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# DEGREE AND INDEX THEORIES FOR NONCOMPACT FUNCTION TRIPLES

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Abstract. We describe a very general procedure how one may extend an arbitrary degree or index theory (originally defined only for compact maps) also for large classes of noncompact maps. We also show how one may obtain degree or index theories relative to some set. Our results even apply to the general setting when one has a combined degree and index theory for function triples. The results are applied to countably condensing perturbations of monotone maps.

## 1. Introduction

The classical degree theory for fixed points of compact maps $q: Y \to Y$ in a Banach space $Y$ was generalized in many respects:

(a) It was generalized to *coincidence degrees* for pairs of maps $F, q: X \to Y$ e.g. by the theory of 0-epi maps $F$ [20], [31], the Mawhin (Nirenberg) degree for Fredholm maps $F$ of zero (nonnegative) index [25], [40], [49] (resp. [26], [43], [44]) or the Skrypnik degree for uniformly monotone maps $F$ [53].

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(b) It was generalized to *index theories* for fixed points of multivalued maps of the form $\Phi = q \circ p^{-1}$ where $p, q\colon \Gamma \to Y$ and the fibres $p^{-1}(y)$ are acyclic or $R_\delta$-sets [17], [36], [38], [52]. Practically all known fixed point theories for multivalued maps [8], [10] fall into this category [3], [28], [37].

(c) Many of the above index theories were generalized to *relative* index theories where $Y$ is replaced by a closed convex subset of a Banach space or, more general, a manifold or even an ANR.

(d) The degree was generalized to the case when $q$ is not compact but only *condensing* with respect to a measure of noncompactness [45], [46], [47], [51] (see also [1], [15]) or, more general, if $q$ possesses a *fundamental set* [35], [62] on which it is compact. These approaches were also generalized to (multivalued) index theories [9], [18], [19], [33], [48], [50], [57], [58], [61], [63].

At a first glance, the difference between coincidence and index theories is only on the method of approach and on somewhat different requirements for $F$, resp. $p$. However, the difference is actually much deeper because, roughly speaking, for degree theories the crucial assumptions have to be formulated on the space $X$ while for index theories the assumptions have to be formulated on the image space $Y$ while the space $\Gamma$ plays practically no role (see the discussion in [60]). It appears now that the right setting to treat all the above theories in a unified manner is by considering function triples

$$Y \overset{F}{\longleftarrow} X \overset{p}{\longleftarrow} \Gamma \overset{q}{\longrightarrow} Y$$

and to look for solutions of the inclusion $F(x) \in q(p^{-1}(x))$, having in mind that the hypotheses on the theory should be formulated on the space $X$. The latter means that it is neither the right approach to consider the problem as a coincidence point problem for the pair $(F \circ p, q)$ nor as a fixed point problem of the multivalued map $q \circ p^{-1} \circ F^{-1}$. (Although, formally, each solution of one of these problems provides a solution of the above inclusion and vice versa, but the *number* of solutions may differ!).

More precisely, we want to assume that there is a degree theory for such function triples which, roughly speaking, in the case $p = \mathrm{id}$ reduces to a coincidence degree for the pair $(F, q)$ and in the case $F = \mathrm{id}$ reduces to a fixed point index for $q \circ p^{-1}$. We will give a formal definition of requirements on such a theory in Section 2. In [23], [24], [37] such a triple-degree (satisfying these requirements) was defined for Fredholm maps $F$ of nonnegative index (and generalized also for noncompact maps). In [56] a general approach was discussed how any coincidence degree can be extended to such a triple-degree in a finite-dimensional setting. In the forthcoming paper [55], we will discuss how to extend this degree to an infinite-dimensional setting under some compactness hypothesis.

The purpose of this paper is to show how such a triple-degree for compact function triples can be extended to a *relative* degree theory and, moreover, also to *noncompact* function triples (using fundamental sets or measures of noncompactness).

Nevertheless, we want to point out that to our knowledge the results are even new when one is only interested in a classical coincidence degree (i.e. for $p = \mathrm{id}$) when $F$ is not the identity. It appears that only in this general framework the approach by fundamental sets can be really understood: Many technical requirements of e.g. [5] take a rather natural form in this setting. For example, in the "classical" case of index theories, a fundamental set $K \subseteq Y$ has to satisfy somewhat strange properties with respect to $\Omega \cap K$ where $\Omega \subseteq Y$ is open. In our general setting it is now clear that one has actually to consider open sets $\Omega \subseteq X$ and that the mentioned properties are actually properties of the function $F\colon \Omega \to Y$ (which in case of index theories is the inclusion).

There is a further novelty in this paper: In [5], [57], [58], we have taken the attitude that one has to know a *relative* index theory before one can seriously discuss the extension to noncompact maps. At a first glance, this is a very natural requirement, because one can then use the index on a fundamental set $K$. However, if one works in general ANRs and thus wants to treat also the case of general (nonconvex) sets $K$, we have seen in [5] that one has to assume that the index is actually defined on a larger family of sets. Therefore, it appears now more natural to treat the two extensions (to the relative case and to the noncompact case) *simultaneously*. Unfortunately, this makes the current paper somewhat technical and lengthy, but this approach seems necessary if one does not want to introduce superfluous and unnatural hypotheses.

Note that, wherever possible, we treat also the case of nonconvex domains. In the presentation, we follow essentially [5], although there are some severe technical differences and although we do not start from the relative case.

The plan of the paper is as follows. In Section 2, we introduce axiomatically what we assume on the given degree theory. In Sections 3 and 4 we define (also in an axiomatic way) what we understand by fundamental sets and how the degree theory is extended. In Section 5 we show how one can actually verify that a set is fundamental. In the remaining sections, we discuss how one can find such a set and consider essentially the case that $Y$ is a closed convex subset of a locally convex space. In particular, we obtain that our extension of the degree applies to countably condensing (noncompact) function triples.

We point out once more that the results are new even in the special case $p = \mathrm{id}$. In the last section, we apply this special case to obtain a continuation principle for noncompact perturbations of monotone maps which is to our knowledge the first result of this kind.

## 2. General assumptions

Throughout this paper, let $X$ and $Y$ be fixed topological spaces. Let $\mathcal{O}$ be a fixed family of open subsets of $X$, and $\mathcal{G}_0$ be a fixed class of topological spaces. Let $\mathcal{F}$ be a fixed class of triples $(F,p,\Omega)$ where $\Omega\in\mathcal{O}$, $F:\overline{\Omega}\to Y$ (not necessarily continuous), and $p:\Gamma\to X$ is continuous with $\Gamma\in\mathcal{G}_0$. If additionally $q:\Gamma\to Y$ then we write $(F,p,q,\Omega)\in\mathcal{T}$.

Given some $(F,p,q,\Omega)\in\mathcal{T}$, we are interested in the coincidence point set

$$\begin{aligned}\operatorname{Coin}_M(F,p,q):=&\{x\in M\mid F(x)\in q(p^{-1}(x))\}\\ =&\{x\in M\mid \exists z: x=p(z),\ F(p(z))=q(z)\}\end{aligned}$$

where $M\subseteq\overline{\Omega}$. If $q$ is a continuous and *compact* map (by the latter we mean in this paper that the range is contained in a compact subset of $Y$), then it is for a large class of maps $F$ and $p$ possible to provide a degree theory for such function triples $(F,p,q)$.

Such a degree theory is the natural topologic tool if one wants to treat inclusions of the type $F(x)\in\Phi(x)$ when $F:X\to Y$ acts e.g. between different Banach spaces and $\Phi:X\multimap Y$ is compact. Roughly speaking, whenever there exists *some* degree theory for $F$ and $p$ is a so-called Vietoris map with $R_\delta$-fibres then such a degree theory for function triples exists [55], [56]. We want to show in this paper how *any* such degree theory can be extended to the noncompact case (i.e. to the case when $\Phi$, resp. $q$, is not necessarily compact).

Let us first make precise what we mean by a degree theory for function triples. We assume for simplicity throughout that for each $\Gamma\in\mathcal{G}_0$ also all closed subsets of $\Gamma$ are contained in $\mathcal{G}_0$. By $\mathcal{T}_0$, we denote the class of all $(F,p,q,\Omega)\in\mathcal{T}$ where the restriction $q|_{p^{-1}(\overline{\Omega})}$ is continuous and compact with

$$\operatorname{Coin}_{\partial\Omega}(F,p,q)=\emptyset.$$

We will assume that we have given a degree for elements of $\mathcal{T}_0$ which assumes values in a fixed semigroup $G$ (additively written).

Note that if $(F,p,q,\Omega)\in\mathcal{T}_0$ then also $(F,p|_{p^{-1}(\overline{\Omega})},q|_{p^{-1}(\overline{\Omega})},\Omega)\in\mathcal{T}_0$ (by our assumption on $\mathcal{G}_0$) and the first of the following properties of the degree implies that the corresponding degree coincides. In this sense, the degree depends only on the restrictions of $p$ and $q$ to $p^{-1}(\overline{\Omega})$. However, although one might be tempted to think so, the degree in general depends not only on the multivalued map $q\circ p^{-1}$ on $\overline{\Omega}$ but also on the actual decomposition $(p,q)$ of this map.

Definition 2.1. We say that Deg is a *compact triple-degree* for $\mathcal{F}$ if it associates to each $(F,p,q,\Omega)\in\mathcal{T}_0$ an element of $G$ such that the following holds:

(a) (Independence from $\Gamma$) If $(F, p_i, q_i, \Omega) \in \mathcal{T}$ $(i = 0, 1)$ are such that there is a surjective homeomorphism $J: p_0^{-1}(\overline{\Omega}) \to p_1^{-1}(\overline{\Omega})$ with $p_0|_{p_0^{-1}(\overline{\Omega})} = p_1 \circ J$ and $q_0|_{p_0^{-1}(\overline{\Omega})} = q_1 \circ J$ then either none or both of $(F, p_i, q_i, \Omega)$ $(i = 0, 1)$ belongs to $\mathcal{T}_0$, and in this case

$$\mathrm{Deg}(F, p_0, q_0, \Omega) = \mathrm{Deg}(F, p_1, q_1, \Omega).$$

(b) (Existence) $\mathrm{Deg}(F, p, q, \Omega) \neq 0$ implies $\mathrm{Coin}_\Omega(F, p, q) \neq \emptyset$.

(c) (Homotopy Invariance in the Third Argument) If $(F, p, \Omega) \in \mathcal{F}$ and $h: [0, 1] \times \Gamma \to Y$ is continuous and compact with $(F, p, h(t, \cdot\,), \Omega) \in \mathcal{T}_0$ for each $t \in [0, 1]$, then

$$\mathrm{Deg}(F, p, h(t, \cdot\,), \Omega) \text{ is independent of } t \in [0, 1].$$

Tacitly writing $F$ also for restrictions of $F$, we can formulate the following properties which Deg may or may not possess.

(d) (Excision) If $(F, p, q, \Omega) \in \mathcal{T}_0$ and $\Omega_0 \in \mathcal{O}$ is contained in $\Omega$ with $\mathrm{Coin}_\Omega(F, p, q) \subseteq \Omega_0$, then we have that $(F, p, q, \Omega_0) \in \mathcal{T}_0$ and

$$\mathrm{Deg}(F, p, q, \Omega_0) = \mathrm{Deg}(F, p, q, \Omega).$$

(e) (Restriction) Under the same assumptions as above we have $(F, p, q, \Omega_0)$ in $\mathcal{T}_0$ and

$$\mathrm{Deg}(F, p, q, \Omega) \neq 0 \;\Rightarrow\; \mathrm{Deg}(F, p, q, \Omega_0) = \mathrm{Deg}(F, p, q, \Omega).$$

(f) (Additivity) If $(F, p, q, \Omega) \in \mathcal{T}_0$ and $\Omega_1, \Omega_2 \in \mathcal{O}$ are disjoint with $\Omega = \Omega_1 \cup \Omega_2$, then $(F, p, q, \Omega_i) \in \mathcal{T}_0$ and

$$\mathrm{Deg}(F, p, q, \Omega) = \mathrm{Deg}(F, p, q, \Omega_1) + \mathrm{Deg}(F, p, q, \Omega_2).$$

In [56], a class of examples of compact triple-degree theories was discussed where it was also observed that triple-degrees which are defined in a purely homotopic manner usually *fail* to satisfy the excision property but satisfy the weaker restriction property (this is why we consider both properties separately). For other examples of triple-degrees see [23], [24], [37].

In applications, one needs an extended notion of "homotopy". To define this, we consider a further class $\mathcal{G}_1$ of topological spaces, assuming once more that for each $\Gamma \in \mathcal{G}_1$ all closed subspaces of $\Gamma$ are contained in $\mathcal{G}_1$.

Let $\mathcal{H}$ be a class of triples $(H, P, \Omega)$ where $\Omega \in \mathcal{O}$, $H: [0, 1] \times \overline{\Omega} \to Y$ and $P: \Gamma \to [0, 1] \times X$ with $\Gamma \in \mathcal{G}_1$. In this case, we define $P_t: P|_{P^{-1}(\{t\} \times X)} \to X$ $(0 \leq t \leq 1)$ by the relation

$$P(z) = (t, P_t(z)) \quad (z \in P^{-1}(\{t\} \times X),\ t \in [0, 1]).$$

We assume that $(H(t, \cdot\,), P_t, \Omega) \in \mathcal{F}$ for each $(H, P, \Omega) \in \mathcal{H}$ and each $t \in [0, 1]$.

By $\mathcal{H}'$, we denote the class of all $(H, P, Q, \Omega)$ where $(H, P, \Omega) \in \mathcal{H}$ and (with the above notation) $Q: \Gamma \to Y$. In this case, we define $Q_t$ $(0 \le t \le 1)$ as the restriction of $Q$ to $P^{-1}(\{t\} \times X)$, and for $M \subseteq X$ we use the notation

$$\mathrm{Coin}_M(H, P, Q) := \bigcup_{t \in [0,1]} \mathrm{Coin}_M(H(t, \cdot\,), P_t, Q_t).$$

By $\mathcal{H}_0$, we denote the class of all $(H, P, Q, \Omega) \in \mathcal{H}'$ where the restriction of $Q$ to $P^{-1}([0, 1] \times \overline{\Omega})$ is continuous and compact, and

$$\mathrm{Coin}_{\partial\Omega}(H, P, Q) = \emptyset.$$

DEFINITION 2.2. We say that the degree Deg is *$\mathcal{H}$-invariant* if it has the following property:

(g) (Homotopy Invariance) If $(H, P, Q, \Omega) \in \mathcal{H}_0$ then

$$\mathrm{Deg}(H(t, \cdot\,), P_t, Q_t, \Omega) \text{ is independent of } t \in [0, 1].$$

REMARK 2.3. If $[0, 1] \times \Gamma \in \mathcal{G}_1$ for each $\Gamma \in \mathcal{G}_0$, then the homotopy invariance of Deg with respect to the third argument can, in view of the independence of $\Gamma$, be equivalently formulated as follows: Deg is $\mathcal{H}(\mathcal{F})$-invariant where $\mathcal{H}(\mathcal{F})$ denotes the class of all $(H, P, \Omega)$ for which there is some $(F, p, \Omega) \in \mathcal{F}$ with $p: \Gamma \to X$ such that $H(t, \cdot\,) = F$ $(0 \le t \le 1)$ and $P: [0, 1] \times \Gamma \to [0, 1] \times X$ is given by $P(t, z) = (t, p(z))$.

## 3. Definition of the degree — global case

Roughly speaking, the idea of the definition of the degree for noncompact function triples is to assume that there is a *fundamental* set on which the triple *is* compact and such that the fundamental set has the property that "everything which is relevant for the degree" happens on this set.

To define fundamental sets, we proceed the *axiomatic* way by *assuming* that Deg has certain required properties on this set. However, in most cases, these properties are hard to verify and they may depend on rather particular properties of the degree under consideration. Therefore, we will in Section 5 describe a "homotopic" condition which can be used to verify that a set is fundamental for *each* degree Deg for $\mathcal{F}$.

We use the notation of the previous section. We point out that in this section we do not require the excision property of Deg: If Deg has the excision property, then many properties that we require now globally need only be satisfied locally. In particular, the notion of fundamental sets and the corresponding homotopic condition can be relaxed in a way which is much more convenient for many applications. However, this local approach will be postponed to Section 4.

Definition 3.1. We put $\mathcal{G} := \mathcal{G}_0 \cup \mathcal{G}_1$. By $\mathrm{AE}_c^0(\mathcal{G}, Y)$ (resp. $\mathrm{ANE}_c^0(\mathcal{G}, Y)$) we denote the family of all $K \subseteq Y$ with the following property: If $\Gamma \in \mathcal{G}$, $A \subseteq \Gamma$ is closed and $f\colon A \to Y$ is continuous and compact with a continuous extension to $\Gamma$ (resp. to a neighbourhood of $A$) and $f(A) \subseteq K$, then $f$ possesses a continuous *compact* extension to $\Gamma$ (resp. to a neighbourhood of $A$) and which assumes only values in $K$.

By $\mathrm{AE}_c(\mathcal{G}, Y)$ (resp. $\mathrm{ANE}_c(\mathcal{G}, Y)$) we denote the family of all $K \subseteq Y$ which satisfy the above property even without the assumption that $f$ has a continuous extension.

Note that we assume only that the range of the map is contained in a compact subset of $Y$ (not necessarily in a compact subset of $K$). However, in most applications $K \subseteq Y$ will be closed and then this distinction is not important.

Proposition 3.2. *Let $Z$ be a locally convex Hausdorff space, and let $\mathcal{G}$ contain only metric spaces.*

(a) *Let $Y \subseteq Z$ have the property that the closed convex hull of each compact subset of $Y$ is compact. If $K \subseteq Y$ is a* (*neighbourhood*) *retract of $Z$* (*and $K$ is closed in $Z$*) *then $K \in \mathrm{AE}_c(\mathcal{G}, Y)$* (*resp. $K \in \mathrm{ANE}_c(\mathcal{G}, Y)$*).

(b) *If $K \subseteq Z$ is convex and has with respect to $Z$ an interior point and a metrizable closure $\overline{K}$, then $K \in \mathrm{AE}_c(\mathcal{G}, Y) \subseteq \mathrm{ANE}_c(\mathcal{G}, Y)$ whenever $\overline{K} \subseteq Y \subseteq Z$.*

Proof. Given some $\Gamma \in \mathcal{G}$, some closed $A \subseteq \Gamma$ and some continuous compact map $f\colon A \to Y$ with $f(A) \subseteq K$, we find for the first claim a convex compact set $C \subseteq Z$ with $f(A) \subseteq C$. By Dugundji's extension theorem [16], we can extend $f$ to a continuous map $f\colon \Gamma \to C$. By hypothesis, there is a retraction $\rho\colon Z_0 \to K$ where $Z_0 := Z$ (resp. $Z_0 \subseteq Z$ is an open neighbourhood of $K$). Let $Z_1 := Z$ (resp. $Z_1 \subseteq Z$ an open neighbourhood of $C \cap K$ with $\overline{Z}_1 \subseteq Z_0$; this is possible since $Z$ is regular and $C \cap K$ is compact). Then $\Gamma_0 := \Gamma$ resp. $\Gamma_0 := f^{-1}(Z_1)$ is an open neighbourhood of $A$, and $\rho \circ f\colon \Gamma_1 \to K$ is a continuous extension of $f|_A$ and has its range in the compact set $\rho(C \cap \overline{Z}_1)$.

For the second claim, we note that by [34, Theorem 4.5] each continuous compact map $f\colon A \to Y$ with closed $A \subseteq \Gamma \in \mathcal{G}$ and $f(A) \subseteq K$ has an extension to a continuous map $f\colon \Gamma \to K$ such that $f(\Gamma)$ is contained in a compact subset $C \subseteq Z$. In particular, $f(\Gamma)$ is contained in the compact set $C \cap \overline{K} \subseteq Y$. □

We point out that due to the application of Dugundji's extension theorem, the proof of Proposition 3.2 makes essential use of the axiom of choice. However, the (countable) axiom of dependent choices suffices if all metric spaces in $\mathcal{G}$ are separable, see [34, Remarks 3.4 and 4.6].

Note that for a Banach space $Z$ each subset $Y \subseteq Z$ automatically has the property required in the first part of Proposition 3.2 in view of Mazur's lemma.

However, if $Z$ is incomplete, this assumption might be a severe restriction for $Y$. Therefore, we consider further tests for the properties of Definition 3.1.

Recall that a metrizable space $K$ is called an AR (resp. ANR) if $K$ is homeomorphic to a (neighbourhood) retract of a convex subset of a locally convex space. See [11] or [30] for the general theory of ARs and ANRs.

PROPOSITION 3.3. *Let $K$ be an* AR*, resp.* ANR.

(a) *If $K \subseteq Y$ is closed and $\mathcal{G}$ contains only $T_4$ (e.g. normal) spaces then $K \in \mathrm{AE}_c(\mathcal{G}, Y)$, resp. $K \in \mathrm{ANE}_c(\mathcal{G}, Y)$.*
(b) *If $K \subseteq Y$ is closed and $Y$ is a $T_4$ space then $K \in \mathrm{AE}^0_c(\mathcal{G}, Y)$, resp. $K \in \mathrm{ANE}^0_c(\mathcal{G}, Y)$.*
(c) *If $K$ is contained in a compact subset of $Y$ and if $\mathcal{G}$ contains only metric spaces, then $K \in \mathrm{AE}_c(\mathcal{G}, Y)$, resp. $K \in \mathrm{ANE}_c(\mathcal{G}, Y)$. This holds even if $K$ is not metrizable.*

PROOF. Concerning (a), note that if $K \subseteq Y$ is closed and $f: A \to K$ is continuous and such that $f(A) \subseteq K$ is contained in a compact subset $C \subseteq Y$, then $C_0 := C \cap K$ is compact and satisfies $f(A) \subseteq C_0 \subseteq K$. Hence, we obtain no weaker statement if we replace $Y$ by $K$. However, if $Y = K$, the claim is a special case of [34, Theorem 4.7].

To see (b), let $\Gamma \in \mathcal{G}$ and $A \subseteq \Gamma$ be closed, and let $F: \Gamma_0 \to Y$ be a continuous extension of a map $f: A \to K$ where $\Gamma_0 := \Gamma$ (resp. $\Gamma_0 \subseteq \Gamma$ is a neighbourhood of $A$) such that $f(A)$ is contained in a compact subset $C \subseteq Y$. Then $C_0 := C \cap K$ is compact and satisfies $f(A) \subseteq C_0 \subseteq K$. Since (a) implies $K \in \mathrm{AE}_c(\{Y\}, K)$ (resp. $K \in \mathrm{ANE}_c(\{Y\}, K)$), we can extend the identity map of $C_0 \subseteq K$ to a continuous map $J: U \to K$ where $U := Y$ (resp. $U \subseteq Y_0$ is a neighbourhood of $C_0$) and such that $J(U)$ is contained in a compact subset of $K$. Then $\Gamma_1 := F^{-1}(U)$ satisfies $\Gamma_1 = \Gamma$ (resp. $\Gamma_1 \subseteq \Gamma$ is a neighbourhood of $A$), and $J \circ F|_{\Gamma_1}: \Gamma_1 \to K$ is a continuous extension of $f$ whose range is contained in a compact subset of $K \subseteq Y$.

For claim (c), let $\Gamma \in \mathcal{G}$ and $A \subseteq \Gamma$ closed, and let $f: A \to K$ be continuous. We find a convex subset $Z$ of a locally metric space, a homoeomorphism $h$ of $K$ onto a subset $K_0 \subseteq Z$ and a retraction $\rho$ of $V := Z$ (resp. of a neighbourhood $V \subseteq Z$ of $K_0$) onto $K_0$. By Dugundji's extension theorem, the map $h \circ f$ has a continuous extension $H: \Gamma \to Z$. Then $U := H^{-1}(V)$ is $\Gamma$ (resp. a neighbourhood of $A$), and $F := h^{-1} \circ \rho \circ H|_U: U \to K$ is a continuous extension of $f$. □

For the proof of (c), we needed the axiom of choice if $\mathcal{G}$ contains also non-separable metric spaces. It is remarkable that in all other cases the (countable) axiom of dependent choices suffices.

For $(F, p, q, \Omega) \in \mathcal{T}$ it will be convenient to use the notations

$$\mathrm{Fix}_M(F, p, q) := F(\mathrm{Coin}_M(F, p, q)) \quad (M \subseteq \overline{\Omega})$$

and for $K \subseteq Y$ also

$$A_M(F,p,K) := \overline{p^{-1}(F^{-1}(K) \cap M)} \quad (M \subseteq \overline{\Omega}).$$

Definition 3.4. Let Deg be a fixed compact triple-degree for $\mathcal{F}$. We call $K \subseteq Y$ a *retraction candidate* for $(F,p,q,\Omega) \in \mathcal{T}$ if the following holds.

(a) $\mathrm{Fix}_\Omega(F,p,q) \subseteq K$.

(b) We have

$$\mathrm{Coin}_{F^{-1}(K)\cap\partial\Omega}(F,p,q) = \emptyset. \tag{3.1}$$

(c) The restriction of $q$ to $A_{\overline{\Omega}}(F,p,K)$ is continuous and compact and assumes its values in $K$.

(d) If $\widetilde{q}_i \colon \Gamma \to K$ $(i = 1,2)$ are extensions of the restriction $q|_{A_{\overline{\Omega}}(F,p,K)}$ such that $(F,p,\widetilde{q}_i,\Omega) \in \mathcal{T}_0$ then

$$\mathrm{Deg}(F,p,\widetilde{q}_1,\Omega) = \mathrm{Deg}(F,p,\widetilde{q}_2,\Omega). \tag{3.2}$$

In the following, $\mathcal{A}$ will always stand for one of the families $\mathrm{AE}_c(\mathcal{G},Y)$, $\mathrm{AE}^0_c(\mathcal{G},Y)$, $\mathrm{ANE}_c(\mathcal{G},Y)$, or $\mathrm{ANE}^0_c(\mathcal{G},Y)$ – the choice being made once and for all. We fix also a subfamily $\mathcal{A}_0 \subseteq \mathcal{A}$.

Definition 3.5. A retraction candidate $K_0$ for $(F,p,q,\Omega)$ is *pre-fundamental* if either $K_0 = \emptyset$ or $K_0 \in \mathcal{A}_0$ and for any retraction candidate $K_1 \in \mathcal{A}_0$ for $(F,p,q,\Omega)$ with $K_0 \cap K_1 \neq \emptyset$ at least one of the sets $K_2 := K_0 \cap K_1$ or $K_2 := K_0 \cup K_1$ belongs to $\mathcal{A}$ and has the following property:

- If $\widetilde{q}_i \colon \Gamma \to K_i$ $(i = 0,2)$ is an extension of the restriction $q|_{A_{\overline{\Omega}}(F,p,K_i)}$ and $(F,p,\widetilde{q}_i,\Omega) \in \mathcal{T}_0$, then

$$\mathrm{Deg}(F,p,\widetilde{q}_0,\Omega) = \mathrm{Deg}(F,p,\widetilde{q}_2,\Omega). \tag{3.3}$$

Note that if we increase $\mathcal{A}_0$, we increase the class of those $(F,p,q,\Omega)$ which possess a pre-fundamental set in $\mathcal{A}_0$. On the other hand, if we consider a fixed pre-fundamental set $K \in \mathcal{A}_0$ it may happen that $K$ is not pre-fundamental with respect to a larger family $\mathcal{A}_0'$.

The above notions will be sufficient to provide an extension of Deg which is homotopy invariant. However, to provide an extension which satisfies the restriction, excision, or additivity properties, we need a further notion. If we use this notion, we will always tacitly assume that $\mathcal{F}$ has the property that for each $(F,p,\Omega) \in \mathcal{F}$ and each $\Omega_0 \in \mathcal{O}$ with $\Omega_0 \subseteq \Omega$ also $(F,p,\Omega_0) \in \mathcal{F}$.

Definition 3.6. A pre-fundamental set $K$ for $(F,p,q,\Omega)$ is *strictly fundamental* if $K$ is pre-fundamental for each $(F,p,q,\Omega_0)$ where $\Omega_0 \in \mathcal{O}$ satisfies $\Omega_0 \subseteq \Omega$ and $\mathrm{Coin}_{F^{-1}(K)\cap\partial\Omega_0}(F,p,q) = \emptyset$.

We call $K$ (*weakly*) *fundamental* if $K$ is pre-fundamental for each $(F,p,q,\Omega_0)$ where $\Omega_0 \subseteq \Omega$ is as above and $\mathrm{Coin}_\Omega(F,p,q) \subseteq \Omega_0$ (resp. $\overline{\mathrm{Coin}_\Omega(F,p,q)} \subseteq \Omega_0$).

The reader who compares the above definitions with [5] (see also [4]) will find that the definitions of "retraction candidate", "pre-fundamental" and "fundamental" have some analogies but that the definition of "strictly fundamental" sets is missing in [4], [5]. This is a mistake in [4], [5], because for "only" fundamental sets the proof of the additivity of the index in [4], [5] is false. We will need strictly fundamental sets also only in the context of the additivity property.

It follows from the very definition of (strictly) fundamental sets:

Proposition 3.7. *If $K$ is* (*strictly*) *fundamental for $(F,p,q,\Omega)$ then $K$ is* (*strictly*) *fundamental for each $(F,p,q,\Omega_0)$ where $\Omega_0 \in \mathcal{O}$ satisfies $\Omega_0 \subseteq \Omega$, $\mathrm{Coin}_{F^{-1}(K)\cap\partial\Omega_0}(F,p,q) = \emptyset$, and $\mathrm{Coin}_\Omega(F,p,q) \subseteq \Omega_0$ (the latter assumption can be dropped if $K$ is strictly fundamental).*

Definition 3.8. By $\mathcal{T}_1$, resp. $\mathcal{T}_1'$, $\mathcal{T}_1''$, we denote the class of all $(F,p,q,\Omega)$ in $\mathcal{T}$ with the following properties:

(a) $(F,p,q,\Omega)$ has a pre-fundamental, resp. fundamental, strictly fundamental set.
(b) Either $\mathcal{A} \subseteq \mathrm{AE}_c(\mathcal{G},Y)$ or assume the following: If a restriction of $q$ to a closed set is continuous, then this restriction has a continuous extension (with values in $Y$).

The latter condition is trivially satisfied if $q$ is continuous. Clearly, $\mathcal{T}_1'' \subseteq \mathcal{T}_1' \subseteq \mathcal{T}_1$.

We also introduce a corresponding definition for homotopies. For $(H,P,\Omega) \in \mathcal{H}$ and $K \subseteq Y$, we introduce the notation

$$A_M(H,P,K) := \overline{P^{-1}(H^{-1}(K) \cap ([0,1] \times M))} \quad (M \subseteq \overline{\Omega}).$$

Definition 3.9. By $\mathcal{H}_1$, resp. $\mathcal{H}_1'$, $\mathcal{H}_1''$, we denote the class of all $(H,P,Q,\Omega)$ where $(H,P,\Omega) \in \mathcal{H}$ and $Q\colon \Gamma \to Y$ (where $\Gamma$ denotes the domain of $P$) are such that the following holds:

(a) $(H(t,\,\cdot\,),P_t,Q_t,\Omega)$ $(0 \le t \le 1)$ has a pre-fundamental, resp. fundamental, strictly fundamental set $K$ which is independent of $t$.
(b) The restriction of $Q$ to $A_{\overline{\Omega}}(H,P,K)$ is continuous and compact and assumes its values in $K$.
(c) Either $\mathcal{A} \subseteq \mathrm{AE}_c(\mathcal{G},Y)$ or assume the following: If a restriction of $Q$ to a closed set is continuous, then this restriction has a continuous extension (with values in $Y$).

Now we can formulate the main result of this section.

Theorem 3.10. *Let $\mathcal{F}$ provide a compact $\mathcal{H}$-invariant triple-degree* Deg. *Let $\mathcal{A}_0 \subseteq \mathcal{A} \subseteq \mathrm{AE}_c^0(\mathcal{G}, Y)$. Then there is a unique degree* DEG *which associates to each $(F, p, q, \Omega) \in \mathcal{T}_1$ an element of $G$ such that the following holds:*

(a) (Normalization) *If $Y \in \mathcal{A}_0$ then each $(F, p, q, \Omega) \in \mathcal{T}_0$ belongs to $\mathcal{T}_1'' \subseteq \mathcal{T}_1' \subseteq \mathcal{T}_1$, and*

$$\mathrm{DEG}(F, p, q, \Omega) = \mathrm{Deg}(F, p, q, \Omega).$$

(b) (Weak Existence) *If $\mathrm{Coin}_\Omega(F, p, q) = \emptyset$ and all pre-fundamental sets for $(F, p, q, \Omega)$ are empty, then $\mathrm{DEG}(F, p, q, \Omega) = 0$.*

(c) (Permanence) $\mathrm{DEG}(F, p, q, \Omega)$ *depends only on $(F, p, q|_{A_\Omega(F,p,K)}, \Omega)$ if $K$ is pre-fundamental for $(F, p, q, \Omega) \in \mathcal{T}_1$. Moreover, if $(F, p, \widetilde{q}, \Omega) \in \mathcal{T}_0$ is such that $\widetilde{q}$ has its range in $K$ and satisfies $\widetilde{q} = q$ on $A_{\overline{\Omega}}(F, p, K)$, then*

$$\mathrm{DEG}(F, p, q, \Omega) = \mathrm{DEG}(F, p, \widetilde{q}, \Omega).$$

*This degree* DEG *possesses automatically the following properties:*

(d) (Independence from $\Gamma$) *If $(F, p_i, q_i, \Omega) \in \mathcal{T}$ $(i = 0, 1)$ are such that there is a surjective homeomorphism $J\colon p_0^{-1}(\overline{\Omega}) \to p_1^{-1}(\overline{\Omega})$ with $p_0|_{p_0^{-1}(\overline{\Omega})} = p_1 \circ J$ and $q_0|_{p_0^{-1}(\overline{\Omega})} = q_1 \circ J$ then either none or both of $(F, p_i, q_i, \Omega)$ $(i = 0, 1)$ belongs to $\mathcal{T}_1$ (or $\mathcal{T}_1'$) and in this case*

$$\mathrm{DEG}(F, p_0, q_0, \Omega) = \mathrm{DEG}(F, p_1, q_1, \Omega).$$

(e) (Existence) $\mathrm{DEG}(F, p, q, \Omega) \neq 0$ *implies* $\mathrm{Coin}_\Omega(F, p, q) \neq \emptyset$.

(f) (Homotopy Invariance) *If $(H, P, Q, \Omega)$ belongs to $\mathcal{H}_1$ (resp. $\mathcal{H}_1'$, $\mathcal{H}_1''$) then $(H(t, \cdot), P_t, Q_t, \Omega)$ belongs to $\mathcal{T}_1$ (resp. $\mathcal{T}_1'$, $\mathcal{T}_1''$) for each $t \in [0, 1]$ and*

$$\mathrm{DEG}(H(t, \cdot), P_t, Q_t, \Omega) \text{ is independent of } t \in [0, 1].$$

*If* Deg *satisfies the excision, restriction, resp. additivity property, then also the restriction of* DEG *to the class $\mathcal{T}_1'$, resp. $\mathcal{T}_1''$, satisfies automatically the corresponding properties:*

(g) (Excision) *If $(F, p, q, \Omega) \in \mathcal{T}_1'$ (resp. $\in \mathcal{T}_1''$) and $\Omega_0 \in \mathcal{O}$ is contained in $\Omega$ with $\mathrm{Coin}_\Omega(F, p, q) \subseteq \Omega_0$ then $(F, p, q, \Omega_0) \in \mathcal{T}_1'$ (resp. $\in \mathcal{T}_1''$) and*

$$\mathrm{DEG}(F, p, q, \Omega_0) = \mathrm{DEG}(F, p, q, \Omega).$$

(h) (Restriction) *Under the same assumptions as above we have $(F, p, q, \Omega_0)$ in $\mathcal{T}_1'$ (resp. in $\mathcal{T}_1''$) and*

$$\mathrm{DEG}(F, p, q, \Omega) \neq 0 \;\Rightarrow\; \mathrm{DEG}(F, p, q, \Omega_0) = \mathrm{DEG}(F, p, q, \Omega).$$

(i) (Additivity) *If* $(F,p,q,\Omega)\in\mathcal{T}_1''$ *and* $\Omega_1,\Omega_2\in\mathcal{O}$ *are disjoint with* $\Omega=\Omega_1\cup\Omega_2$*, then* $(F,p,q,\Omega_i)\in\mathcal{T}_1''$ $(i=1,2)$ *and*

$$\mathrm{DEG}(F,p,q,\Omega)=\mathrm{DEG}(F,p,q,\Omega_1)+\mathrm{DEG}(F,p,q,\Omega_2).$$

The homotopy invariance and the independence of $\Gamma$ imply as in Remark 2.3:

COROLLARY 3.11. *The above degree* DEG *satisfies for each* $(F,p,\Omega)\in\mathcal{F}$ *with* $[0,1]\times p^{-1}(\overline{\Omega})\in\mathcal{G}=\mathcal{G}_0\cup\mathcal{G}_1$ *the following property:*

(j) (Homotopy Invariance in the Third Argument) *Let* $h\colon[0,1]\times\Gamma\to Y$ *satisfy:*
  (j1) $(F,p,h(t,\,\cdot\,),\Omega)\in\mathcal{T}$ *has a pre-fundamental set* $K$ *which is independent of* $t\in[0,1]$.
  (j2) *The restriction of* $h$ *to* $[0,1]\times A_{\overline{\Omega}}(F,p,K)$ *is continuous and compact.*
  (j3) *Either* $\mathcal{A}\subseteq\mathrm{AE}_c(\mathcal{G},Y)$ *or assume the following: If a restriction of* $h$ *to a closed set is continuous, then this restriction has a continuous extension.*

  *Then* $\mathrm{DEG}(F,p,h(t,\,\cdot\,),\Omega)$ *is independent of* $t\in[0,1]$.

Before we turn to the proof, let us note that the permanence property means that our extended degree is actually a relative degree (on fundamental sets).

PROOF. Let $(F,p,q,\Omega)\in\mathcal{T}_1$ be arbitrary and $K$ be pre-fundamental. If $K=\emptyset$, we have necessarily $\mathrm{Coin}_\Omega(F,p,q)=\emptyset$ and we define

$$\mathrm{DEG}(F,p,q,\Omega):=0. \tag{3.4}$$

Otherwise, $\emptyset\neq K\in\mathcal{A}_0\subseteq\mathcal{A}$. Hence, we can extend the restriction of $q$ to the closed set $A_{\overline{\Omega}}(F,p,K)\subseteq\Gamma$ to a continuous compact map $\widetilde{q}\colon\Gamma\to Y$ with range in $K$. We denote the system of all such functions $\widetilde{q}$ by $B_\Omega(F,p,q,K)$. Note that we have for each $\widetilde{q}\in B_\Omega(F,p,q,K)$ and $M\subseteq\overline{\Omega}$

$$\mathrm{Coin}_M(F,p,\widetilde{q})=\mathrm{Coin}_{M\cap F^{-1}(K)}(F,p,\widetilde{q})=\mathrm{Coin}_{M\cap F^{-1}(K)}(F,p,q). \tag{3.5}$$

For $M:=\partial\Omega$, we obtain in particular $(F,p,\widetilde{q},\Omega)\in\mathcal{T}_0$, and so we can define

$$\mathrm{DEG}(F,p,q,\Omega):=\mathrm{Deg}(F,p,\widetilde{q},\Omega)\quad(\widetilde{q}\in B_\Omega(F,p,q,K)). \tag{3.6}$$

Since $K$ is a retraction candidate, this definition does not depend on the particular choice of $\widetilde{q}\in B_\Omega(F,p,q,K)$. Let us show that it does not depend on the particular choice of $K$ either and that it does not collide with (3.4).

To see the latter, we note first that in case $\mathrm{Coin}_\Omega(F,p,q)=\emptyset$ the right-hand side of (3.6) vanishes: This follows from (3.5), applied for $M=\overline{\Omega}$ and the existence property of Deg. (This argument implies also the existence property of DEG.)

Now if $K_0$ and $K_1$ are two pre-fundamental sets and $\widetilde{q}_i \in B_\Omega(F,p,q,K_i)$ $(i=0,1)$, we have $K_i \supseteq \mathrm{Fix}_\Omega(F,p,q)$. By what we just observed we may assume that the latter set is nonempty. Hence, $K_0 \cap K_1 \neq \emptyset$ and so at least one of the sets $K_2 := K_0 \cap K_1$ or $K_2 := K_0 \cup K_1$ has the property of Definition 3.5. Note that we have

$$\begin{aligned} &A_{\overline{\Omega}}(F,p,K_2) \subseteq A_{\overline{\Omega}}(F,p,K_0) \cap A_{\overline{\Omega}}(F,p,K_1) \\ \text{resp.}\quad &A_{\overline{\Omega}}(F,p,K_2) = A_{\overline{\Omega}}(F,p,K_0) \cup A_{\overline{\Omega}}(F,p,K_1). \end{aligned} \tag{3.7}$$

In particular, the restriction of $q$ to $A_{\overline{\Omega}}(F,p,K_2) \subseteq A_{\overline{\Omega}}(F,p,K_0) \cup A_{\overline{\Omega}}(F,p,K_1)$ is continuous and compact with values in $K_2 \in \mathcal{A}$ and thus possesses an extension to a continuous compact map $\widetilde{q}_2\colon \Gamma \to Y$ with values in $K_2$, i.e. $\widetilde{q}_2 \in B_\Omega(F,p,q,K_2)$.

An analogous calculation as above shows that this implies $(F,p,\widetilde{q}_2,\Omega) \in \mathcal{T}_0$. Since $K_i$ $(i=0,1)$ is pre-fundamental, we obtain from (3.3) that

$$\mathrm{Deg}(F,p,\widetilde{q}_i,\Omega) = \mathrm{Deg}(F,p,\widetilde{q}_2,\Omega) \quad (i=0,1)$$

which implies that (3.6) is indeed independent of the particular choice of $K$.

We have seen that DEG has even the existence property. The permanence property is clear by construction (because the definition is independent of the particular choice of $\widetilde{q}$ and $K$ and because $B_\Omega(F,p,q,K) \neq \emptyset$).

If $(F,p,q,\Omega) \in \mathcal{T}_0$ then $K := Y$ is a retraction candidate and thus clearly pre-fundamental if $Y \in \mathcal{A}_0$ (put $K_2 := K_0 \cup K_1$ in Definition 3.6). The same argument for $\Omega_0 \subseteq \Omega$ shows that $K$ is even strictly fundamental. In particular, $(F,p,q,\Omega) \in \mathcal{T}_1''$, and in (3.6) the choice $\widetilde{q} := q$ is admissible. This proves the normalization property.

The uniqueness of DEG follows from our definition: If all pre-fundamental sets $K$ for $(F,p,q,\Omega) \in \mathcal{T}_1$ are empty, then the weak existence property implies that we must have (3.4). So assume that $K \neq \emptyset$. Then $B_\Omega(F,p,q,K)$ contains some function $\widetilde{q}$ (as we have seen), and by the permanence and normalization, we see that (3.6) is the only possible definition of DEG.

To see the independence from $\Gamma$, let $(F,p_i,q_i,\Omega) \in \mathcal{T}$ $(i=0,1)$. Each retraction candidate $K \subseteq Y$ for $(F,p_0,q_0,\Omega)$ is a retraction candidate for $(F,p_1,q_1,\Omega)$ (and thus vice versa). In fact, noting that $J$ is a homeomorphism of closed subspaces and thus $J(\overline{M}) = \overline{J(M)}$ and $J^{-1}(\overline{N}) = \overline{J^{-1}(N)}$, one can verify straightforwardly that $\widetilde{q}_1 \in B_\Omega(F,p_1,q_1,K)$ if and only if $\widetilde{q}_1 \circ J$ is the restriction of some $\widetilde{q}_0 \in B_\Omega(F,p_0,q_0,K)$, and since deg is independent of $\Gamma$, the corresponding degrees coincide. It follows analogously that $K$ is pre-fundamental for $(F,p_0,q_0,\Omega)$ if and only if $K$ is pre-fundamental for $(F,p_1,q_1,\Omega)$ (and the corresponding degrees coincide). The same argument applies for each $\Omega_0 \subseteq \Omega$ and thus also a corresponding statement for fundamental sets holds.

We now prove the homotopy invariance. Let $K$ and $A_{\overline{\Omega}}(H,P,K)$ be as in Definition 3.9. The restriction of $Q$ to $A_{\overline{\Omega}}(H,P,K)$ is by assumption continuous and has its range in $K$. The case $K=\emptyset$ is trivial, and otherwise we have $K\in\mathcal{A}_0\subseteq\mathcal{A}$ and thus find an extension $\widetilde{Q}$ of this restriction to a continuous compact map with values in $K$. Since clearly $\widetilde{Q}_t\in B_\Omega(H(t,\,\cdot\,),P_t,Q_t,K)$, we have

$$\operatorname{DEG}(H(t,\,\cdot\,),P_t,Q_t,\Omega)=\operatorname{Deg}(H(t,\,\cdot\,),P_t,\widetilde{Q}_t,\Omega),$$

and the claim follows from the homotopy invariance of Deg.

To see that the excision, resp. restriction property holds, let $K$ be (strictly) fundamental for $(F,p,q,\Omega)\in\mathcal{T}_1'$, and let $\Omega_0\in\mathcal{O}$ satisfy

$$\operatorname{Coin}_\Omega(F,p,q)\subseteq\Omega_0\subseteq\Omega.$$

Note that

$$\operatorname{Coin}_{F^{-1}(K)\cap\partial\Omega_0}(F,p,q)\subseteq\operatorname{Coin}_{F^{-1}(K)\cap\partial\Omega}(F,p,q)\cup(\Omega_0\cap\operatorname{Coin}_{\partial\Omega_0}(F,p,q))=\emptyset,$$

and so Proposition 3.7 implies that $K$ is (strictly) fundamental for $\Omega_0$. Hence, $(F,p,q,\Omega_0)$ belongs to $\mathcal{T}_1'$ (resp. to $\mathcal{T}_1''$). Since the case $\operatorname{Coin}_\Omega(F,p,q,\Omega)=\emptyset$ is trivial, we may assume that $K\neq\emptyset$ and find some $\widetilde{q}\in B_\Omega(F,p,q,K)$ with (3.6). Since $K$ is in particular pre-fundamental for $(F,p,q,\Omega_0)$, and since clearly $\widetilde{q}\in B_{\Omega_0}(F,p,q,K)$ and thus (repeating an earlier argument) $(F,p,\widetilde{q},\Omega_0)\in\mathcal{T}_0$, we obtain by the definition of DEG that

$$\operatorname{DEG}(F,p,q,\Omega_0)=\operatorname{Deg}(F,p,\widetilde{q},\Omega_0).$$

Together with (3.6), it follows that the excision, resp. restriction property of DEG is a consequence of the corresponding property of Deg.

To prove the additivity, let $(F,p,q,\Omega)\in\mathcal{T}_1''$ with a fundamental set $K$, and let $\Omega_1,\Omega_2\in\mathcal{O}$ be disjoint with $\Omega=\Omega_1\cup\Omega_2$. Since $\partial\Omega_i\subseteq\partial\Omega$ $(i=1,2)$, we obtain from Proposition 3.7 that $K$ is fundamental for $(F,p,q,\Omega_i)$. Hence, $(F,p,q,\Omega_i)\in\mathcal{T}_1''$, and for each $\widetilde{q}\in B_\Omega(F,p,q,K)$ we have also $\widetilde{q}\in B_{\Omega_i}(F,p,q,K)$. We obtain

$$\operatorname{DEG}(F,p,q,\Omega_i)=\operatorname{Deg}(F,p,\widetilde{q},\Omega_i)\quad(i=1,2).$$

Together with (3.6), we find that the additivity of DEG follows from the additivity of Deg. □

## 4. Definition of the degree — local case

Roughly speaking, the approach of this section works under the following assumptions (we will actually need slightly less).

(a) $\mathcal{T}_0$ provides a compact triple-degree Deg which satisfies the excision property.

(b) $\mathcal{O}$ is *solid* in the open sets, i.e. if $\Omega_1 \subseteq \Omega_2$ are open with $\Omega_2 \in \mathcal{O}$, then $\Omega_1 \in \mathcal{O}$.
(c) $X$ is normal (resp. regular), and the closure of $\mathrm{Coin}_\Omega(F, p, q)$ contains no point of $\partial\Omega$ (resp. the closure is a compact subset of $\Omega$).
(d) The function $p^{-1}$ is upper semicontinuous or, equivalently, $p$ is a closed map.

It is natural to conjecture that, under these assumptions, one can relax the notion of fundamental sets such that all properties for fundamental sets are required only "in sufficiently small neighbourhoods of $\overline{\mathrm{Coin}_\Omega(F, p, q)}$". In particular, instead of requiring $\mathcal{A} \subseteq \mathrm{AE}^0_c(\mathcal{G}, Y)$ as in the previous section, we will require now $\mathcal{A} \subseteq \mathrm{ANE}^0_c(\mathcal{G}, Y)$. As in the previous section, we fix a family $\mathcal{A}_0 \subseteq \mathcal{A}$.

Definition 4.1. We say that $K \subseteq Y$ is *locally* (*strictly*) *fundamental* for $(F, p, q, \Omega) \in \mathcal{T}$ if there is some $\Omega_0 \in \mathcal{O}$ with

$$\overline{\mathrm{Coin}_\Omega(F, p, q)} \subseteq \Omega_0 \subseteq \Omega \tag{4.1}$$

such that $K$ is weakly (resp. strictly) fundamental for $(F, p, q, \Omega_0)$.

The very definition implies:

Proposition 4.2. *Let* $(F, p, q, \Omega) \in \mathcal{T}$, *and let* $\Omega_0 \in \mathcal{O}$ *satisfy* (4.1). *Then* $K \subseteq Y$ *is locally* (*strictly*) *fundamental for* $(F, p, q, \Omega)$ *if and only if* $K$ *is locally* (*strictly*) *fundamental for* $(F, p, q, \Omega_0)$.

Definition 4.3. We say that a closed set $C \subseteq X$ is $\mathcal{O}$-*normal* if for each nonempty closed $C_0 \subseteq C$ and each open $U \subseteq X$ with $C_0 \subseteq U$ there is some $\Omega_0 \in \mathcal{O}$ with $\overline{\Omega}_0 \subseteq U$ and $C_0 \subseteq \Omega_0$.

Example 4.4. Let $\mathcal{O}$ be solid in the open sets.

(a) If $\overline{\Omega}$ is a $T_4$ space (e.g. normal) then each closed subset is $\mathcal{O}$-normal.
(b) If $\overline{\Omega}$ is a $T_3$ space (e.g. regular) then each compact subset is $\mathcal{O}$-normal.

Definition 4.5. By $\mathcal{T}_2$ (resp. $\mathcal{T}_2'$) we denote the class of all $(F, p, q, \Omega) \in \mathcal{T}$ such that the following holds for some $\Omega_0 \in \mathcal{O}$ with $\Omega_0 \subseteq \Omega$.

(a) The set $C := \overline{\mathrm{Coin}_\Omega(F, p, q)}$ is contained in $\Omega_0$ and $\mathcal{O}$-normal.
(b) $p|_{p^{-1}(\Omega_0)}$ is a closed map, i.e. for each closed $A \subseteq \overline{p^{-1}(\Omega_0)}$ we have $\overline{p(A)} \cap \Omega_0 \subseteq p(A)$.
(c) $(F, p, q, \Omega)$ has a locally (strictly) fundamental set $K \supseteq F(C)$.
(d) Either $\mathcal{A} \subseteq \mathrm{ANE}_c(\mathcal{G}, Y)$ or assume the following: If a restriction of $q$ to a closed set $A \subseteq p^{-1}(\overline{\Omega}_0)$ is continuous, then this restriction has a continuous extension $\widetilde{q}\colon \Gamma_0 \to Y$ to some neighbourhood $\Gamma_0$ of $A$.

Clearly, $\mathcal{T}_2' \subseteq \mathcal{T}_2$. Concerning homotopies, we define:

Definition 4.6. By $\mathcal{H}_2$ (resp. $\mathcal{H}_2'$) we denote the class of all $(H,P,Q,\Omega)$ where $(H,P,\Omega)\in\mathcal{H}$ and $Q\colon\Gamma\to Y$ (where $\Gamma$ denotes the domain of $P$) are such that the following holds for some $K\subseteq Y$ and some $\Omega_0\in\mathcal{O}$ with $\Omega_0\subseteq\Omega$.

(a) The set $C:=\overline{\mathrm{Coin}_\Omega(H,P,Q)}$ is contained in $\Omega_0$ and $\mathcal{O}$-normal.
(b) $P|_{P^{-1}([0,1]\times\Omega_0)}$ is a closed map, i.e. for each closed $A\subseteq\overline{P^{-1}([0,1]\times\Omega_0)}$ we have $\overline{P(A)}\cap([0,1]\times\Omega_0)\subseteq P(A)$.
(c) $K\supseteq H(C)$ is weakly (resp. strictly) fundamental for each of the quadruples $(H(t,\,\cdot\,),P_t,Q_t,\Omega_0)$ $(0\le t\le 1)$.
(d) The restriction of $Q$ to $A_{\Omega_0}(H,P,K)$ is continuous and compact and assumes only values in $K$.
(e) Either $\mathcal{A}\subseteq\mathrm{ANE}_c(\mathcal{G},Y)$ or assume the following: If a restriction of $q$ to a closed set $A\subseteq P^{-1}([0,1]\times\overline{\Omega}_0)$ is continuous, then this restriction has a continuous extension $\widetilde{Q}\colon\Gamma_0\to Y$ to some neighbourhood $\Gamma_0$ of $A$.

The main theorem of this section now reads as follows.

Theorem 4.7. *Let $\mathcal{F}$ provide a compact $\mathcal{H}$-invariant triple-degree* Deg *which satisfies the excision property. Then there is a degree* DEG *which associates to each $(F,p,q,\Omega)\in\mathcal{T}_2$ an element of $G$ such that the following holds:*

(a) (Normalization) *If $Y\in\mathcal{A}_0$ then each $(F,p,q,\Omega)\in\mathcal{T}_0$ belongs to $\mathcal{T}_2'\subseteq\mathcal{T}_2$, and*
$$\mathrm{DEG}(F,p,q,\Omega)=\mathrm{Deg}(F,p,q,\Omega).$$
(b) (Weak Existence) *If $\mathrm{Coin}_\Omega(F,p,q)=\emptyset$ and all locally fundamental sets for $(F,p,q,\Omega)$ are empty, then $\mathrm{DEG}(F,p,q,\Omega)=0$.*
(c) (Permanence) *Put $C:=\overline{\mathrm{Coin}_\Omega(F,p,q,\Omega)}$. If $K\supseteq F(C)$ is locally fundamental for $(F,p,q,\Omega)\in\mathcal{T}_1$ then $\mathrm{DEG}(F,p,q,\Omega)$ depends only on $(F,p,q|_{A_\Omega(F,p,K)},\Omega)$. Moreover, if $C\subseteq\Omega_0\subseteq\Omega$ and $(F,p|_{p^{-1}(\overline{\Omega}_0)},\widetilde{q},\Omega_0)$ in $\mathcal{T}_0$ is such that $\widetilde{q}$ has its range in $K$ and satisfies $\widetilde{q}=q$ on $A_{\Omega_0}(F,p,K)$ then*
$$\mathrm{DEG}(F,p,q,\Omega)=\mathrm{DEG}(F,p,\widetilde{q},\Omega_0).$$

*This degree* DEG *possesses automatically the following properties:*

(d) (Independence from $\Gamma$) *If $(F,p_i,q_i,\Omega)\in\mathcal{T}$ $(i=0,1)$ are such that there is a surjective homeomorphism $J\colon p_0^{-1}(\overline{\Omega})\to p_1^{-1}(\overline{\Omega})$ with $p_0|_{p_0^{-1}(\overline{\Omega})}=p_1\circ J$ and $q_0|_{p_0^{-1}(\overline{\Omega})}=q_1\circ J$ then either none or both of $(F,p_i,q_i,\Omega)$ $(i=0,1)$ belongs to $\mathcal{T}_2$ (or $\mathcal{T}_2'$) and in this case*
$$\mathrm{DEG}(F,p_0,q_0,\Omega)=\mathrm{DEG}(F,p_1,q_1,\Omega).$$
(e) (Existence) *If $\mathrm{Coin}_\Omega(F,p,q)=\emptyset$, then $\mathrm{DEG}(F,p,q,\Omega)=0$.*

(f) (Homotopy Invariance) *If* $(H,P,Q,\Omega)$ *belongs to* $\mathcal{H}_2$ *(resp.* $\mathcal{H}_2'$*) then* $(H(t,\,\cdot\,),P_t,Q_t,\Omega)$ *belongs to* $\mathcal{T}_2$ *(resp.* $\mathcal{T}_2'$*) for each* $t\in[0,1]$ *and*

$$\mathrm{DEG}(H(t,\,\cdot\,),P_t,Q_t,\Omega) \text{ is independent of } t\in[0,1].$$

(g) (Excision) *If* $(F,p,q,\Omega)\in\mathcal{T}_2$ *(resp.* $\in\mathcal{T}_2'$*) and* $\Omega_0\in\mathcal{O}$ *is contained in* $\Omega$ *with* $\mathrm{Coin}_\Omega(F,p,q)\subseteq\Omega_0$ *then* $(F,p,q,\Omega_0)\in\mathcal{T}_2$ *(resp.* $\in\mathcal{T}_2'$*) and*

$$\mathrm{DEG}(F,p,q,\Omega_0)=\mathrm{DEG}(F,p,q,\Omega). \tag{4.2}$$

*If* Deg *is additive, then also the restriction of* DEG *to* $\mathcal{T}_2'$ *is additive:*

(h) (Additivity) *If* $(F,p,q,\Omega)\in\mathcal{T}_2'$ *and* $\Omega_1,\Omega_2\in\mathcal{O}$ *are disjoint with* $\Omega=\Omega_1\cup\Omega_2$*, then* $(F,p,q,\Omega_i)\in\mathcal{T}_2'$ $(i=1,2)$ *and*

$$\mathrm{DEG}(F,p,q,\Omega)=\mathrm{DEG}(F,p,q,\Omega_1)+\mathrm{DEG}(F,p,q,\Omega_2).$$

Corollary 4.8. *The above degree* DEG *satisfies for each* $(F,p,\Omega)\in\mathcal{F}$ *with* $[0,1]\times p^{-1}(\overline{\Omega})\in\mathcal{G}=\mathcal{G}_0\cup\mathcal{G}_1$ *the following property.*

(i) (Homotopy Invariance in the Third Argument) *Let* $h\colon[0,1]\times\Gamma\to Y$ *be such that* $(F,p,h(t,\,\cdot\,),\Omega)\in\mathcal{T}$ *and the following holds for some* $K\subseteq Y$ *and some* $\Omega_0\in\mathcal{O}$ *with* $\Omega_0\subseteq\Omega$*:*

(i1) $C:=\overline{\bigcup_{t\in[0,1]}\mathrm{Coin}_\Omega(F,p,h(t,\,\cdot\,))}$ *is contained in* $\Omega_0$ *and* $\mathcal{O}$*-normal.*

(i2) $p|_{p^{-1}(\Omega_0)}$ *is a closed map.*

(i3) $K\supseteq F(C)$ *is weakly (resp. strictly) fundamental for each of the quadruples* $(F,p,h(t,\,\cdot\,),\Omega_0)$ $(0\le t\le 1)$*.*

(i4) *The restriction of* $h$ *to* $[0,1]\times A_{\Omega_0}(F,p,K)$ *is continuous and compact and assumes only values in* $K$*.*

(i5) *Either* $\mathcal{A}\subseteq\mathrm{ANE}_c(\mathcal{G},Y)$ *or assume: If a restriction of* $h$ *to a closed subset* $A\subseteq[0,1]\times p^{-1}(\overline{\Omega}_0)$ *is continuous, then this restriction has a continuous extension* $\widetilde{h}\colon\Gamma_0\to Y$ *to some neighbourhood* $\Gamma_0$ *of* $A$*.*

*Then* $(F,p,h(t,\,\cdot\,),\Omega)$ *belongs to* $\mathcal{T}_2$ *(resp.* $\mathcal{T}_2'$*) for each* $t\in[0,1]$ *and*

$$\mathrm{DEG}(F,p,h(t,\,\cdot\,),\Omega) \text{ is independent of } t\in[0,1].$$

Proof. Let $(F,p,q,\Omega)\in\mathcal{T}_2$ be arbitrary, $C:=\overline{\mathrm{Coin}_\Omega(F,p,q)}$, and $K\supseteq F(C)$ locally fundamental. If $K=\emptyset$, we put

$$\mathrm{DEG}(F,p,q,\Omega):=0. \tag{4.3}$$

If $K\neq\emptyset$, we denote by $B_\Omega(F,p,q,K)$ the system of all pairs $(\widetilde{q},\Omega_0)$ satisfying the requirement of the permanence property, i.e. $(F,p|_{p^{-1}(\overline{\Omega}_0)},\widetilde{q},\Omega_0)\in\mathcal{T}_0$, $C\subseteq\Omega_0\subseteq\Omega$ and such that $\widetilde{q}$ has its range in $K$ and satisfies $\widetilde{q}=q$ on $A_{\Omega_0}(F,p,K)$.

Let us first show that $B_\Omega(F,p,q,K)$ is not empty. Choose some $\Omega_1\in\mathcal{O}$ with $C\subseteq\Omega_1\subseteq\Omega$ such that $K$ is weakly fundamental for $(F,p,q,\Omega_1)$, in particular

$K \in \mathcal{A}_0 \subseteq \mathcal{A}$. Since $C$ is $\mathcal{O}$-normal, we may assume without loss of generality that $\Omega_1$ is contained in the set $\Omega_0$ of Definition 4.5. The restriction of $q$ to the closed set $A_{\overline{\Omega}_1}(F,p,K) \subseteq \Gamma$ has an extension to a continuous compact map $\widetilde{q}\colon \Gamma_0 \to Y$ with range in $K$ where $\Gamma_0 \supseteq A_{\overline{\Omega}}(F,p,K)$ is open in the space $p^{-1}(\overline{\Omega}_1)$. Since $C \subseteq F^{-1}(K)$, we have

$$p^{-1}(C) \subseteq A_{\overline{\Omega}_1}(F,p,K) \subseteq \Gamma_0.$$

Since $p^{-1}$ is upper semicontinuous on $\Omega_1$, we find some open $U \subseteq \Omega_1$ with $C \subseteq U$ and $p^{-1}(U) \subseteq \Gamma_0$. Since $C$ is $\mathcal{O}$-normal, we thus find some $\Omega_0 \in \mathcal{O}$ with $\overline{\Omega}_0 \subseteq U$ and $C \subseteq \Omega_0$. In particular, the restriction of $\widetilde{q}$ to $p^{-1}(\overline{\Omega}_0) \subseteq \Gamma_0$ is continuous and compact and assumes its values in $K$. Moreover,

$$\begin{aligned} \text{(4.4)} \qquad \operatorname{Coin}_M(F,p,\widetilde{q}) &= \operatorname{Coin}_{M\cap F^{-1}(K)}(F,p,\widetilde{q}) \\ &= \operatorname{Coin}_{M\cap F^{-1}(K)}(F,p,q) \quad (M \subseteq \overline{\Omega}_0), \end{aligned}$$

and so (put $M = \partial\Omega_0$) we have $(F,p,\widetilde{q},\Omega_0) \in \mathcal{T}_0$. Hence we have found some $(\widetilde{q},\Omega_0) \in B_\Omega(F,p,q,K)$, as claimed. We now prove that we can define

$$\text{(4.5)} \qquad \operatorname{DEG}(F,p,q,\Omega) := \operatorname{Deg}(F,p,\widetilde{q},\Omega_0) \quad ((\widetilde{q},\Omega_0) \in B_\Omega(F,p,q,K)),$$

i.e. that the right-hand side of (4.5) is independent of the particular choice of $(\widetilde{q},\Omega_0)$ and $K$ and that the definition does not collide with (4.3). To see this, note first that the same calculation as above (put $M = \Omega_0$ in (4.4)) shows that

$$\text{(4.6)} \qquad \operatorname{Coin}_\Omega(F,p,q) = \operatorname{Coin}_{\Omega_0}(F,p,\widetilde{q}) \quad ((\widetilde{q},\Omega_0) \in B_\Omega(F,p,q,K)).$$

In case $C = \emptyset$, it follows that the right-hand side of (4.5) vanishes by the existence property of Deg (independent of the choice of $(\widetilde{q},\Omega_0)$ and $K$). This proves the existence property of DEG and that the definition (4.5) is compatible with (4.3).

Assume now that $K_i \supseteq F(C)$ are two locally fundamental sets and $(\widetilde{q}_i,\Omega_i) \in B_\Omega(F,p,q,K_i)$ $(i = 0,1)$. Since the case $C = \emptyset$ was treated above, we may thus assume that $K_0 \cap K_1 \neq \emptyset$.

Put $K_2 := K_0 \cup K_1$ and $K_3 := K_0 \cap K_1$. If $K_j \in \mathcal{A}$ ($j = 2$ or $j = 3$) we can in view of (3.7) extend the restriction $q|_{A_{\overline{\Omega}}(F,p,K_j)}$ to a continuous compact map $\widetilde{q}_j\colon \Gamma' \to Y$ with range in $K_j$ where $\Gamma' \supseteq A_{\overline{\Omega}}(F,p,K_j)$ is open. A similar argument as in the beginning of the proof shows that we find some $\Omega_j \in \mathcal{O}$ with $C \subseteq \Omega_j \subseteq \Omega_0 \cap \Omega_1$ such that $(F,p,\widetilde{q}_j,\Omega_j) \in \mathcal{T}_0$. If $K_j \notin \mathcal{A}$, we put $\Omega_j := \Omega$ $(j = 2,3)$.

Since $K_i$ $(i = 0,1)$ is locally fundamental, there is some $\Omega_{4+i} \in \mathcal{O}$ with $C \subseteq \Omega_{4+i} \subseteq \Omega$ such that $K_i$ is weakly fundamental for $(F,p,q,\Omega_{4+i})$. Since $C$ is $\mathcal{O}$-normal, we find some $\Omega' \in \mathcal{O}$ with $C \subseteq \Omega' \subseteq \Omega_i$ $(i = 0,\dots,5)$. Note that (4.6) implies that $(F,p,\widetilde{q}_i,\Omega') \in \mathcal{T}_0$ $(i = 0,1)$ and, by the excision property,

$$\operatorname{Deg}(F,p,\widetilde{q}_i,\Omega_i) = \operatorname{Deg}(F,p,\widetilde{q}_i,\Omega') \quad (i = 0,1).$$

The sets $K_0$ and $K_1$ are both pre-fundamental for $(F,p,q,\Omega')$. In particular, at least one of the sets $K_2=K_0\cup K_1$ or $K_3=K_0\cap K_1$ must have the property of Definition 3.5 which means that we have either for $j=2$ or $j=3$ that $K_j\in\mathcal{A}$ and

$$\operatorname{Deg}(F,p,\widetilde{q}_i,\Omega')=\operatorname{Deg}(F,p,\widetilde{q}_j,\Omega')\quad(i=0,1).$$

Combining the above formulas, we thus proved that

$$\operatorname{Deg}(F,p,\widetilde{q}_0,\Omega_0)=\operatorname{Deg}(F,p,\widetilde{q}_1,\Omega_1),$$

and so (4.5) is indeed independent of the particular choice of $(\widetilde{q},\Omega_0)$ or $K$.

The existence property has been proved above. The independence of $\Gamma$ and the normalization property is established by the same arguments as in the proof of Theorem 3.10. The permanence property follows immediately from (4.5) and the fact that there is some $(\widetilde{q},\Omega_0)\in B_\Omega(F,p,q,K)$ with $\overline{\Omega}_0\subseteq\Omega$.

The uniquenes of $\operatorname{DEG}(F,p,q,\Omega)$ is trivial if all locally fundamental sets are empty, because then $\operatorname{Coin}_\Omega(F,p,q)=\emptyset$ and (4.3) must hold by the weak existence property. However, otherwise, since $B_\Omega(F,p,q,K)\neq\emptyset$, we must have (4.5) by the permanence and normalization property.

To prove the excision property, let $K$ be locally (strictly) fundamental for $(F,p,q,\Omega)\in\mathcal{T}_2$. By Proposition 4.2, $K$ is locally (strictly) fundamental for $(F,p,q,\Omega_0)$, and so $(F,p,q,\Omega)\in\mathcal{T}_2$. The formula (4.2) is an immediate consequence of (4.5), because we find some $(\widetilde{q},\Omega_1)\in B_\Omega(F,p,q,K)$ with $\Omega_1\subseteq\Omega_0$.

To prove the homotopy invariance, let $C$ and $K\supseteq H(C)$ be as in Definition 4.6. Since $C$ is $\mathcal{O}$-normal, we find some $\Omega_0\in\mathcal{O}$ such that $\overline{\Omega}_0$ is contained in the set $\Omega_0$ of Definition 4.6, and $C\subseteq\Omega_0$. In particular,

$$\operatorname{Coin}_{\partial\Omega_0}(H(t,\,\cdot\,),P_t,Q_t)=\emptyset,$$

and the restriction of $Q$ to $A_{\overline{\Omega}_0}(H,P,K)$ is continuous and compact with values in $K$. The definition now immediately implies that $K$ is locally (strictly) fundamental for $(H(t,\,\cdot\,),P_t,Q_t,\Omega)$ and that $(H(t,\,\cdot\,),P_t,Q_t,\Omega)$ belongs to $\mathcal{T}_2$ (resp. $\mathcal{T}_2'$) for each $t\in[0,1]$ (note here that $\overline{\operatorname{Coin}_\Omega(H(t,\,\cdot\,),P_t,Q_t)}$ is a closed subset of $C$ and thus $\mathcal{O}$-normal). In view of the excision property of DEG, it suffices to prove that

$$\operatorname{DEG}(H(t,\,\cdot\,),P_t,Q_t,\Omega_0)\text{ is independent of }t\in[0,1].$$

Since the case $C=\emptyset$ is trivial (all corresponding degrees are 0), we assume that $K\neq\emptyset$ and thus $K\in\mathcal{A}_0$. Since $K$ is pre-fundamental for each $(H(t,\,\cdot\,),P_t,Q_t,\Omega_0)$ $(0\le t\le 1)$, the continuous and compact restriction of $Q$ to the closed set $A_{\overline{\Omega}_0}(H,P,K)$ attains its values in $K\in\mathcal{A}$ and thus has an extension to a continuous compact map $\widetilde{Q}$ on some neighbourhood $U$ of $A_{\overline{\Omega}_0}(H,P,K)$ with values in $K$. Since $P^{-1}$ is upper semicontinuous, $P^{-1}([0,1]\times C)\subseteq U$, and $C$ is $\mathcal{O}$-normal, we find some $\Omega_1 \in \mathcal{O}$ with $C \subseteq \Omega_1 \subseteq \Omega_0$ such that $P^{-1}([0,1] \times \overline{\Omega}_1) \subseteq U$. We have

$$\mathrm{Coin}_{H(t,\,\cdot\,)^{-1}(K)\cap\partial\Omega_1}(H(t,\,\cdot\,), P_t, \widetilde{Q}_t) \subseteq C \cap \partial\Omega_1 = \emptyset,$$

and so $(H(t,\,\cdot\,), P_t, \widetilde{Q}_t, \Omega_1) \in \mathcal{T}_0$ which implies

$$(\widetilde{Q}_t, \Omega_1) \in B_\Omega(H(t,\,\cdot\,), P_t, Q_t, \Omega_0).$$

Hence, we have by (4.5)

$$\mathrm{DEG}(H(t,\,\cdot\,), P_t, Q_t, \Omega_0) = \mathrm{Deg}(H(t,\,\cdot\,), P_t, \widetilde{Q}_t, \Omega_1) \quad (0 \le t \le 1).$$

Since Deg is $\mathcal{H}$-invariant, the right-hand side is independent of $t$.

Now we prove the additivity for $(F,p,q,\Omega) \in \mathcal{T}_2'$. Let $\Omega_i \in \mathcal{O}$ $(i = 1,2)$ be disjoint with $\Omega = \Omega_1 \cup \Omega_2$, $C_i := \overline{\mathrm{Coin}_{\Omega_i}(F,p,q)}$ $(i = 1,2)$ and

$$C := \overline{\mathrm{Coin}_\Omega(F,p,q)} = C_1 \cup C_2.$$

Let $K \supseteq F(C)$ be locally strictly fundamental. Choose $(\widetilde{q}, \Omega_0) \in B_\Omega(F,p,q,K)$. In particular, $K$ is strictly fundamental for $(F,p,q,\Omega_0)$. Note that $C_i$ are closed subsets of $C$ and thus also $\mathcal{O}$-normal. Moreover, since $C \subseteq \Omega = \Omega_1 \cup \Omega_2$ and $C_i$ is a closure of a subset of $\Omega_i$, it follows that $C_i \subseteq \Omega_i$ $(i = 1,2)$. Hence, $C_i \subseteq \Omega_{i,0} := \Omega_0 \cap \Omega_i$ $(i = 1,2)$. Since $\overline{\Omega}_0 \subseteq \Omega$ we have also $\overline{\Omega}_{i,0} \subseteq \Omega_i$ $(i = 1,2)$. In particular, $\mathrm{Coin}_{\partial\Omega_{i,0}}(F,p,q) = \emptyset$. Hence, $K$ is strictly fundamental for $(F,p,q,\Omega_{i,0})$ and thus locally strictly fundamental for $(F,p,q,\Omega_i)$. It follows that $(F,p,q,\Omega_i) \in \mathcal{T}_2'$ and, moreover, $(\widetilde{q}, \Omega_{i,0}) \in B_{\Omega_i}(F,p,q,K)$. Hence,

$$\mathrm{DEG}(F,p,q,\Omega_i) = \mathrm{Deg}(F,p,\widetilde{q},\Omega_{i,0}) \quad (i = 1,2)$$

and $\mathrm{DEG}(F,p,q,\Omega) = \mathrm{Deg}(F,p,\widetilde{q},\Omega_0)$. Since $\Omega_0 = \Omega_{1,0} \cup \Omega_{2,0}$, the additivity of DEG now follows from the addditivity of Deg. □

## 5. Homotopic tests for fundamental sets

In the previous sections, we have seen how a compact degree can be extended to certain noncompact $(F,p,q,\Omega) \in \mathcal{T}$. However, the crucial assumption was that one can find a fundamental, resp. a locally fundamental set $K$. In this section, we discuss, how one can verify that a given set $K$ has this property. Necessary conditions are, roughly speaking, that $K$ is invariant under the multivalued map $q \circ p^{-1} \circ F^{-1}$, that $K$ contains all fixed points of this map, and that this map is compact on $K$. If $K \in \mathrm{AE}_c(\mathcal{G}, Y)$, resp. $K \in \mathrm{ANE}_c(\mathcal{G}, Y)$, this already implies that $K$ is a retraction candidate as the following Proposition 5.1 shows. It is remarkable that these hypotheses are independent of the particular choice of the degree Deg. This is why we speak in this section of *homotopic* tests.

Proposition 5.1. *Let* $(F,p,q,\Omega)\in\mathcal{T}$ *and* $K\subseteq Y$ *contain* $\mathrm{Fix}_\Omega(F,p,q)$. *Suppose that* (3.1) *holds and that the restriction of* $q$ *to* $A_{\overline{\Omega}}(F,p,K)$ *is continuous and compact and assumes its values in* $K$. *Finally, let* $[0,1]\times p^{-1}(\overline{\Omega})\in\mathcal{G}$. *Then for each compact triple-degree* Deg *for* $\mathcal{F}$ *the set* $K$ *is a retraction candidate for* $(F,p,q,\Omega)$, *if at least one of the following conditions is satisfied:*

(a) $K\in\mathrm{AE}_c(\mathcal{G},Y)$,
(b) $K\in\mathrm{ANE}_c(\mathcal{G},Y)$, Deg *satisfies the excision property,* $C:=\overline{\mathrm{Coin}_\Omega(F,p,q)}$ *is contained in* $\Omega$ *and* $\mathcal{O}$*-normal,* $F(C)\subseteq K$, *and there is a neighbourhood* $\Omega_0\subseteq\Omega$ *of* $C$ *such that* $p|_{p^{-1}(\Omega_0)}$ *is a closed map.*

Proof. We have to verify the last property of Definition 3.4. Thus, let $\Gamma:=p^{-1}(\overline{\Omega})$ and $\widetilde{q}_i\colon\Gamma\to K$ $(i=0,1)$ be extensions of the restriction $q|_{A_{\overline{\Omega}}(F,p,K)}$ such that $(F,p,\widetilde{q}_i,\Omega)\in\mathcal{T}_0$. We define

$$h(t,z):=\begin{cases}\widetilde{q}_i(z) & \text{if } t=i\in\{0,1\},\\ q(z) & \text{if } z\in A_{\overline{\Omega}}(F,p,K).\end{cases}$$

Then $h$ is a continuous compact map defined on a closed subset $A$ of $\Pi:=[0,1]\times p^{-1}(\overline{\Omega})\in\mathcal{G}$ with values in $K$. Hence, $h$ has an extension to $\Pi_0:=\Pi$ (resp. to a neighbourhood $\Pi_0\subseteq\Pi$ of $A$) such that $h$ is continuous and compact and $h(\Pi_0)\subseteq K$. The latter implies for each $t\in[0,1]$

$$\begin{aligned}\mathrm{Coin}_{\overline{\Omega}}(F,p,h(t,\,\cdot\,)) &= \mathrm{Coin}_{\overline{\Omega}\cap F^{-1}(K)}(F,p,h(t,\,\cdot\,))\\ &= \mathrm{Coin}_{\overline{\Omega}\cap F^{-1}(K)}(F,p,q)\\ &= \mathrm{Coin}_{\Omega\cap F^{-1}(K)}(F,p,q)=\mathrm{Coin}_\Omega(F,p,q)\end{aligned}\tag{5.1}$$

where we used the definition of $h$ on $A_{\overline{\Omega}}(F,p,K)$ for the second, and (3.1) for the third equality. In case $\Pi_0=\Pi$, we conclude by the homotopy invariance of Deg that

$$\mathrm{Deg}(F,p,\widetilde{q}_0,\Omega)=\mathrm{Deg}(F,p,\widetilde{q}_1,\Omega)\tag{5.2}$$

and are done. This proves in particular the claim under the first hypothesis (i.e. if $K\in\mathrm{AE}_c(\mathcal{G},Y)$). For the other hypothesis, note that $F(C)\subseteq K$ implies $p^{-1}(C)\subseteq A_{\overline{\Omega}}(F,p,q)$. Since $[0,1]$ is compact and for each $t_0\in[0,1]$ there is an open neighbourhood $\Gamma_0\subseteq\Gamma$ of $p^{-1}(C)$ such that $\{t\}\times\Gamma_0\subseteq\Pi_0$ for all $t$ in a neighbourhood of $t_0$, we conclude that even $[0,1]\times\Gamma_0\subseteq\Gamma$ for some open neighbourhood $\Gamma_0\subseteq\Gamma$ of $p^{-1}(C)$. Let $\Omega_0$ be as in the hypothesis. By the upper semicontinuity of $p^{-1}|_{\Omega_0}$, we may assume that $p^{-1}(\Omega_0)\subseteq\Gamma_0$. Since $C$ is $\mathcal{O}$-normal, we find some $\Omega_1\in\mathcal{O}$ with $C\subseteq\Omega_1$ and $\overline{\Omega}_1\subseteq\Omega_0\subseteq\Omega$. Note that $h$ is defined, continuous and compact on $[0,1]\times\overline{\Omega}_1$. By (5.1), we conclude that $(F,p,h(t,\,\cdot\,),\Omega_1)\in\mathcal{T}_0$, and so the homotopy invariance of Deg implies

$$\mathrm{Deg}(F,p,h(0,\,\cdot\,),\Omega_1)=\mathrm{Deg}(F,p,h(1,\,\cdot\,),\Omega_1).$$

Moreover, the excision property of Deg implies in view of (5.1) that

$$\operatorname{Deg}(F,p,h(i,\,\cdot\,),\Omega_1)=\operatorname{Deg}(F,p,h(i,\,\cdot\,),\Omega)=\operatorname{Deg}(F,p,\widetilde{q}_i,\Omega)\quad (i=0,1).$$

Combining these equalities, we find (5.2), as required. □

Proposition 5.1 is a very convenient tool in order to verify that a given set $K$ is a retraction candidate. We would like to have a similar (homotopic) tool to verify that $K$ is pre-fundamental, i.e. a tool to verify (3.3). This condition is always satisfied if, roughly speaking, one of the sets $K_0$ and $K_2$ is a strong deformation retract of the other and the deformation can be chosen such that it avoids certain coincidences outside the smaller set.

Definition 5.2. Let $(F,p,q,\Omega)\in\mathcal{T}$, and let $K_1\subseteq K_2\subseteq Y$. Then we call $K_1$ an $(F,p,q,\Omega)$-*deformation retract* of $K_2$ if there is a continuous map $R\colon[0,1]\times\overline{K}_2\to Y$ such that the following holds.

(a) $R([0,1]\times K_2)\subseteq K_2$, $R(0,\,\cdot\,)=\mathrm{id}$, and $R(1,K_2)\subseteq K_1$.
(b) $R(1,\,\cdot\,)=\mathrm{id}$ on $K_1\cap q(A_{\overline{\Omega}}(F,p,K_1))$.
(c) $R(t,\,\cdot\,)=\mathrm{id}$ on $K_1\cap q(p^{-1}(F^{-1}(K_1)\cap\partial\Omega))\subseteq K_1\cap q(A_{\overline{\Omega}}(F,p,K_1))$ for each $t\in[0,1]$.
(d) $\operatorname{Fix}_{F^{-1}(K_2)\cap\partial\Omega}(F,p,R(t,q(\,\cdot\,)))\subseteq K_1$ for each $t\in[0,1]$.

If we can even replace $\partial\Omega$ by $\overline{\Omega}$ in the last two requirements, then we call $K_1$ a *strong* $(F,p,q,\Omega)$-*deformation retract*.

Proposition 5.3. *Let $K_1$ be an $(F,p,q,\Omega)$-deformation retract of $K_2$. Let $\widetilde{q}_i$ $(i=1,2)$ be extensions of the restriction of $q$ to $A_{\overline{\Omega}}(F,p,K_i)$ which assume only values in $K_i$ and such that $(F,p,\widetilde{q}_i,\Omega)\in\mathcal{T}_0$. Then we have for each compact triple-degree* Deg *(for $\mathcal{F}$) for which $K_1$ is a retraction candidate for $(F,p,q,\Omega)$ that*

$$\operatorname{Deg}(F,p,\widetilde{q}_1,\Omega)=\operatorname{Deg}(F,p,\widetilde{q}_2,\Omega).$$

Proof. With $R$ as in Definition 5.2, consider the homotopy

$$h(t,z):=R(t,\widetilde{q}_2(z)).$$

Since $R$ is defined and continuous on the compact set $[0,1]\times\overline{\widetilde{q}_2(p^{-1}(\overline{\Omega}))}$, the homotopy $h$ is compact. We claim that $(F,p,h(t,\,\cdot\,),\Omega)\in\mathcal{T}_0$ for each $t\in[0,1]$.

In fact, suppose on the contrary that there are $t\in[0,1]$, $x\in\partial\Omega$, and $z\in p^{-1}(x)$ with $y:=F(x)=h(t,z)$. Since $\widetilde{q}_2(z)\in K_2$, we have $F(x)=h(t,z)\in K_2$, and so $x\in F^{-1}(K_2)$. In particular, we have $z\in A_{\overline{\Omega}}(F,p,K_2)$ which implies $\widetilde{q}_2(z)=q(z)$, and so $F(x)=R(t,q(z))$. Since $x\in F^{-1}(K_2)\cap\partial\Omega$, our assumption on $R$ thus implies $F(x)\in K_1$. Hence, $z\in A_{\overline{\Omega}}(F,p,K_1)$ which implies $q(z)=\widetilde{q}_1(z)\in K_1$. Clearly, $q(z)\in q(p^{-1}(F^{-1}(K_1)\cap\partial\Omega))$. Our assumption on $R$ thus implies $R(t,q(z))=q(z)$, and so $F(x)=q(z)=\widetilde{q}_1(z)$. Hence, $x\in\operatorname{Coin}_{\partial\Omega}(F,p,\widetilde{q}_1)$, a contradiction to our assumption $(F,p,\widetilde{q}_1,\Omega)\in\mathcal{T}_0$.

This contradiction shows that indeed $(F, p, h(t, \cdot\,), \Omega) \in \mathcal{T}_0$ for each $t \in [0, 1]$, and so the homotopy invariance of Deg implies

$$\operatorname{Deg}(F, p, \widetilde{q}_2, \Omega) = \operatorname{Deg}(F, p, h(0, \cdot\,), \Omega) = \operatorname{Deg}(F, p, h(1, \cdot\,), \Omega).$$

The function $h(1, \cdot\,)$ attains its values in $K_1$ and is (by our choice of $R$) an extension of $q|_{A_{\overline{\Omega}}(F,p,K_1)}$. Since $K_1$ is a retraction candidate, this implies

$$\operatorname{Deg}(F, p, \widetilde{q}_1, \Omega) = \operatorname{Deg}(F, p, h(1, \cdot\,), \Omega),$$

and the claim follows. □

Corollary 5.4. *Let $K_0$ be a retraction candidate for $(F, p, q, \Omega) \in \mathcal{T}$ and either be empty or belong to $\mathcal{A}_0$. Then $K_0$ is pre-fundamental for $(F, p, q, \Omega)$ if for each retraction candidate $K_1 \in \mathcal{A}_0$ for $(F, p, q, \Omega)$ with $K_0 \cap K_1 \neq \emptyset$ at least one of the following holds:*

(a) *$K_0 \cap K_1$ belongs to $\mathcal{A}$ and is a retraction candidate for $(F, p, q, \Omega)$ and an $(F, p, q, \Omega)$-deformation retract of $K_0$.*
(b) *$K_0 \cup K_1$ belongs to $\mathcal{A}$, and $K_0$ is an $(F, p, q, \Omega)$-deformation retract of $K_0 \cup K_1$.*

It will be convenient to introduce some further notions.

Definition 5.5. We call $K \subseteq Y$ *union-admissible* if for each $K' \in \mathcal{A}_0$ with $K \cap K' \neq \emptyset$ we have $K \cup K' \in \mathcal{A}$.

As an example, we prove (using Dugundji's extension theorem and thus the axiom of choice if $Y$ is not separable):

Proposition 5.6. *Let $Y$ be a metrizable convex subset of a locally convex space, and suppose that each $K \in \mathcal{A}_0$ is closed (in $Y$) and convex. Let one of the following assumptions be satisfied.*

(a) *$\mathcal{G}$ contains only $T_4$ spaces.*
(b) *$\mathcal{A} = \mathrm{AE}^0_c(\mathcal{G}, Y)$ or $\mathcal{A} = \mathrm{ANE}^0_c(\mathcal{G}, Y)$.*

*Then each $K \in \mathcal{A}_0 \cup \{\emptyset\}$ is union-admissible.*

Proof. Each $K \in \mathcal{A}_0$ is an AR. Moreover, for each $K' \in \mathcal{A}_0$ for which the intersection $K \cap K' \in \mathcal{A}_0$ is nonempty, this intersection is also convex and thus also an AR. Since $K$ and $K'$ are closed in $K \cup K'$, it follows that $K \cup K'$ is an AR, see e.g. [11, § 6]. By Proposition 3.3, we conclude $K \cup K' \subseteq \mathcal{A}$. □

Definition 5.7. We call $K \subseteq Y$ a (*strong*) *deformation candidate* for $(F, p, q, \Omega) \in \mathcal{T}$ if the following holds:

(a) $K$ is either empty or belongs to $\mathcal{A}_0$ and is a (strong) $(F, p, q, \Omega)$-deformation retract of $Y$.
(b) $\mathrm{Fix}_\Omega(F, p, q) \subseteq K$.

The following theorem is a summary of our previous observations.

Theorem 5.8. *Let* $(F,p,q,\Omega)\in\mathcal{T}$ *and* $K\subseteq Y$ *satisfy:*

(a) *$K$ is a (strong) deformation candidate for* $(F,p,q,\Omega)$.
(b) *$K$ is union-admissible.*
(c) $[0,1]\times p^{-1}(\overline{\Omega})\in\mathcal{G}$ *and* $K\in\mathrm{AE}_c(\mathcal{G},Y)$.
(d) $\mathrm{Coin}_{F^{-1}(K)\cap\partial\Omega}(F,p,q)=\emptyset$.
(e) *The restriction of $q$ to $A_{\overline{\Omega}}(F,p,K)$ is continuous and compact and assumes its values in $K$.*

*Then $K$ is pre-fundamental* (*resp. strictly fundamental*) *for* $(F,p,q,\Omega)$.

For some applications the invariance condition $q(A_{\overline{\Omega}}(F,p,K))\subseteq K$ is too restrictive. Instead, one would like to require only the invariance $q(A_{\Omega}(F,p,K))\subseteq K$ or, even better, an invariance of the type

$$q(A_{\Omega_0}(F,p,K))\subseteq K \tag{5.3}$$

where $\Omega_0\subseteq\Omega$ is an arbitrary small neighbourhood of $C:=\mathrm{Coin}_{\Omega}(F,p,q)$.

The weaker assumption (5.3) is in the attitude of the "pushing assumption" which was apparently first introduced in [21], [22] and further employed in [3] and [2]. Moreover, the dropping of the closure in (5.3) simplifies some considerations in connection with compactness assumptions on countable sets as we will see in Section 7. This is indeed possible in the local setting of Section 4. We summarize our previous observations in this special case.

Theorem 5.9. *Let* $(F,p,q,\Omega)\in\mathcal{T}$ *and* $K\subseteq Y$. *Let* Deg *satisfy the excision property and suppose that there is some* $\Omega_0\in\mathcal{O}$ *with* $\Omega_0\subseteq\Omega$ *such that the following holds.*

(a) *$K$ is a strong deformation candidate for* $(F,p,q,\Omega_0)$.
(b) *$K$ is union-admissible.*
(c) $[0,1]\times p^{-1}(\overline{\Omega})\in\mathcal{G}$ *and* $K\in\mathrm{ANE}_c(\mathcal{G},Y)$.
(d) $C:=\overline{\mathrm{Coin}_{\Omega}}(F,p,q)$ *is contained in $\Omega_0$ and $\mathcal{O}$-normal.*
(e) $F(C)\subseteq K$.
(f) $p|_{p^{-1}(\Omega_0)}$ *is a closed map.*
(g) *The restriction of $q$ to $A_{\Omega_0}(F,p,K)$ is continuous and compact and assumes its values in $K$.*

*Then $K$ is locally strictly fundamental for $(F,p,q,\Omega)$; in particular, if $\mathcal{A}\subseteq\mathrm{ANE}_c(\mathcal{G},Y)$, then $(F,p,q,\Omega)\in\mathcal{T}_2'\subseteq\mathcal{T}_2$. More precisely, $K$ is strictly fundamental for $(F,p,q,\Omega_0')$ whenever $\Omega_0'\in\mathcal{O}$ satisfies $C\subseteq\Omega_0'$ and $\overline{\Omega}_0'\subseteq\Omega_0$.*

The last remark in Theorem 5.9 is important in connection with the homotopy invariance of DEG (where $\Omega_0$ has to be chosen independent of $t$).

## 6. Finding small deformation candidates

By the previous section, we reduced the main difficulty of the degree for noncompact $(F,p,q,\Omega)\in\mathcal{T}$ to the existence of (strong) deformation candidates which are sufficiently small and invariant under the multivalued map $q\circ p^{-1}\circ F^{-1}$. In this section, we discuss how one can prove the existence of such sets $K$. The idea is to define $K$ as the intersection of "hull candidates".

We assume in this section that we have given a fixed *hull function* $c\colon 2^Y\to 2^Y$ with the property

$$M_1\subseteq M_2\subseteq Y \;\Rightarrow\; M_1\subseteq c(M_1)\subseteq c(M_2)\subseteq Y \tag{6.1}$$

We think of $c$ as a function which associates to each set $M\subseteq Y$ a corresponding "hull". The most important case will be when $Y$ is a convex subset of a topological vector space and $c(M):=\operatorname{conv}(M)$ denotes the convex hull of $M$.

Definition 6.1. Let $c$ satisfy (6.1). Then we call $K\subseteq Y$ a *hull candidate* for $(F,p,q,\Omega)\in\mathcal{T}$ if

$$F(x)\in c(q(p^{-1}(x))\cup K) \;\Rightarrow\; F(x)\in K \quad (x\in\overline{\Omega}) \tag{6.2}$$

and if either $K=\emptyset$ or $K\in\mathcal{A}_0$.

Definition 6.2. We call a hull candidate $K$ *proper* if either $K=\emptyset$ or if there is a continuous function $R\colon[0,1]\times Y\to Y$ with the following properties:

(a) $R(0,\,\cdot\,)=\mathrm{id}$, and $R(1,Y)\subseteq K$.
(b) $R(t,\,\cdot\,)=\mathrm{id}$ on $K\cap q(A_{\overline{\Omega}}(F,p,K))$ for each $t\in[0,1]$.
(c) $R(\lambda,y)\in c(\{y\}\cup K)$ for each $(\lambda,y)\in[0,1]\times F(\overline{\Omega})$.

Proposition 6.3. *Each proper hull candidate $K$ for $(F,p,q,\Omega)$ is a strong deformation candidate and satisfies even*

$$\operatorname{Fix}_{\overline{\Omega}}(F,p,q)\subseteq K.$$

Proof. Let $K$ be a nonempty hull candidate and $R$ be the corresponding function. On each open $\Omega_0\subseteq\Omega$, the function has all properties of Definition 5.2 with $K_1:=K$ and $K_2:=Y$. Only the last of these properties requires a proof. Thus, let $\lambda\in[0,1]$ and $y\in\operatorname{Fix}_{\overline{\Omega}}(F,p,R(\lambda,q(\,\cdot\,)))$. This means that $y=F(x)\in R(\lambda,q(p^{-1}(x)))$ for some $x\in\overline{\Omega}$. Consequently, $y=F(x)\in c(q(p^{-1}(x))\cup K)$, and so (6.2) implies $y\in K$, as required.

Similarly, $y=F(x)\in\operatorname{Fix}_{\overline{\Omega}}(F,p,q)$ implies

$$F(x)\in q(p^{-1}(x))\subseteq c(q(p^{-1}(x))\cup K),$$

and so $y\in K$ by (6.2). $\square$

Proposition 6.4. *Let $Y$ be a convex subset of a topological vector space, and let $c :=$ conv. Let $K \subseteq Y$ be convex and a retract of $Y$ or empty. If $K$ is a hull candidate for $(F, p, q, \Omega) \in \mathcal{T}$, then $K$ is automatically a proper hull candidate and thus a strong deformation candidate.*

Proof. Let $r: Y \to K$ be the retraction. The required map is then given by $R(\lambda, y) := (1 - \lambda)y + \lambda r(y)$. □

Assume now that $\mathcal{A}_0 \cup \{\emptyset\}$ is closed under intersections and that $Y \in \mathcal{A}_0$. In this case, we can define a function $c_{\mathcal{A}_0}: 2^Y \to 2^Y$ by

$$c_{\mathcal{A}_0}(M) = \bigcap \{K \in \mathcal{A}_0 : M \subseteq K\}.$$

If there is some relation between $c_{\mathcal{A}_0}$ (i.e. the set $\mathcal{A}_0$) and the function $c$, one may expect that one can consider intersections of hull candidates to obtain a minimal hull candidate. The following result is the main observation in this connection: The relation between $c$ and $c_{\mathcal{A}_0}$ is expressed by assumption (6.5). In connection with homotopies it is important to find sets which are simultaneously hull candidates for a whole class of sets $(F_i, p_i, q_i, \Omega_i) \in \mathcal{T}$ (with $i$ from some index set $I$). For most applications, the function $G$ in the following result will usually be just

$$G(K) = \bigcup_{i \in I} q_i(A_{\overline{\Omega}_i}(F_i, p_i, K)).$$

However, in some cases, one might want to find "small" hull candidates $K$ which contain e.g. even the closure of the above set (or a slightly smaller set) – therefore, we consider a more general class of functions $G$. Moreover, in many applications of degree theory it is desirable to find fundamental sets which contain a certain given set $V \subseteq Y$. We thus formulate rather general:

Proposition 6.5. *Let $\mathcal{A}_0 \cup \{\emptyset\}$ be closed under intersections, let $Y \in \mathcal{A}_0$, and suppose that the function $c: 2^Y \to 2^Y$ satisfies (6.1). Let $(F_i, p_i, q_i, \Omega_i) \in \mathcal{T}$ $(i \in I)$, and let $G: \mathcal{A}_0 \cup \{\emptyset\} \to 2^Y$ satisfy*

$$G(K) \supseteq q_i(p_i^{-1}(F_i^{-1}(K) \cap \overline{\Omega}_i)) \quad (K \in \mathcal{A}_0,\ i \in I). \tag{6.3}$$

*Assume also that $G$ is monotone, i.e.*

$$K_1 \subseteq K_2 \ \Rightarrow\ G(K_1) \subseteq G(K_1) \quad (K_1, K_2 \in \mathcal{A}_0 \cup \{\emptyset\}). \tag{6.4}$$

*Then, for each $V \subseteq Y$, there exists a smallest set $K_0 \subseteq Y$ which contains $V$, is simultaneously a hull candidate for each $(F_i, p_i, q_i, \Omega_i)$ $(i \in I)$, and satisfies $G(K_0) \subseteq K_0$. Moreover, if*

$$c(K) = K \quad (K \in \mathcal{A}_0 \cup \{\emptyset\}) \tag{6.5}$$

*then this smallest set $K_0$ satisfies automatically*

$$K_0 = c_{\mathcal{A}_0}(G(K_0) \cup V). \tag{6.6}$$

Proof. Let $\mathcal{K}$ be the family of all sets $K$ with the above properies. Then $Y \in \mathcal{K}$, and so $K_0 := \bigcap \mathcal{K}$ exists. Then $V \cup G(K) \subseteq K$. Moreover, if $F_i(x) \in c(q_i(p_i^{-1}(x)) \cup K_0)$, then we have for each $K \in \mathcal{K}$ that $F_i(x) \in c(q_i(p_i^{-1}(x)) \cup K)$ and so, since $K$ is a hull candidate for $(F_i, p_i, q_i, \Omega_i)$, also $F_i(x) \in K$. Since either $K_0 = \emptyset$ or $K_0 \in \mathcal{A}_0$, it follows that $K_0$ is a hull candidate for $(F_i, p_i, q_i, \Omega_i)$. This proves $K_0 \in \mathcal{K}$, i.e. there exists indeed a smallest set $K_0 \in \mathcal{K}$.

Clearly, $K_1 := c_{\mathcal{A}_0}(G(K_0) \cup V) \subseteq c_{\mathcal{A}_0}(K_0) = K_0$. Hence, $G(K_1) \subseteq G(K_0) \subseteq G(K_1)$. Moreover, if $F_i(x) \in c(q_i(p_i^{-1}(x)) \cup K_1)$ then $F_i(x) \in c(q_i(p_i^{-1}(x)) \cup K_0)$, and so $F_i(x) \in K_0$. It follows that $F_i(x) \in K_0$, i.e. $x \in F_i^{-1}(K_0)$, and so

$$F_i(x) \in c(q_i(p_i^{-1}(F_i^{-1}(K_0))) \cup K_1) \subseteq c(G(K_0) \cup K_1) = c(K_1) = K_1.$$

Since either $K_1 = \emptyset$ or $K_1 \in \mathcal{A}_0$, we have shown that the set $K_1$ is a hull candidate for $(F_i, p_i, q_i, \Omega_i)$. Hence, $K_1 \in \mathcal{K}$ which implies $K_1 \supseteq K$, and so $K = K_1$. $\square$

The crucial observation is now that in many cases the relation (6.6) alone implies the compactness of the restriction $q_i|_{A_{\overline{\Omega}_i}(F_i, q_i, K_0)}$ and thus (under natural additional assumptions) that $K_0$ is fundamental. We discuss this now.

## 7. The convex case

We are going to apply the previous results for the case that $Y$ is a convex subset of a locally convex space. The most important special case reads as follows. (Note that without additional assumptions, we need the general axiom of choice in the form of Dugundji's extension theorem for the following proof).

We consider the following situation: Let $Y$ be a metrizable convex subset of a locally convex space, and let $\mathcal{A}_0$ be the family of all closed (in $Y$) convex subsets of $Y$. Assume that $\mathcal{G} = \mathcal{G}_0$ is the class of all spaces $\Gamma$ with the property that $[0,1] \times \Gamma$ is a $T_4$ space.

Theorem 7.1. *Consider the above situation. Let $(F_i, p_i, q_i, \Omega_i) \in \mathcal{T}$ $(i \in I)$ and let $G\colon \mathcal{A}_0 \cup \{\emptyset\} \to 2^Y$ satisfy* (6.4) *and*

$$G(K) \supseteq q_i(A_{\overline{\Omega}_i}(F_i, p_i, K)) \quad (K \in \mathcal{A}_0).$$

*Assume that there is some $V \subseteq Y$ such that for each $K \in \mathcal{A}_0$ the relation*

$$K = Y \cap \overline{\operatorname{conv}}(G(K) \cup V) \tag{7.1}$$

*implies that the restriction $q_i|_{A_{\overline{\Omega}_i}(F_i, p_i, K)}$ is continuous and compact and satisfies*

$$\operatorname{Coin}_{F_i^{-1}(K) \cap \partial\Omega_i}(F_i, p_i, q_i) = \emptyset \quad (i \in I).$$

*Then there is some* $K \subseteq Y$ *satisfying* (7.1) *which is strictly fundamental for each* $(F_i, p_i, q_i, \Omega_i)$ *for* $i \in I$.

Proof. Let $c(M) := \operatorname{conv}(M)$ (or $c(M) := Y \cap \overline{\operatorname{conv}}(M)$), and let $K := K_0$ be the set of Proposition 6.5. By Proposition 6.4, $K$ is a strong deformation candidate, and by Proposition 5.6, $K$ is union-admissible. Theorem 5.8 thus implies that $K$ is strictly fundamental for $(F_i, p_i, q_i, \Omega_i)$. □

Concerning the degree of Section 4, we obtain similarly:

Theorem 7.2. *Consider the situation described in front of Theorem* 7.1. *Let* $(F_i, p_i, q_i, \Omega) \in \mathcal{T}$ $(i \in I)$, *and assume that*

$$C := \overline{\bigcup_{i \in I} \operatorname{Coin}_\Omega(F_i, p_i, q_i)}$$

*is* $\mathcal{O}$*-normal and contained in* $\Omega$. *Let* $\Omega_0 \in \mathcal{O}$ *be given with* $C \subseteq \Omega_0 \subseteq \Omega$. *Let* $G: \mathcal{A}_0 \cup \{\emptyset\} \to 2^Y$ *satisfy* (6.4) *and*

$$G(K) \supseteq q_i(A_\Omega(F_i, p_i, K)) \quad (K \in \mathcal{A}_0).$$

*Assume that there is some* $V \subseteq Y$ *such that for each* $K \in \mathcal{A}_0$ *with* (7.1) *the restriction* $q_i|_{A_{\Omega_0}(F_i,p_i,K)}$ *is continuous and compact. Then there is some* $K \subseteq Y$ *with* (7.1) *which is locally strictly fundamental for each* $(F_i, p_i, q_i, \Omega)$ $(i \in I)$. *More precisely, for each* $\Omega_0' \in \mathcal{O}$ *with* $C \subseteq \Omega_0'$ *and* $\overline{\Omega}_0' \subseteq \Omega_0$ *there is some* $K$ *with* (7.1) *which is strictly fundamental for each* $(F_i, p_i, q_i, \Omega_0')$ $(i \in I)$.

Note that in most applications of Theorem 7.1, resp. Theorem 7.2, the functions $q_i$ are continuous. In this case the only essential assumption is that the relation (7.1) for some $K \in \mathcal{A}_0$ implies that $q_i(A_{\overline{\Omega}_i}(F_i, p_i, K))$ is contained in a compact subset of $Y$. Note that in view of (6.3) this set is always contained in $K$. In particular, the essential assumption is satisfied if (7.1) implies that $K$ has a compact closure in $Y$. This condition is satisfied, in particular, if $Y$ is a closed convex subset of a Banach or Fréchet space, $\overline{V}$ is compact, and $G$ has a bounded range and is condensing with respect to the Hausdorff or Kuratowski measure of noncompactness.

However, in many applications, in particular for integral operators of vector functions, one can obtain sharp estimates for measures of noncompactness only on countable sets, see e.g. [6], [42], [29], [39], [33], [59]. We thus aim to replace (7.1) by a similar equality for countable sets. Such results are in the attitude of [13], [14], [32], [41], [54], [57], [58]. Two essentially different approaches in this direction are possible, depending on whether we are considering countable subsets of $X$ or of $Y$. Let us first consider the former case. For this case, we obtain the following results corresponding to Theorems 7.1 and 7.2.

We consider the following situation: Let $Y$ be a closed convex metrizable subset of a locally convex space, and let $\mathcal{A}_0$ be the family of all closed convex subsets of $Y$. Assume that $\mathcal{G} = \mathcal{G}_0$ is the class of all spaces $\Gamma$ with the property that $[0,1]\times\Gamma$ is a $T_4$ space. Let $(H,P,Q,\Omega)\in\mathcal{H}'$ and assume that $\overline{\Omega}$ is a metric space.

The result concerning the local situation of Section 4 (i.e. corresponding to Theorem 7.2) is simpler: Essentially, we are able to replace (7.1) by (7.3) for countable subsets $C_0 \subseteq X$.

Theorem 7.3. *Suppose in the above situation that*

$$C := \overline{\operatorname{Coin}_\Omega(H,P,Q)} = \overline{\bigcup_{t\in[0,1]} \operatorname{Coin}_\Omega(H(t,\,\cdot\,),P_t,Q_t)}$$

*is contained in* $\Omega$ *and* $\mathcal{O}$*-normal. Let* $\Omega_0\in\mathcal{O}$ *satisfy* $C\subseteq\Omega_0\subseteq\Omega$ *and be such that* $Q$ *is continuous on* $P^{-1}([0,1]\times\Omega_0)$ *and that* $Q(P^{-1}(t,x))$ *is separable for each* $(t,x)\in[0,1]\times\Omega_0$. *Assume also that* $V\subseteq Y$ *is separable,* $V\supseteq H([0,1]\times C)\setminus\overline{\operatorname{conv}}(Q(P^{-1}([0,1]\times C)))$,

$$\overline{\operatorname{conv}}(Q(P^{-1}([0,1]\times\Omega_0))\cup V)\setminus H([0,1]\times\Omega_0) \text{ is closed,} \tag{7.2}$$

*and that* $P|_{P^{-1}([0,1]\times\Omega_0)}$ *is a closed map. Suppose that for each countable* $C_0\subseteq[0,1]\times\Omega_0$ *with*

$$\overline{H(C_0)} = \overline{H([0,1]\times\Omega_0)\cap\operatorname{conv}(Q(\overline{P^{-1}(C_0)})\cup V)} \tag{7.3}$$

*the set* $\overline{Q(P^{-1}(C_0))}$ *is compact. Then* $(H,P,Q,\Omega)\in\mathcal{H}_2'\subseteq\mathcal{H}_2$ *and the set* $K$ *of Definition* 4.6 *can be chosen such that it contains* $V$.

Note that usually (e.g. if $\operatorname{Coin}_\Omega(H,P,Q)$ is closed)

$$H([0,1]\times C)\setminus\overline{\operatorname{conv}}(Q(P^{-1}([0,1]\times C)))=\emptyset,$$

and in this case the choice $V=\emptyset$ is admissible.

Remark 7.4. Actually, we find in Theorem 7.3 for each $\Omega_0'\in\mathcal{O}$ with $C\subseteq\Omega_0'$ and $\overline{\Omega_0'}\subseteq\Omega_0$ some $K\supseteq V$ which is strictly fundamental for $(H(t,\,\cdot\,),P_t,Q_t,\Omega_0')$ $(0\le t\le 1)$ and such that the restriction $Q|_{A_{\Omega_0}(H,P,K)}$ is compact and assumes its values in $K$.

Since $\Omega_0$ is open, the assumption (7.2) is rather natural: In fact, for many classes $\mathcal{F}$ which provide a triple-degree it is the case that $F$ is an open map for each $(F,p,\Omega)\in\mathcal{F}$, and then of course $H([0,1]\times\Omega_0)$ is open for each $(H,P,\Omega)\in\mathcal{F}$. However, the requirement (7.2) is actually not essential for Theorem 7.3: It merely simplifies the equality (7.3) slightly. Without this requirement, we have to be more careful with the closures in (7.3) and have to consider the two inclusions separately:

THEOREM 7.5. *An analogous result to Theorem 7.3 (and Remark 7.4) holds without the assumption (7.2) provided that we replace (7.3) by the pair of inclusions:*

$$\overline{H(C_0)} \subseteq \overline{H([0,1]\times\Omega_0)\cap\overline{\mathrm{conv}}(Q(P^{-1}(C_0))\cup V)}, \tag{7.4}$$

$$\overline{H(C_0)} \supseteq H([0,1]\times\Omega_0)\cap\mathrm{conv}(Q(\overline{P^{-1}(C_0)})\cup V). \tag{7.5}$$

Theorem 7.5 has the following analogue to the global situation of Section 3 (i.e. corresponding to Theorem 7.1).

THEOREM 7.6. *Consider the situation described in front of Theorem 7.3, and assume that $Q$ is continuous and $\mathrm{Coin}_{\partial\Omega}(H,P,Q)=\emptyset$. Let $Q(P^{-1}(t,x))$ be separable for each $(t,x)\in[0,1]\times\overline{\Omega}$. Assume also that $V\subseteq Y$ is separable. Suppose that for each countable $C_0\subseteq[0,1]\times\overline{\Omega}$ with*

$$\overline{H(C_0)} \subseteq \overline{H([0,1]\times\overline{\Omega})\cap\overline{\mathrm{conv}}(Q(P^{-1}(C_0))\cup V)}, \tag{7.6}$$

$$\overline{H(C_0)} \supseteq H([0,1]\times\overline{\Omega})\cap\mathrm{conv}(Q(\overline{P^{-1}(C_0)})\cup V), \tag{7.7}$$

*the set $\overline{Q(P^{-1}(C_0))}$ is compact. Then $(H,P,Q,\Omega)\in\mathcal{H}_1''\subseteq\mathcal{H}_1'\subseteq\mathcal{H}_1$, and the set $K$ of Definition 3.9 can be chosen such that it contains $V$.*

The proof of the above theorems follows from the following result which is proved completely analogously to [58, Proposition 4.1 and Corollary 4.2]:

LEMMA 7.7. *Let $D$ be a metric space, and $Y$ be a closed convex metrizable subset of a locally convex space $Z$. Let $H\colon D\to Z$ and $\Phi\colon D\to 2^Z$ be such that $\Phi(x)$ is separable for each $x\in D$. Let $V\subseteq Z$ be separable, and $M\subseteq D$ satisfy*

$$H^{-1}(\overline{\mathrm{conv}}(\Phi(M)\cup V)) = M. \tag{7.8}$$

*Then for each countable $C_1\subseteq M$ there is some countable $C_0\supseteq C_1$ with $C_0\subseteq M$ such that*

$$H(D)\cap\mathrm{conv}(\Phi(C_0)\cup V)\subseteq\overline{H(C_0)}\subseteq\overline{H(D)\cap\overline{\mathrm{conv}}(\Phi(C_0)\cup V)}. \tag{7.9}$$

*In particular, if for each countable $C_0\subseteq M$ with (7.9) the set $\overline{\Phi(C_0)}$ is compact, then $\overline{\Phi(M)}$ is compact. If $\overline{\mathrm{conv}}(\Phi(M)\cup V)\setminus H(D)$ is closed, then (7.9) can be equivalently rewritten as*

$$\overline{H(C_0)} = \overline{\mathrm{conv}(\Phi(C_0)\cup V)\cap H(D)}. \tag{7.10}$$

PROOF OF THEOREM 7.6. Apply Theorem 7.1 with

$$(F_t,p_t,q_t,\Omega_t) := (H(t,\cdot),P_t,Q_t,\Omega) \quad\text{and}\quad G(K) := Q(A_{\overline{\Omega}}(H,P,K)).$$

We are done, if we can show that the relation $K = \overline{\operatorname{conv}}(G(K) \cup V)$ implies that $\overline{G(K)}$ is compact. To see this, we apply Lemma 7.7 with $D := [0,1] \times \overline{\Omega}$, $\Phi(W) := Q(\overline{P^{-1}(W)})$, and $M := H^{-1}(K)$. Since $Q$ is continuous, we have

$$\Phi(W) \subseteq \overline{Q(P^{-1}(W))} \quad (W \subseteq D). \tag{7.11}$$

In particular, our assumptions imply that $\Phi(t,x)$ is separable. Moreover,

$$\overline{P^{-1}(M)} = A_{\overline{\Omega}}(H,P,K),$$

and so

$$\Phi(M) = Q(A_{\overline{\Omega}}(H,P,K)) = G(K),$$

in particular

$$\overline{\operatorname{conv}}(\Phi(M) \cup V) = \overline{\operatorname{conv}}(G(K) \cup V) = K.$$

Hence, (7.8) holds. In view of (7.11), the inclusions (7.6) and (7.7) are just reformulations of (7.9). Lemma 7.7 thus implies that $\overline{\Phi(M)} = \overline{G(K)}$ is compact, as required. □

Proof of Theorems 7.3 and 7.5. The proofs are similar to the above proof of Theorem 7.6: Apply Theorem 7.2 and put $D := [0,1] \times \Omega_0$ and $G(K) := Q(A_{\Omega_0}(H,P,K))$ in the above arguments; for Theorem 7.3 note that the set $\overline{\operatorname{conv}}(\Phi(M) \cup V) \setminus H(D)$ is closed if (7.2) holds and that (7.3) (resp. (7.6) and (7.7)) is a reformulation of (7.10) (resp. of (7.9)). □

If we want conditions in terms of countable subsets of $Y$ (not of $X$), we have the following two results (corresponding to the local, resp. global situation of Section 3, resp. 4). Note that if $Y_n$ are "sufficiently large", the inclusions (7.12) and (7.13) together come pretty close to the equality

$$C_0 = \overline{\operatorname{conv}}(Q(P^{-1}(H^{-1}(C_0) \cap ([0,1] \times \overline{\Omega}))) \cup V)$$

which is a "countable" version of (7.1).

Theorem 7.8. *Consider the situation described in front of Theorem 7.3, and assume that $Q$ is continuous and that $\operatorname{Coin}_{\partial\Omega}(F,P,Q) = \emptyset$. Let $V \subseteq Y$ and let $Y_n \subseteq Y$ $(n = 1,2,\dots)$ be such that $Y_n \cap V$ and $Y_n \cap Q(P^{-1}(\overline{H^{-1}(y)} \cap ([0,1] \times \overline{\Omega})))$ are separable for each $y \in Y$ and each $n$. Suppose that for each countable $C_0 \supseteq Y$ with*

$$C_0 \subseteq \overline{\operatorname{conv}}(Q(P^{-1}(H^{-1}(C_0) \cap ([0,1] \times \overline{\Omega}))) \cup V), \tag{7.12}$$

$$\overline{Y_n \cap C_0} \supseteq Y_n \cap \operatorname{conv}(Q(\overline{P^{-1}(H^{-1}(C_0)) \cap ([0,1] \times \overline{\Omega})}) \cup V), \tag{7.13}$$

*for $n = 1,2,\dots,$ the set $\overline{Q(P^{-1}(H^{-1}(C_0) \cap ([0,1] \times \overline{\Omega})))}$ is compact. Then $(H,P,Q,\Omega) \in \mathcal{H}_1'' \subseteq \mathcal{H}_1' \subseteq \mathcal{H}_1$, and the set $K$ of Definition 3.9 can be chosen such that it contains $V$.*

Theorem 7.9. *Consider the situation described in front of Theorem* 7.3. *Suppose that*

$$C := \overline{\mathrm{Coin}_\Omega(H,P,Q)} = \overline{\bigcup_{t\in[0,1]} \mathrm{Coin}_\Omega(H(t,\,\cdot\,),P_t,Q_t)} \tag{7.14}$$

*is contained in* $\Omega$ *and* $\mathcal{O}$*-normal. Let* $\Omega_0 \in \mathcal{O}$ *satisfy* $C \subseteq \Omega_0 \subseteq \Omega$ *and be such that* $Q$ *is continuous on* $P^{-1}([0,1]\times\Omega_0)$ *and that* $P|_{P^{-1}([0,1]\times\Omega_0)}$ *is a closed map. Let* $V \subseteq Y$, $V \supseteq H(C)\setminus\overline{\mathrm{conv}}(Q(P^{-1}(C)))$, *and let* $Y_n \subseteq Y$ $(n=1,2,\dots)$ *be such that* $Y_n\cap V$ *and* $Y_n \cap Q(P^{-1}(\overline{H^{-1}(y)}\cap([0,1]\times\Omega_0)))$ *are separable for each* $y\in Y$ *and each* $n$. *Suppose that for each countable* $C_0 \supseteq Y$ *with*

$$C_0 \subseteq \overline{\mathrm{conv}}(Q(P^{-1}(H^{-1}(C_0)\cap([0,1]\times\Omega_0)))\cup V), \tag{7.15}$$

$$\overline{Y_n\cap C_0} \supseteq Y_n \cap \mathrm{conv}(Q(\overline{P^{-1}(H^{-1}(C_0))\cap([0,1]\times\Omega_0)})\cup V), \tag{7.16}$$

*for* $n=1,2,\dots$, *the set* $\overline{Q(P^{-1}(H^{-1}(C_0)\cap([0,1]\times\Omega_0)))}$ *is compact. Then* $(H,P,Q,\Omega)\in\mathcal{H}_2'\subseteq\mathcal{H}_2$ *and the set* $K$ *of Definition* 4.6 *can be chosen such that it contains* $V$. *Actually even the property of Remark* 7.4 *holds.*

Theorems 7.8 and 7.9 follow from the following Lemma 7.10 which can be proved analogously to the proof of [58, Proposition 4.3].

In this lemma, we use for $H\colon D\to 2^Y$ and $M\subseteq Y$ the notation

$$H^{-1}(M) := \{x\in D : H(x)\subseteq M\}.$$

Lemma 7.10. *Let* $D$ *be a metric space, and* $Y$ *be a closed convex metrizable subset of a locally convex space. Let* $H,\Phi\colon D\to 2^Y$ *and* $K,V\subseteq Y$ *satisfy*

$$K = \overline{\mathrm{conv}}(\Phi(H^{-1}(K))\cup V). \tag{7.17}$$

*Let* $Y_n\subseteq Y$ $(n=1,2,\dots)$ *be such that* $Y_n\cup V$ *and* $Y_n\cap\Phi(\overline{H^{-1}(y)})$ *are separable for each* $y\in K$ *and each* $n$. *Then for each countable* $C_1\subseteq K$ *there is some countable* $C_0\supseteq C_1$ *with* $C_0\subseteq K$ *such that*

$$C_0 \subseteq \overline{\mathrm{conv}}(\Phi(H^{-1}(C_0))\cup V), \tag{7.18}$$

$$\overline{Y_n\cap C_0} \supseteq Y_n\cap\mathrm{conv}(\Phi(H^{-1}(C_0))\cup V) \quad (n=1,2,\dots). \tag{7.19}$$

*In particular, if for each countable* $C_0\subseteq K$ *with* (7.18) *and* (7.19) *the set* $\overline{\Phi(H^{-1}(C_0))}$ *is compact, then* $\overline{\Phi(H^{-1}(K))}$ *is compact.*

Proof of Theorem 7.8. Using the notation of the proof of Theorem 7.6, in particular, $G(K) := Q(A_{\overline{\Omega}}(H,P,K))$, we have to show that the relation $K = \overline{\mathrm{conv}}(G(K)\cup V)$ implies that $\overline{G(K)}$ is compact. To see this, we apply Lemma 7.10 with $D := [0,1]\times\overline{\Omega}$ and$\Phi(W) := Q(\overline{P^{-1}(W)})$. Since $\Phi(H^{-1}(K)) = G(K)$, assumption (7.17) is just a reformulation of (7.1). Moreover, in view of (7.11), the inclusions (7.12) and (7.13) are reformulations of (7.18) and (7.19), respectively. Lemma 7.7 thus implies that $\overline{\Phi(H^{-1}(K))} = \overline{G(K)}$ is compact, as required. □

Proof of Theorem 7.9. The proof is similar to the above proof of Theorem 7.8: Apply Theorem 7.2 and put

$$D := [0,1] \times \Omega_0 \quad \text{and} \quad G(K) := Q(A_{\Omega_0}(H,P,K))$$

in the above arguments, noting that (7.15) and (7.16) are reformulations of (7.18) and (7.19), respectively. □

## 8. Applications for countably condensing maps

Recall that the *Hausdorff measure of noncompactness* of a set $M$ in a pseudometric space is defined as the infimum of all $\varepsilon > 0$ such that $M$ has a finite $\varepsilon$-net $N$ in the space, i.e. $\operatorname{dist}(x,N) \le \varepsilon$ for each $x \in M$. Similarly, the *Kuratowski measure of noncompactness* is the infimum of all $\varepsilon > 0$ such that $M$ can be covered by finitely many sets of diameter at most $\varepsilon$. A natural generalization of these measures of noncompactness in $Y$ is the following definition in the sense of [1], [51]:

Definition 8.1. Let $U$ be a partially ordered set. We call $\gamma: 2^Y \to U$ a *monotone $V$-measure of noncompactness* (on $Y$) if

(a) $\gamma(\overline{\operatorname{conv}} M) = \gamma(M)$ for each $M \subseteq Y$.
(b) $\gamma(M_1) \le \gamma(M_2)$ for each $M_1 \subseteq M_2 \subseteq Y$.
(c) $\gamma(M \cup V) = \gamma(M)$ for each $M \subseteq Y$.

In case $V = \emptyset$, we simply speak of a *monotone measure of noncompactness*.

Example 8.2. Let $Y$ be a convex subset of a locally convex metrizable space $Z$, and let $(\|\cdot\|_n)_{n\in\mathbb{N}}$ be a corresponding countable family of continuous seminorms in $Z$ generating the topology. Let $U := [0,\infty]^{\mathbb{N}}$, and put $\gamma(M) := (\gamma_n(M))_n$ where $\gamma_n$ denotes either the Kuratowski measure of noncompactness with respect to $\|\cdot\|_n$ or the Hausdorff measure of noncompactness in either the space $Y$ or $Z$ with respect to $\|\cdot\|_n$. Then $\gamma$ is a monotone $V$-measure of noncompactness for each precompact set $V$.

Definition 8.3. Let $A$ be some set, $\Phi: A \to 2^Y$ and $F: A \to Y$. We call $\Phi$ *$V$-countably $F$-condensing on $A$* if for each countable $C_0 \subseteq A$ for which $\overline{\Phi(C_0)}$ is not compact there is some partially ordered set $U$ and a monotone $V$-measure of noncompactness $\gamma: 2^Y \to U$ such that

$$\gamma(\Phi(C_0)) \not\ge \gamma(F(C_0)).$$

If $A \subseteq Y$, we call $G: A \to 2^Y$ *$V$-countably condensing on $A$* if $G$ is $V$-countably $i$-condensing on $A$ where $i: A \to Y$ is the inclusion map.

Example 8.4. Consider the situation of Example 8.2. Let $F: A \to Y$ and $\Phi: A \to 2^Y$ be such that for each $n \in \mathbb{N}$ there is a function $\widetilde{\gamma}_n: 2^A \to [0,\infty]$ with

$$\gamma_n(F(C_0)) \geq L_n\widetilde{\gamma}_n(C_0) \quad (C_0 \subseteq A \text{ countable}), \tag{8.1}$$

$$\gamma_n(\Phi(C_0)) \leq \ell_n\widetilde{\gamma}_n(C_0) \quad (C_0 \subseteq A \text{ countable}), \tag{8.2}$$

where $0 \leq \ell_n < L_n \leq \infty$. Assume that either one of the sets $\Phi(A)$ or $F(A)$ is bounded (i.e. bounded with respect to each of the seminorms $\|\cdot\|_n$) or that $\widetilde{\gamma}_n(C_0) < \infty$ ($n \in \mathbb{N}$) for each countable $C_0 \subseteq A$. If $Y$ is complete (or if the relation $\widetilde{\gamma}_n(C_0) = 0$ ($n \in \mathbb{N}$) for some countable $C_0 \subseteq A$ implies that $\overline{\Phi(C_0)}$ is compact), then $\Phi$ is $V$-countably $F$-condensing on $A$ for each $V \subseteq Y$ for which $\overline{V}$ is a compact subset of $Y$.

In particular, let $A \subseteq Y$ and $G: A \to 2^Y$ satisfy for each $n \in \mathbb{N}$

$$\gamma_n(G(C_0)) \leq q_n\gamma_n(C_0) \quad (C_0 \subseteq A \text{ countable}) \tag{8.3}$$

with some $q_n \in [0,1)$. Let either $A$ or $G(A)$ be bounded. If $Y$ is complete (or $A$ is complete and $G$ sends compact sets into compact sets) then $G$ is $V$-countably condensing on $A$ for each $V \subseteq Y$ with compact $\overline{V} \subseteq Y$.

Note that none of the conditions (8.1), (8.2) or (8.3) changes under compact perturbations of the maps. In particular, (8.3) holds if $G$ is a single-valued compact perturbation of a contraction (with respect to each $\|\cdot\|_n$), and similarly (8.2) holds if $\Phi$ is a compact perturbation of Lipschitz map (and $\widetilde{\gamma}_n$ denotes a corresponding measure of noncompactness). But also for many multivalued maps good estimates of the type (8.2) are known, see e.g. [33]. Roughly speaking, the conditions (8.1) and (8.2) mean that "$\Phi$ is more compact than $F$ is proper".

Proposition 8.5. *The compactness assumption of Theorem* 7.5 (*resp. Theorem* 7.6) *is satisfied if* $\Phi := Q \circ P^{-1}$ *is* $V$*-countably* $H$*-condensing on* $[0,1] \times \Omega_0$ (*resp. on* $[0,1] \times \overline{\Omega}$).

Proof. The inclusion (7.4), resp. (7.6), implies that

$$H(C_0) \subseteq \overline{\text{conv}}(\Phi(C_0) \cup V).$$

We thus have for each monotone $V$-measure $\gamma$ of noncompactness

$$\gamma(H(C_0)) \leq \gamma(\overline{\text{conv}}(\Phi(C_0) \cup V)) = \gamma(\Phi(C_0) \cup V) = \gamma(\Phi(C_0)).$$

Hence, $\overline{\Phi(C_0)}$ is compact. □

Proposition 8.6. *The compactness assumption of Theorem* 7.8*, resp. Theorem* 7.9*, is satisfied if*

$$G(y) := Q(P^{-1}(H^{-1}(y) \cap ([0,1] \times \overline{\Omega}))),$$

*respectively*

$$G(y) := Q(P^{-1}(H^{-1}(y) \cap ([0,1] \times \Omega_0))),$$

*is $V$-countably condensing on $Y$.*

Proof. For each monotone $V$-measure $\gamma$ of noncompactness the relation (7.12), resp. (7.15), implies $\gamma(C_0) \le \gamma(\overline{\mathrm{conv}}(G(C_0) \cup V)) = \gamma(G(C_0))$. □

## 9. An application

As pointed out in the introduction, the theory presented in this paper is new even in the case $p = \mathrm{id}$. As new application of this case, we prove a continuation theorem.

Let $X$ be a real reflexive separable Banach space with dual space $X^*$. Let $\Omega \subseteq X$ be open and bounded.

Definition 9.1. We say that $F: \overline{\Omega} \to X^*$ is a *Skrypnik map*, if the following holds:

(a) $F(\overline{\Omega})$ is bounded.
(b) $F$ is demicontinuous, i.e. $\overline{\Omega} \ni x_n \to x$ implies $F(x_n) \rightharpoonup F(x)$.
(c) The relations $\overline{\Omega} \ni x_n \rightharpoonup x$ and $\limsup_{n\to\infty} \langle F(x_n), x_n - x\rangle \le 0$ imply that there is a subsequence $x_{n_k} \to x$.

Let $\mathcal{T}$ be the class of all $(F, \mathrm{id}, \Omega)$ as above. The Skrypnik degree theory [53] implies that $\mathcal{T}$ provides a compact triple-degree with values in $G = \mathbb{Z}$. Moreover, as has been proved in [56], this degree is additive and satisfies the exhaustion property.

Theorem 9.2 (Continuation Theorem for Skrypnik Maps). *Let $X$ be a real reflexive separable Banach space, $0 \in \Omega \subseteq X$ open and bounded, and $F: \overline{\Omega} \to X^*$ be a Skrypnik map. Let $h: [0,1] \times X \to X^*$ be continuous and such that the following holds:*

(a) *$q = h(0, \cdot\,)$ is compact and has one of the following properties:*
  (a1) *$\langle F(x) - q(x), x\rangle \ge 0$ for all $x \in \Omega$.*
  (a2) *There is some open ball $B_r \subseteq \Omega$ around $0$ which contains $\{x \in \overline{\Omega} : F(x) = q(x)\}$ and such that $F - q$ is odd on $B_r$.*
(b) *There are no $(t, x) \in [0,1] \times \partial\Omega$ with $F(x) = h(t, x)$.*
(c) *For each countable $C \subseteq \Omega$ for which $\overline{H([0,1] \times C)}$ is not compact, there is a monotone measure $\gamma$ of noncompactness with $\gamma(H([0,1] \times C)) \not\ge \gamma(F(C))$.*

*Then $F(x) = H(1, x)$ has a solution $x \in \Omega$.*

Proof. Put $H(t, x) := F(x)$ and note that

$$D := \{(t, x) \in [0,1] \times \overline{\Omega} : H(t, x) = h(t, x)\}$$

is closed. Indeed, if $(t_n, x_n) \in D$ converges to $(t, x)$, then $H(t_n, x_n) = h(t_n, x_n) \to h(t, x)$, and the demicontinuity of $H$ implies $(t, x) \in D$. Since $[0, 1]$ is compact, the projection of $[0, 1] \times \overline{\Omega}$ to the second component is a closed map, see e.g. [12, Proposition I.8.2]. In particular,

$$C := \bigcup_{t \in [0,1]} \{x \in \overline{\Omega} : H(t, x) = h(t, x)\}$$

is closed. By hypothesis, $C \subseteq \Omega$. Apply now Theorem 7.5 with $\Omega_0 := \Omega$, $P := \mathrm{id}$, $Q := h$, and $V := \emptyset$. In view of Proposition 8.5, we have $(H, P, Q, \Omega) \in \mathcal{H}_2' \subseteq \mathcal{H}_2$. By the homotopy invariance and normalization of the degree of Theorem 4.7, we thus have

$$\mathrm{DEG}(F, \mathrm{id}, h(1, \cdot\,), \Omega) = \mathrm{Deg}(F, \mathrm{id}, q, \Omega).$$

The right-hand side is the Skrypnik degree of the map $F - q$ on $\Omega$ (see e.g. [56]). Our hypotheses on $q$ imply that this degree is 1, resp. odd, see Theorem 1.3.4 and Theorem 1.3.5 in [53]. In particular, $F(x) = H(1, x)$ has a solution by the existence property of Deg. □

Corollary 9.3 (Leray–Schauder Principle for Monotone Maps). *Let $X$ be a real reflexive separable Banach space, $0 \in \Omega \subseteq X$ open and bounded, and $F: \overline{\Omega} \to X^*$ be demicontinuous with bounded range and such that there is a nondecreasing function $r: [0, \infty) \to [0, \infty)$ with*

$$\langle F(x) - F(y), x - y\rangle \geq r(\|x - y\|) \quad (x, y \in \Omega).$$

*Suppose also that $F(0) = 0$. Let $G: \overline{\Omega} \to X^*$ be continuous with the following properties:*

(a) *The Leray–Schauder boundary condition holds:*

$$\lambda F(x) \neq G(x) \quad (x \in \partial\Omega,\ \lambda \geq 1).$$

(b) *$G$ is $\{0\}$-countably $F$-condensing on $\Omega$.*

*Then $F(x) = G(x)$ has a solution $x \in \overline{\Omega}$.*

Proof. Apply Theorem 9.2 with $H(t, x) := tG(x)$. The monotonicity assumption implies that $F$ is a Skrypnik map (see e.g. [56]) and that in view of $F(0) = 0$ the first assumption of Theorem 9.2 holds. Note that for each monotone $\{0\}$-measure $\gamma$ of noncompactness the equality

$$\gamma(H([0, 1] \times C)) = \gamma(\mathrm{conv}(\{0\} \cup G(C))) = \gamma(G(C))$$

holds. Hence, if $F(x) = G(x)$ has no solution in $\partial\Omega$, Theorem 9.2 implies that $F(x) = G(x)$ has a solution in $\Omega$. □

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