbation of the Schwarzschild metric has been examined but from a different point of view.

Our technique has an analytical character and does not deal with geometrical aspects of the subject. We concentrate our analysis on examination of a hyperbolic system by a reduction of the problem to a case of "small solutions".

Theorem 1.1 will be a conclusion of Theorem 2.1 stated in the next section. A key element of the proof will be an analysis of the behavior of the speed of propagation for system (1.2)–(1.3).

2. Analytical statement

The following section introduces the analytical background to state our result from the theory of PDEs point of view. We study a spacetime manifold generated by an initial submanifold which is required to be asymptotically flat. Since we assume that the support of $T_{\mu\nu}$ is localized (see (1.6)), we concentrate our attention on the vacuum equations, i.e. the Einstein system takes the following form

$$G_{\mu\nu} = 0. \tag{2.1}$$

Since the Einstein tensor is given by the following definition

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R,$$

where $R_{\mu\nu}$ is the Ricci tensor and $R = g^{\mu\nu} R_{\mu\nu}$ is the scalar curvature. For not degenerated metrics system (2.1) is equivalent to the following set of equations (see [2], [12])

$$R_{\mu\nu} = 0. \tag{2.2}$$

Our spacetime is required to be a pseudo-Riemannian manifold with a metric $g_{\alpha\beta}$ with the signature $(-+++)$. It follows that at least locally we have

$$-g_{00} > a, \quad b|X|^2 \leq g_{kl} X^k X^l \leq c|X|^2 \quad \text{for } X \in \mathbb{R}^3, \tag{2.3}$$

where $a, b, c > 0$ and $k, l = 1, 2, 3$.

Let us recall, the Ricci tensor is given by the Christoffel symbols

$$R_{\alpha\beta} = \partial_\sigma \Gamma^\sigma_{\alpha\beta} - \partial_\alpha \Gamma^\sigma_{\beta\sigma} + \Gamma^\sigma_{\alpha\beta} \Gamma^\rho_{\rho\sigma} - \Gamma^\rho_{\alpha\sigma} \Gamma^\sigma_{\beta\rho},$$

where Christoffel symbols are given by the metric coefficients as follows

$$\Gamma^\gamma_{\beta\mu} = \frac{1}{2} g^{\alpha\gamma} (g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}).$$

The general form of the Ricci tensor $R_{\alpha\beta}$ is complex and to see a hyperbolic character of equations (2.2) it is better to look on them in a special coordinate system. The easiest way is to take the harmonic coordinates, called wavelike, too. They are defined by contracted Christoffel symbols

$$\Gamma^\mu = g^{\alpha\beta}\Gamma^\mu_{\alpha\beta}.$$

And we say that the metric is harmonic if and only if

$$\Gamma^\mu = 0 \quad \text{for } \mu = 0, 1, 2, 3.$$

Classical results guarantee that the extension of this type of the coordinates [2] exists as far as the metric exists.

An advantage of the chosen setting is the form of the Ricci tensor. In the harmonic coordinates it reads

$$R_{\mu\nu} = -\frac{1}{2} g^{\alpha\beta} \frac{\partial^2}{\partial x^\alpha \partial x^\beta} g_{\mu\nu} + H_{\mu\nu},$$

where $H_{\mu\nu} = H_{\mu\nu}(g_{\alpha\beta}, (\partial g_{\alpha\beta}/\partial x^\gamma))$ and term H_μ is a bilinear operator with respect to the first derivative of the metric coefficients, i.e. symbolically we have

$$H \sim \widetilde{H}(g) \cdot Dg \cdot Dg, \tag{2.4}$$

where D denotes the whole gradient operator, i.e. $D = (\partial_0, \partial_1, \partial_2, \partial_3)$.

Hence the Einstein equations in the harmonic gauge read

$$-\frac{1}{2} g^{\alpha\beta} \frac{\partial^2}{\partial x^\alpha \partial x^\beta} g_{\mu\nu} + H_{\mu\nu} = 0. \tag{2.5}$$

To start the investigation we describe the initial data. At the beginning we require

$$g_{\mu\nu}|_{x^0=0} = \eta_{\mu\nu} + O(r^{-\delta}), \quad \left.\frac{\partial g_{\mu\nu}}{\partial x^0}\right|_{x^0=0} = O(r^{-1-\delta}) \tag{2.6}$$

for a fixed $\delta > 0$. Moreover, due to our harmonic gauge we assume additionally the following initial conditions

$$\Gamma^\mu|_{x^0=0} = 0, \quad \left.\frac{\partial \Gamma^\mu}{\partial x^0}\right|_{x^0=0} = 0 \tag{2.7}$$

which will guarantee preserving of harmonic coordinates. In general, these conditions are required to be satisfied only for

$$\overline{x} = (x^1, x^2, x^3) \in \mathbb{R}^3 \setminus B(0, R)$$

for sufficiently large R.

Our aim is to establish an existence result in a domain

$$\mathcal{M} = ((0, \infty) \times \mathbb{R}^3) \setminus S,$$

where $S = \{x \in (0,\infty) \times \mathbb{R}^3 : \kappa x^0 \leq r + M\}$, ($M > 0$ will be chosen letter), $r = |\overline{x}| = ((x^1)^2 + (x^2)^2 + (x^3)^2)^{1/2}$ and κ is a bound of the maximal speed of the propagation to system (2.5), i.e.

$$\kappa > s^* = \sup\{\text{speed of propagation in } \mathcal{M} \text{ to } (2.5)\}. \tag{2.8}$$

This type of results is possible to prove for hyperbolic systems, since we are able to control the speed of propagation of information. Any modification of equations (2.5) in sector S will follow no changes of solutions in domain $\mathcal{M}$.

Let us introduce the following auxiliary (cut off) function.

$$w(x) = \begin{cases} 1 & \text{for } \kappa x^0 \geq r + M, \\ 0 & \text{for } \kappa_* x^0 \leq r + M_0, \\ \in [0,1] & \text{for } \kappa_* x^0 - M_0 < r < \kappa x^0 - M, \end{cases} \tag{2.9}$$

where κ is given by (2.8) and numbers κ_*, M_0 and M satisfies the below relations

$$1 < \kappa_* < \kappa \quad \text{and} \quad M_0 < M. \tag{2.10}$$

Moreover, $w(\,\cdot\,)$ is sufficiently smooth and $|Dw| \leq c(1 + (\kappa - \kappa_*)x^0)^{-1}$. Also to simplify the examination we set $\kappa_* < 2$.

Applying function $w(\,\cdot\,)$ we modify searched functions $g_{\alpha\beta}$ as follows

$$d_{\alpha\beta} = \eta_{\alpha\beta} + w h_{\alpha\beta},$$

where $h_{\alpha\beta}$ are given as solutions to the following system being a modification of equations (2.5)–(2.6)

$$-\frac{1}{2} d^{\alpha\beta} \frac{\partial^2}{\partial x^\alpha \partial x^\beta} h_{\mu\nu} + w H_{\mu\nu}(\eta_{\alpha\beta} + h_{\alpha\beta}, Dh_{\alpha\beta}) = 0 \quad \text{in } \mathbb{R}^3 \times (0,T), \tag{2.11}$$

$$h_{\mu\nu}|_{x^0=0} = h^0_{\mu\nu} = w g^0_{\mu\nu} \quad \text{on } \mathbb{R}^3, \tag{2.12}$$

$$h_{\mu\nu,0}|_{x^0=0} = h^1_{\mu\nu} = w g^1_{\mu\nu} \quad \text{on } \mathbb{R}^3. \tag{2.13}$$

Analysis of the above system will give information for problem (2.5)–(2.6) in set $\mathcal{M}$. A key idea is that smallness of solutions $h_{\mu\nu}$ will control the maximal speed of propagation in set $\mathcal{M}$. It is possible, since choosing suitable large M_0 we obtain smallness of data $h^0_{\mu\nu}$ and $h^1_{\mu\nu}$, although (2.12) can be arbitrary large. As a consequence of these considerations we get the following relations between solutions to systems (2.5)–(2.6) and (2.11)–(2.13) as follows

$$\begin{aligned} h_{\mu\nu} &= g_{\mu\nu} - \eta_{\mu\nu} \quad && \text{in } \mathcal{M}, \\ d_{\alpha\beta} &= g_{\alpha\beta} && \text{in } \mathcal{M}, \end{aligned} \tag{2.14}$$

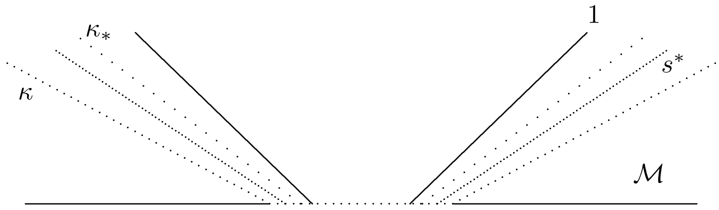

FIGURE 2.1

where $g_{\mu\nu}$ is the solution to problem (2.5)–(2.6) in the harmonic gauge, see Figure 2.1.

To begin the statement of our results we introduce some notations and relations to precise our mathematical background. First, let us define vector fields related to the Minkowski spacetime. We introduce

$$T_\mu = \partial_\mu \qquad T = \{T_\mu : \mu = 0,1,2,3\}, \tag{2.15}$$

$$L_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu \qquad L = \{L_{\mu\nu} : 0 \le \mu < \nu \le 3\}, \tag{2.16}$$

$$S = x^\mu\partial_\mu. \tag{2.17}$$

The whole set of the above vectors fields are denoted by A and

$$A = (T, L, S) = \{\Gamma_a, a \in I\}, \tag{2.18}$$

where I is a finite set of appropriate indices.

Next, we define Banach spaces $G^m(\mathbb{R}^3; 0, T)$ by the following norm

$$||u||_{G^m(\mathbb{R}^3;0,T)} = \sup_{0\le t\le T} ||u(t,\cdot)||_{\overline{G}^m(\mathbb{R}^3)}, \tag{2.19}$$

where

$$||u||_{\overline{G}^m(\mathbb{R}^3)} = \sum_{0\le l\le m} \sum_{\Gamma\in A} \left(\int_{\mathbb{R}^3} |\Gamma_1 \dots \Gamma_l u(x)|^2 \, dx \right)^{1/2}, \tag{2.20}$$

where the sum extended over all vector fields $\Gamma_1, \dots, \Gamma_l$ belonging to set A.

The kernel of the paper is the following result.

THEOREM 2.1. *Let $\delta > 0$, κ, κ_* fulfill* (2.10), *and initial data* (2.12)–(2.13) *satisfy the following regularity conditions:*

$$h^0_{\alpha\beta}(1+r)^\delta \in L_\infty(\mathbb{R}^3) \quad \text{and} \quad \nabla h^0_{\alpha\beta},\ h^1_{\alpha\beta} \text{ such that } Dh_{\alpha\beta}|_{x^0=0} \in \overline{G}^2(\mathbb{R}^3).$$

If

$$||h^0_{\alpha\beta}(1+r)^\delta||_{L_\infty(\mathbb{R}^3)} + ||Dh_{\alpha\beta}|_{x^0=0}||_{\overline{G}^2(\mathbb{R}^3)} \le X_0 \tag{2.21}$$

and X_0 is sufficiently small, then there exists a global in time solution to problem (2.11)–(2.13) *such that:*

$$(2.22) \qquad Dh_{\alpha\beta} \in G^2(\mathbb{R}^3; 0, \infty) \quad \text{and} \quad h_{\alpha\beta}(1+r)^{\delta} \in L_\infty(\mathbb{R}^3 \times (0,\infty)),$$

where $\nabla = (\partial_1, \partial_2, \partial_3)$ *and* $D = (\partial_0, \partial_1, \partial_2, \partial_3)$.

Although, the general theory for nonlinear wave equations with nonlinearity of the second order as (2.4) does not imply global in time existence of solutions [8], we are able to obtain a priori estimates (2.22) using standard techniques. Our approach is effective, since the character of nonlinear terms reduces our considerations only to analysis on the support of function w. This modification allows to apply the whole information which can be obtained from analysis in spaces G^m (see Propositions 3.1–3.3 in the next section). Comparing Theorem 2.1 to results from [6] we apply here techniques from [9], [10] for the nonlinear wave equations which essentially extends the standard energy methods.

The required regularity of initial data is not optimal. However in our approach it is better to work with integer order of derivatives, because definitions (2.19) and (2.20) for fractional derivatives would be more complex and the energy method could be less effective. The sharp result in the L_2-framework has been proved in [7].

Proof of Theorem 1.1. Suppose Theorem 2.1 is proved, we show Theorem 1.1. Choosing $\varepsilon > 0$, we take $\kappa = 1 + \varepsilon$, it follows that we find so large M_0 and M in (2.9) that X_0 is sufficiently small. Then solutions given by Theorem 2.1 generate a spacetime with the maximal speed of propagation less than $1+\varepsilon$. It can be possible as we have assumptions (2.6) with extra restrictions on regularity of initial data. Then the basic features of hyperbolic systems imply that on domain $\mathcal{M}$ we have (2.14). Let us note that there is no un-uniqueness effects, since the regularity of solutions is sufficiently large. Theorem 1.1 has been proved. $\square$

In the next section we introduce definitions and auxiliary results for spaces G^m. Section 4 proves the a priori bound, which via local existence results will show Theorem 2.1.

3. Preliminaries

In this section we introduce some auxiliary results and notations necessary to prove Theorem 2.1. First, let us recall some results for spaces G^m from [9].

Proposition 3.1. *Let* $u \in G^2(\mathbb{R}^3; 0, \infty)$*, then for any* $x \in (0,\infty) \times \mathbb{R}^3$ *holds*

$$(3.1) \qquad |u(x)| \le c(1 + |r - x^0|)^{-1/2}(1 + |r + x^0|)^{-1}||u||_{G^2(\mathbb{R}^3;0,\infty)}.$$

Proposition 3.2. *Using notation* (2.15)–(2.17), *we have*

$$\partial_\mu = (x^\nu L_{\mu\nu} + x_\mu S)\langle S, S\rangle^{-1},$$

where $\langle S, S\rangle = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$. *It follows that for sufficiently smooth functions* u *the following estimate is valid*

$$|Du(x)| \le c|x^0 - r|^{-1}\Big(|Su| + \sum_{0\le\mu<\nu\le 3} |L_{\mu\nu}u|\Big) = c|x^0 - r|^{-1}|\Lambda u|. \tag{3.2}$$

Proposition 3.3. *Let* $u \in G^1(\mathbb{R}^3; 0, \infty)$, *then* $u \in L_\infty(0, \infty; L_4(\mathbb{R}^3))$ *and*

$$||Wu(x^0, \cdot)||_{L_4(\mathbb{R}^3)} \le c(1 + (\kappa_* - 1)x^0)^{-3/4}||u||_{G^1(\mathbb{R}^3,0,\infty)}, \tag{3.2}$$

where

$$W = \begin{cases} 1 & \textit{for } x \in \operatorname{supp} w, \\ 0 & \textit{for } x \in \mathbb{R}^3 \times (0,\infty) \setminus \operatorname{supp} w. \end{cases} \tag{3.4}$$

Proof. Since Proposition 3.3 is not proved in [9], we show it here. Imbedding $u \in L_4(\mathbb{R}^3)$ is trivial. To prove estimate (3.2) we apply the Marcinkiewicz interpolation theorem (for general theory, see [16]). By Proposition 3.1 we see that

$$||Wu||_{L_\infty(\mathbb{R}^3)} \le c(1 + (\kappa_* - 1)x^0)^{-3/2}||u||_{G^2}. \tag{3.4}$$

Thus, operator T, being an embedding $u \to u(x^0, \cdot)$, is bounded as a map

$$T\colon G^2 \to L_\infty,$$

where the norm is described by Proposition 3.1. Moreover, we consider the trivial embedding

$$T\colon G^0 \to L_2. \tag{3.5}$$

Applying the Marcinkiewicz interpolation theorem, we get boundedness of the following operator

$$T\colon (G^2, G^0)_{1/2,2} \to (L_\infty, L_2)_{1/2,2} \tag{3.6}$$

with the following norm

$$||T||_{L((G^2,G^0)_{1/2,2};(L_\infty,L_2)_{1/2,2})} = ||T||^{1/2}_{L(G^2,L_\infty)}||T||^{1/2}_{L(G^0,L_2)}. \tag{3.7}$$

Since the norm of (3.5) is equal one, from (3.4) and (3.7) we conclude the following bound of the norm of operator (3.6)

$$||T||_{L((G^2,G^0)_{1/2,2};(L_\infty,L_2)_{1/2,2})} \le c(1 + (\kappa_* - 1)x^0)^{-3/4}.$$

To finish the proof, let us note that

$$(L_\infty, L_2)_{1/2,2} = L_{4,2} \subset L_4,$$

where $L_{4,2}$ is the standard Lorentz space, moreover we have

$$(G^2, G^0)_{1/2,2} = G^1.$$

Thus (3.3) follows from (3.7). The proof of Proposition 3.3 is finished. $\square$

Let us recall commutator rules for the standard d'Alambert operator (where $\square = -\partial_0^2 + \partial_1^2 + \partial_2^2 + \partial_3^2$, see also (2.15)–(2.17))

$$\begin{aligned} [\square, T_\mu] &= 0 \qquad \text{for} \quad 0 \le \mu \le 3, \\ [\square, L_{\mu\nu}] &= 0 \qquad \text{for} \quad 0 \le \mu < \nu \le 3, \\ [\square, S] &= 2\square. \end{aligned}$$

Main considerations will be concentrated on system (2.11)–(2.13), hence we distinguish the following operator

$$\square_d = d^{\alpha\beta} \frac{\partial^2}{\partial x^\alpha \partial x^\beta}.$$

For the above operator the following commutator rules hold (which can be stated symbolically as follows)

$$[\square_d, T_\mu] \sim (Dd)D^2 \qquad \text{for } 0 \le \mu \le 3, \tag{3.8}$$

$$[\square_d, L_{\mu\nu}] \sim (wh)D^2 + (L_{\mu\nu}d)D^2 \qquad \text{for } 0 \le \mu < \nu \le 3, \tag{3.9}$$

$$[\square_d, S] \sim (wh)D^2 + (Sd)D^2 + 2\square_d. \tag{3.10}$$

For any function u defined on $(0,\infty) \times \mathbb{R}^3$ we introduce

$$E_d[u] = \left(\int_{\mathbb{R}^3} (d^{00} u_{,0}^2 + d^{kl} u_{,k} u_{,l})\, dx \right)^{1/2}. \tag{3.11}$$

The above quantity defines function spaces where solutions will be looked for. Within our considerations metric coefficients $d^{\alpha\beta}$ are required to fulfill (2.3) but with fixed and controlled constants, i.e.

$$-d^{00} \ge \frac{1}{2}, \quad \frac{1}{2}|X|^2 \le d^{kl} X_k X_l \le 2|X|^2 \quad \text{for } X \in \mathbb{R}^3.$$

The above restrictions will be easily satisfied by solutions as we will be able to control smallness of functions $h_{\mu\nu}$. We can replace numbers $1/2$ and 2 by $1-\varepsilon$ and $1+\varepsilon$, however it would not change the final result.

As an elementary corollary of above facts we see that quantity (3.11) is equivalent to the following norm

$$E_d[u] \simeq ||Du(t,\cdot)||_{L_2(\mathbb{R}^3)}.$$

The main quantity, which controls the norm of solutions, is the following

$$\mathcal{X} = \sup_{0 \le t < \infty} (E_d^2[h(t,\cdot)] + E_d^2[\Gamma h(t,\cdot)] + E_d^2[\Gamma\Gamma h(t,\cdot)])^{1/2} + X_0, \tag{3.12}$$

where X_0 described the norm of initial data, see (2.21), and

$$E_d^2[\Gamma h] = \sum_{\Gamma_a \in A} E_d^2[\Gamma_a h], \quad E_d^2[\Gamma\Gamma h] = \sum_{\Gamma_a, \Gamma_b \in A} E_d^2[\Gamma_a \Gamma_b h] \tag{3.13}$$

denote sums over all possible indices. Finiteness of quantity $\mathcal{X}$ implies that $Dh \in G^2(\mathbb{R}^3; 0, \infty)$.

4. A priori bound

Here we prove the main result. We will find the a priori bound on quantity $\mathcal{X}$ to control solutions of (2.11)–(2.13) globally in time. To conclude Theorem 2.1 it will be enough to have an a priori estimate, since by the local existence results (see [7], [12]), controlling norms of solutions we are able to prolong the domain of the lifespan. Hence we skip the part of the proof concerning the issue of existence. For our purpose we apply the energy method, which in our case is split into three steps.

The first energy estimate. Multiplying (2.11) by $h_{\mu\nu,0}$, integrating over $\mathbb{R}^3$, next integrating by parts the l.h.s., we obtain the following inequality

$$\frac{1}{4}\frac{d}{dx^0} E_d^2[h_{\mu\nu}] \le \int_{\mathbb{R}^3} (|wH_{\mu\nu}(\eta_{\alpha\beta} + h_{\alpha\beta}, Dh_{\alpha\beta})h_{\mu\nu,0}| + |Dh|^2|Dd|)\, dx$$
$$= I_1 = I_{11} + I_{12}.$$

To obtain the above inequality we used relation (3.8).

Recalling the form of terms $H_{\mu\nu}$, the estimation for term I_{11} is reduced to analyze the following integral consider in the form

$$\int_{\mathbb{R}^3} W|Dh||Dh|^2\, dx, \tag{4.1}$$

where W is given by (3.3). To apply Proposition 3.1 we estimate term (4.1) as follows

$$I_{11} \le \sup_{x\in\mathbb{R}^3} |W|Dh|| \int_{\mathbb{R}^3} |Dh|^2\, dx.$$

Since supp $W = \{\kappa_* x^0 \ge r + M_0\}$ relation (3.1) implies that

$$I_{11} \le c(1 + (\kappa_* - 1)x^0)^{-3/2} \mathcal{X} \int_{\mathbb{R}^3} |Dh|^2\, dx, \tag{4.2}$$

where $\mathcal{X}$ is given by (3.12), which implies that $Dh \in G^2$.

To treat I_{12} note that

$$I_{12} \le \int_{\mathbb{R}^3} W|Dh|\,|Dh|^2\, dx + \int_{\mathbb{R}^3} |Dw|\,|h|\,|Dh|^2\, dx = I_{121} + I_{122}.$$

Term I_{121} is just (4.1), so we examine I_{122}.

The crucial point is to analyze the behavior of the function h with respect to time. By the form of I_{122} we consider only points from the support of function w. For this purpose we examine the following representation of sought functions

$$h_{\alpha\beta} = h^0_{\alpha\beta} + \int_0^{x^0} h_{\alpha\beta,0}\, dt.$$

We are interested only in finding a suitable estimate on $Wh_{\alpha\beta}$, so we may apply here assumption (2.21) and get

$$\sup_{\overline{x}\in\mathbb{R}^3} |Wh^0_{\alpha\beta}| \le \frac{X_0}{(1+x^0)^\delta}. \tag{4.3}$$

Thus, to find the desired bound on $h_{\alpha\beta}$, we estimate the following integral

$$W\int_0^{x^0} h_{\alpha\beta,0}\, dt$$

and, by Proposition 3.1, we have

$$\le cW\int_0^{x^0} (1+(r-t))^{-1/2}(1+r+t)^{-1}\mathcal{X}\, dt,$$

by properties of the support of W, we find

$$\le c\mathcal{X}W\int_0^{x^0} (1+r+t)^{-3/2}\, dt \le c\mathcal{X}\frac{1}{(1+x^0)^{1/2}}. \tag{4.4}$$

From (4.3) and (4.4) we conclude (taking $\delta < 1/2$) that

$$\sup_{\overline{x}\in\mathbb{R}^3} |Wh(x)| \le c(1+x^0)^{-\delta}\mathcal{X}. \tag{4.5}$$

By features of function w given by (2.9) we deduce that

$$\sup_{\overline{x}\in\mathbb{R}^3} |Dw| \le c(1+(\kappa_*-1)x^0)^{-1}. \tag{4.6}$$

Hence by (4.5) and (4.6) we obtain that

$$I_{122} \le c(1+(\kappa_*-1)x^0)^{-1-\delta}\int_{\mathbb{R}^3} |Dh|^2\, dx. \tag{4.7}$$

Thus, summing over indices μ and ν, by (4.2) and (4.7) we get

$$\frac{d}{dx^0}E^2_d[h] \le c(1+(\kappa_*-1)x^0))^{-1-\delta}\mathcal{X}E^2_d[h], \tag{4.8}$$

and by the Gronwall inequality we obtain

$$\begin{aligned}\sup_{0\le t\le\infty} E^2_d[h(t)] &\le E^2_{d_0}[h|_{x^0=0}]\exp\left\{c\int_0^\infty (1+(\kappa_*-1)t)^{-1-\delta}\mathcal{X}\, dt\right\}\\ &\le E^2_{d_0}[h|_{x^0=0}]\exp\{c\mathcal{X}\},\end{aligned}$$

where $d_0 = d|_{x^0=0}$.

The second energy estimate. Let $\Gamma \in A$, then by (3.8)–(3.10) we have

$$-\frac{1}{2}\Box_d(\Gamma h_{\mu\nu}) = \Gamma(wH_{\mu\nu}) + \frac{1}{2}[\Box_d, \Gamma]h_{\mu\nu}. \tag{4.9}$$

Multiplying (4.9) by $(\Gamma h_{\mu\nu})_{,0}$, we obtain

$$\frac{1}{4}\frac{d}{dx^0}E_d^2[\Gamma h_{\mu\nu}] \le \int_{\mathbb{R}^3}\Big(|\Gamma(wH_{\mu\nu}(\Gamma h_{\mu\nu})_{,0}| + \frac{1}{2}|[\Box_d, \Gamma]h_{\mu\nu}(\Gamma h_{\mu\nu})_{,0}|\Big)\,dx \\ = I_2 = I_{21} + I_{22}. \tag{4.10}$$

Taking the first term of the r.h.s. of (4.10) we split it into two terms

$$I_{21} \le \int_{\mathbb{R}^3} |(\Gamma w)H_{\mu\nu}(\Gamma h_{\mu\nu})_{,0}|\,dx \\ + \int_{\mathbb{R}^3} |w(\Gamma H_{\mu\nu})(\Gamma h_{\mu\nu})_{,0}|\,dx = I_{211} + I_{212}. \tag{4.11}$$

The second term of (4.11) is treated the same as term I_1 in the first energy estimate, i.e. we have

$$I_{212} \le c(1 + (\kappa_* - 1)x^0)^{-3/2}\mathcal{X}E_d^2[\Gamma h]. \tag{4.12}$$

To consider the first term we note that by the definition of the cut off function (2.9) we have globally in $(0,\infty) \times \mathbb{R}^3$ for a certain constant the following bound

$$|\Gamma w| \le c \sim O(1/(\kappa - \kappa_*)),$$

which follows from properties of the support of w.

To examine I_{211} we see that by (4.11) and Proposition 3.1 we have

$$\begin{aligned} I_{211} &\le c\int_{\mathbb{R}^3} W|Dh|\,|Dh|\,|(\Gamma h)_{,0}|\,dx \\ &\le c\sup_{\overline{x}\in\mathbb{R}^3}|W|Dh||\int_{\mathbb{R}^3}|Dh|\,|(\Gamma h)_{,0}|\,dx \\ &\le c(1 + (\kappa_* - 1)x^0)^{-3/2}\mathcal{X}E_d[h]E_d[\Gamma h]. \end{aligned} \tag{4.13}$$

To finish the estimation for the second energy estimate we find a bound for the last term of the r.h.s. of (4.10)

$$\begin{aligned} I_{22} &\le c\int_{\mathbb{R}^3}|[\Box_d, \Gamma]h_{\mu\nu}(\Gamma h)_{,0}|\,dx \\ &\le \int_{\mathbb{R}^3} W(|(\Gamma h)D^2h(\Gamma h)_{,0}| + |(wh)D^2h(\Gamma h)_{,0}| + |\Box_d h(\Gamma h)_{,0}|)\,dx \\ &= I_{221} + I_{222} + I_{223}. \end{aligned}$$

Take I_{221}. By the definition of elements of set A (see (2.18)), we note that

$$|\Gamma h| \le c(r + x^0)|Dh|. \tag{4.14}$$

Moreover, by Proposition 3.2, we have

$$|D^2h| \le c(1+(r-x^0))^{-1}|\Lambda Dh|. \tag{4.15}$$

By properties of the support of function w we see that for a certain constant the following pointwise estimate holds

$$\left| W\frac{r+x^0}{1+r-x^0} \right| \le c. \tag{4.16}$$

Hence we conclude from (4.14), (4.15) and (4.16) the following inequality

$$I_{221} \le c\int_{\mathbb{R}^3} W|Dh|\,|\Lambda Dh|\,|(\Gamma h)_{,0}|\,dx.$$

Repeating steps for the first energy estimate we get

$$I_{221} \le c(1+(\kappa_*-1)x^0)^{-3/2}\mathcal{X}E_d^2[\Gamma h]. \tag{4.17}$$

Term I_{221} has a similar structure as I_{122}. Applying (4.5) and (4.15) we get

$$\begin{aligned} I_{222} &\le \int_{\mathbb{R}^3} |hD^2h(\Gamma h)_{,0}|\,dx \\ &\le c(1+x^0)^{-\delta}\mathcal{X}\int_{\mathbb{R}^3} W(1+(r-x^0))^{-1}(\Lambda Dh)(\Gamma h)_{,0}\,dx \\ &\le c(1+x^0)^{-\delta}\mathcal{X}(1+(\kappa_*-1)x^0)^{-1}E_d^2(\Gamma h). \end{aligned}$$

The third term I_{223} is reduced to the first one since $-(1/2)\Box_d h = wH$.

Summing (4.12), (4.13), (4.17) and (4.18) we obtain the following differential inequality

$$\frac{d}{dx^0}E_d^2[\Gamma h] \le c(1+(\kappa_*-1)x^0)^{-1-\delta}\mathcal{X}E_d[\Gamma h](E_d[\Gamma h]+E_d[h]). \tag{4.19}$$

The third energy estimate. Differentiating (4.9) by $\Gamma \in A$ we obtain

$$-\frac{1}{2}\Box_d(\Gamma\Gamma h_{\mu\nu})_{,0} = \Gamma\left(\Gamma(wH_{\mu\nu}) + \frac{1}{2}[\Box_d,\Gamma]h_{\mu\nu}\right) + \frac{1}{2}[\Box_d,\Gamma]\Gamma h_{\mu\nu}. \tag{4.20}$$

Multiplying (4.20) by $(\Gamma\Gamma h_{\mu\nu})_{,0}$, integrating over $\mathbb{R}^3$ we get

$$\begin{aligned} \frac{1}{4}\frac{d}{dx^0}E_d^2[\Gamma\Gamma h_{\mu\nu}] &\le \int_{\mathbb{R}^3} |\Gamma(\Gamma(wH_{\mu\nu}))(\Gamma\Gamma h_{\mu\nu})_{,0}|\,dx \\ &+ \int_{\mathbb{R}^3}\frac{1}{2}|\Gamma([\Box_d,\Gamma]h_{\mu\nu})(\Gamma\Gamma h_{\mu\nu})_{,0}|\,dx + \int_{\mathbb{R}^3}\frac{1}{2}|[\Box_d,\Gamma](\Gamma h_{\mu\nu})(\Gamma\Gamma h_{\mu\nu})_{,0}|\,dx \\ &= I_3 = I_{31}+I_{32}+I_{33}. \end{aligned}$$

Take I_{31}. Here, we have

$$\begin{aligned} I_{31} \le \int_{\mathbb{R}^3} W|\Gamma(\Gamma w)|\,|H_{\mu\nu}|\,|(\Gamma\Gamma h_{\mu\nu})_{,0}|\,dx + 2\int_{\mathbb{R}^3} W|\Gamma w|\,|\Gamma H_{\mu\nu}|\,|(\Gamma\Gamma h_{\mu\nu})_{,0}|\,dx \\ + \int_{\mathbb{R}^3} W|\Gamma\Gamma H_{\mu\nu}|\,|(\Gamma\Gamma h_{\mu\nu})_{,0}|dx = I_{311}+I_{312}+I_{313}. \end{aligned}$$

To analyze I_{311}, we note that by features of function w, we have globally the following bound

$$|\Gamma\Gamma w| \leq c \sim O(1/(\kappa - \kappa_*)).$$

Hence I_{311} is treated as I_{211} and we get

$$I_{311} \leq c(1 + (\kappa_* - 1)x^0)^{-3/2} \mathcal{X} E_d[h] E_d[\Gamma\Gamma h]. \tag{4.21}$$

The same we have for I_{312}, since

$$I_{312} \leq c \int_{\mathbb{R}^3} W |Dh|\, |\Gamma Dh|\, |(\Gamma\Gamma h_{\mu\nu})_{,0}|\, dx, \tag{4.22}$$

thus $I_{312} \leq c(1 + (\kappa_* - 1)x^0)^{-3/2} \mathcal{X} E[\Gamma h] E[\Gamma\Gamma h]$.

The last term delivers a new type of nonlinearity. Let us note that I_{313} is bounded as follows

$$\begin{aligned} I_{313} \leq c \int_{\mathbb{R}^3} & W |Dh|\, |\Gamma\Gamma Dh|\, |(\Gamma\Gamma h_{\mu\nu})_{,0}|\, dx \\ & + c \int_{\mathbb{R}^3} W |\Gamma Dh|\, |\Gamma Dh|\, |(\Gamma\Gamma h_{\mu\nu})_{,0}|\, dx = I_{3131} + I_{3132}. \end{aligned} \tag{4.23}$$

The first term can be estimated as follows

$$I_{3131} \leq c(1 + (\kappa_* - 1)x^0)^{-3/2} \mathcal{X} E_d^2[\Gamma\Gamma h]. \tag{4.24}$$

To analyze the second term of the r.h.s. of (4.23) we apply Proposition 3.3. Since we can assume that $\Gamma Dh \in G^1$, applying the Hölder inequality to integral I_{3132}, we deduce the following estimate

$$\begin{aligned} I_{3132} & \leq c \|\Gamma Dh\|^2_{L_4(\mathbb{R}^3)} \|\Gamma\Gamma Dh\|_{L_2(\mathbb{R}^3)} \\ & \leq c(1 + (\kappa_* - 1)x^0)^{-3/2} (E_d[\Gamma h] + E_d[\Gamma\Gamma h])^2 E_d[\Gamma\Gamma h] \\ & \leq c(1 + (\kappa_* - 1)x^0)^{-3/2} \mathcal{X} (E_d[\Gamma h] + E_d[\Gamma\Gamma h]) E_d[\Gamma\Gamma h]. \end{aligned} \tag{4.25}$$

Thus, from (4.21), (4.22), (4.24) and (4.25) we conclude the following differential inequality

$$\frac{d}{dx^0} E_d^2[\Gamma\Gamma h] \leq c(1 + (\kappa_* - 1)x^0)^{-1-\delta} \mathcal{X} (E_d^2[h] + E_d^2[\Gamma h] + E_d^2[\Gamma\Gamma h]). \tag{4.26}$$

The above estimate finished the last third step and we show the a priori bound.

A priori estimate. Summing up three energy inequalities (4.8), (4.19) and (4.26) we obtain

$$\begin{aligned} & \frac{d}{dx^0} \{E_d^2[h] + E_d^2[\Gamma h] + E_d^2[\Gamma\Gamma h]\} \\ & \qquad \leq c(1 + (\kappa_* - 1)x^0)^{-1-\delta} \mathcal{X} \{E_d^2[h] + E_d^2[\Gamma h] + E_d^2[\Gamma\Gamma h]\}. \end{aligned} \tag{4.27}$$

Applying to (4.27) the Gronwall inequality we get the following bound

$$\sup_{0\le t<\infty}\{E_d^2[h]+E_d^2[\Gamma h]+E_d^2[\Gamma\Gamma h]\} \le \{E_{d^0}^2[h^0]+E_{d^0}^2[\Gamma h^0]+E_{d^0}^2[\Gamma\Gamma h^0]\}\exp\{\pi_0\mathcal{X}\}. \tag{4.28}$$

Since π_0 in (4.28) is an absolute constant, we can require to initial data satisfy the following bound (smallness condition (2.21))

$$3\{E_{d^0}^2[h^0]+E_{d^0}^2[\Gamma h^0]+E_{d^0}^2[\Gamma\Gamma h^0]\}+X_0^2\le(1/\pi_0)^2, \tag{4.29}$$

then from (4.28) and (4.29), if we assume suitable smallness of $\mathcal{X}$ ($\le 1/\pi_0$ — we can analyze (4.28) only on a time interval where $\mathcal{X}\le 1/\pi_0$), we get

$$\sup_{0\le t<\infty}\{E_d^2[h]+E_d^2[\Gamma h]+E_d^2[\Gamma\Gamma h]\}\le 3\{E_{d^0}^2[h^0]+E_{d^0}^2[\Gamma h^0]+E_{d^0}^2[\Gamma\Gamma h^0]\},$$

what, by (3.13) and (4.29), closes the estimation, since we conclude

$$\sup_{0\le t<\infty}\{E_d^2[h(t,\cdot)]+E_d^2[\Gamma h(t,\cdot)]+E_d^2[\Gamma\Gamma h(t,\cdot)]\}+X_0^2\le(1/\pi_0)^2.$$

which guarantees that $\mathcal{X}\le 1/\pi_0$ (in particular, for a fixed time we get (4.29) as a new initial data and we can continue this procedure to obtain the global in time estimate). Finally, by the definition of $\mathcal{X}$, we obtain

$$\mathcal{X}\le 4X_0, \tag{4.30}$$

which finishes the proof of the a priori bound showing inclusions (2.22). By previous remarks and bound (4.30) we conclude the thesis of Theorem 2.1.

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