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SANDWICH PAIRS IN p-LAPLACIAN PROBLEMS

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Abstract. We solve boundary value problems for the p-Laplacian using the notion of sandwich pairs.

1. Introduction

The notion of sandwich pairs introduced by Schechter [6] is based upon the sandwich theorem for complementing subspaces by Silva [10] and Schechter [7], [8]. It is a useful tool in solving semilinear elliptic boundary value problems.

Definition 1.1. We say that a pair of subsets A, B of a Banach space W forms a *sandwich pair* if for any $\Phi \in C^1(W, \mathbb{R})$,

$$\infty < b := \inf_B \Phi \le \sup_A \Phi =: a \ < \infty \tag{1.1}$$

implies that there is a sequence $(u_j) \subset W$ such that

$$\Phi(u_j) \to c, \quad \Phi'(u_j) \to 0 \tag{1.2}$$

for some $c \in [b, a]$.

Recall that a sequence satisfying (1.2) is called a Palais–Smale sequence at the level c and Φ satisfies the compactness condition $(\mathrm{PS})_c$ if every such sequence

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has a convergent subsequence (then the limit is a critical point of Φ with the critical value c).

The sandwich pairs used in the literature so far have been formed using the eigenspaces of a semilinear operator and are therefore unsuitable for dealing with quasilinear problems where there are no eigenspaces to work with. In this paper we construct new sandwich pairs that are tailor-made for problems of the form

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 1$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian of u, $1 < p < \infty$, and f is a Carathéodory function on $\Omega \times \mathbb{R}$ with subcritical growth. Solutions of (1.3) coincide with the critical points of the C^1 functional

$$\Phi(u) = \int_\Omega |\nabla u|^p - p\,F(x,u), \tag{1.4}$$

where $F(x,t) = \int_0^t f(x,s)\,ds$, defined on the Sobolev space $W_0^{1,p}(\Omega)$.

Consider the nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

Its eigenvalues coincide with the critical values of the C^1 functional

$$I(u) = \frac{1}{\int_\Omega |u|^p} \tag{1.6}$$

on the unit sphere S in $W_0^{1,p}(\Omega)$. Let Γ_l be the class of odd continuous maps γ from the unit sphere S^{l-1} in R^l to S and set

$$\lambda_l := \inf_{\gamma \in \Gamma_l} \max_{u \in \gamma(S^{l-1})} I(u). \tag{1.7}$$

Then $0 < \lambda_1 < \lambda_2 \leq \ldots \to \infty$ are eigenvalues of (1.5) (cf. Drábek and Robinson [4]).

Setting $H(x,t) = p\,F(x,t) - tf(x,t)$, we shall prove

Theorem 1.2. *If*

$$(\lambda_l + \varepsilon)\,|t|^p - V(x) \leq pF(x,t) \leq \lambda_{l+1}|t|^p + V(x) \tag{1.8}$$

for some $l, \varepsilon > 0$, *and* $V \in L^1(\Omega)$, *and*

$$H(x,t) \leq C\,(|t|+1), \quad \overline{H}(x) := \varlimsup_{|t|\to\infty} \frac{H(x,t)}{|t|} < 0 \quad \text{a.e.} \tag{1.9}$$

then (1.3) *has a solution.*

THEOREM 1.3. *If*

$$\lambda_l|t|^p - V(x) \leq pF(x,t) \leq (\lambda_{l+1} - \varepsilon)|t|^p + V(x) \tag{1.10}$$

for some $l, \varepsilon > 0$*, and* $V \in L^1(\Omega)$*, and*

$$H(x,t) \geq -C(|t|+1), \quad \underline{H}(x) := \varliminf_{|t|\to\infty} \frac{H(x,t)}{|t|} > 0 \quad a.e. \tag{1.11}$$

then (1.3) *has a solution.*

Similar resonance problems have been studied by Perera [5] when

$$f(x,t)/|t|^{p-2}t \to \alpha_\pm(x) \in L^\infty(\Omega) \quad \text{as } t \to \pm\infty$$

and by Arcoya and Orsina [1], Bouchala and Drábek [2], and Drábek and Robinson [4] for the special case $\alpha_\pm(x) \equiv \lambda_l$.

We emphasize that the resonance is considered only with respect to the specific variational eigenvalues given by (1.7) and not with respect to other possible nonvariational eigenvalues or variational eigenvalues which are given by different methods (via Krasnosel'skiĭ genus, Lusternik–Schnirelman category, etc.) that need not coincide with those given by (1.7).

2. Sandwich pairs

First we prove a general result concerning sandwich pairs.

PROPOSITION 2.1. Γ *be the class of maps* $\gamma \in C(W \times [0,1], W)$ *such that*

(a) $\gamma_0 = \mathrm{id}$,

(b) $\sup_{(u,t)\in W\times[0,1]} \|\gamma_t(u) - u\| < \infty$,

where $\gamma_t = \gamma(\,\cdot\,,t)$*. Assume that for any* $\gamma \in \Gamma$,

$$\gamma_1(A) \cap B \neq \emptyset. \tag{2.1}$$

Then A*,* B *forms a sandwich pair.*

PROOF. Assume (1.1), and set

$$c := \inf_{\gamma\in\Gamma} \sup_{u\in\gamma_1(A)} \Phi(u). \tag{2.2}$$

Since the identity $\gamma_t(u) = u$ is in Γ, $c \leq a$. Since (2.1) holds for any $\gamma \in \Gamma$, we have $c \geq b$. We claim that there is a sequence satisfying (1.2). If not, c is not a critical value of Φ and Φ satisfies $(\mathrm{PS})_c$ since there are no Palais–Smale sequences at the level c, so there are $\varepsilon > 0$ and $\eta \in \Gamma$ such that $\eta_1(\Phi_{c+\varepsilon}) \subset \Phi_{c-\varepsilon}$ where $\Phi_{c\pm\varepsilon} = \{u \in W : \Phi(u) \leq c \pm \varepsilon\}$ (see e.g. Brezis and Nirenberg [3]). Take $\gamma \in \Gamma$ such that $\gamma_1(A) \subset \Phi_{c+\varepsilon}$ and define $\widetilde{\gamma} \in \Gamma$ by

$$\widetilde{\gamma}(u,t) = \begin{cases} \gamma(u,2t) & \text{for } 0 \leq t \leq 1/2, \\ \eta(\gamma_1(u), 2t-1) & \text{for } 1/2 < t \leq 1. \end{cases}$$

Then $\widetilde{\gamma}_1(A) \subset \Phi_{c-\varepsilon}$, contradicting (2.2). □

We now apply Proposition 2.1 to our specific situation. Let A_1, B_1 be a pair of nonempty subsets of the unit sphere S in a Banach space W such that $\operatorname{dist}(A_1, B_1) > 0$. We say that A_1 links B_1 if for any $\Phi \in C^1(S, \mathbb{R})$,

$$-\infty < \sup_{A_1} \Phi =: a_0 < b_0 := \inf_{B_1} \Phi < +\infty$$

implies that there is a sequence $(u_j) \subset S$ such that (1.1) holds for some $c \geq b_0$. We refer to Schechter [9] for the proof of the following proposition.

Proposition 2.2. *A_1 links B_1 in S if for any $\varphi \in C(CA_1, S)$ such that $\varphi(\,\cdot\,, 0) = \mathrm{id}_{A_1}$,*

$$\varphi(CA_1) \cap B_1 \neq \emptyset, \tag{2.3}$$

where $CA_1 = (A_1 \times [0,1])/(A_1 \times \{1\})$ is the cone on A_1.

Proposition 2.3. *If A_1 and B_1 satisfy the hypotheses of Proposition 2.2 in S, then*

$$A = \pi^{-1}(A_1) \cup \{0\}, \quad B = \pi^{-1}(B_1) \cup \{0\} \tag{2.4}$$

forms a sandwich pair, where $\pi \colon W \setminus \{0\} \to S$, $u \mapsto u/\|u\|$ is the radial projection onto S.

Proof. If not, using $\operatorname{dist}(A_1, B_1) > 0$ and (b), take an $R \geq 1$ so large that $\gamma(RA_1 \times [0,1]) \cap B = \emptyset$, where $RA_1 = \{Ru : u \in A_1\}$, and define $\psi \in C(CA_1, W \setminus B)$ by

$$\psi(u,t) = \begin{cases} (1 - 3t + 3Rt)u & \text{for } 0 \leq t \leq 1/3, \\ \gamma(Ru, 3t-1) & \text{for } 1/3 < t \leq 2/3, \\ \gamma_1(3(1-t)Ru) & \text{for } 2/3 < t \leq 1. \end{cases}$$

Then $\varphi = \pi \circ \psi \in C(CA_1, S \setminus B_1)$ and $\varphi(\,\cdot\,, 0) = \mathrm{id}_{A_1}$, contradicting (2.3). □

3. Proofs

Proof of Theorem 1.2. By (1.7), there is a $\gamma \in \Gamma_l$ such that $I \leq \lambda_l + \varepsilon/2$ on $A_1 = \gamma(S^{l-1})$. Let $B_1 = \{u \in S : I(u) \geq \lambda_{l+1}\}$. Since $\lambda_l + \varepsilon/2 < \lambda_{l+1}$ by (1.8), A_1 and B_1 are disjoint. Since A_1 is compact and B_1 is closed, it follows that $\operatorname{dist}(A_1, B_1) > 0$. We claim that A_1 links B_1 in S. Given $\varphi \in C(CA_1, S)$ such that $\varphi(\,\cdot\,, 0) = \mathrm{id}_{A_1}$, writing $x \in S^l$ as $(x', x_{l+1}) \in \mathbb{R}^l \oplus \mathbb{R}$, define $\overline{\gamma} \in \Gamma_{l+1}$ by

$$\overline{\gamma}(x) = \begin{cases} \varphi(\gamma(x'/|x'|), x_{l+1}) & \text{for } 0 \leq x_{l+1} < 1, \\ \varphi(A_1 \times \{1\}) & \text{for } x_{l+1} = 1, \\ \overline{\gamma}(x', -x_{l+1}) & \text{for } x_{l+1} < 0. \end{cases}$$

Then $\overline{\gamma}(S^l) \cap B_1 \neq \emptyset$ by the definition of λ_{l+1}, so (2.3) holds as B_1 is symmetric. So A, B given by (2.4) forms a sandwich pair by Proposition 2.3. Let Φ be given by (1.4). Since

$$\int_\Omega |\nabla u|^p \geq \lambda_{l+1} \int_\Omega |u|^p, \quad u \in B$$

and

$$\int_\Omega |\nabla u|^p \leq (\lambda_l + \varepsilon) \int_\Omega |u|^p, \quad u \in A,$$

(1.8) implies

$$-\int_\Omega V \leq \inf_B \Phi \leq \sup_A \Phi \leq \int_\Omega V,$$

so there is a sequence $(u_j) \subset W_0^{1,p}(\Omega)$ satisfying (1.2).

We claim that (u_j) is bounded and hence has a convergent subsequence by a standard argument. If $\rho_j = \|u_j\| \to \infty$, a subsequence of $\widetilde{u}_j = u_j/\rho_j$ converges to some $\widetilde{u}$ weakly in $W_0^{1,p}(\Omega)$, strongly in $L^p(\Omega)$, and a.e. in Ω. Then

$$\int_\Omega \frac{H(x,u_j)}{\rho_j} = \frac{\Phi'(u_j)\, u_j/p - \Phi(u_j)}{\rho_j} \to 0$$

by (1.2) and

$$\overline{\lim} \int_\Omega \frac{H(x,u_j)}{\rho_j} \leq \int_\Omega \overline{\lim}\, \frac{H(x,u_j)}{|u_j|}\, |\widetilde{u}_j| \leq \int_\Omega \overline{H}(x)|\widetilde{u}| \leq 0$$

by (1.9). Since $\overline{H} < 0$ a.e. it follows that $\widetilde{u} = 0$ a.e. Now passing to the limit in

$$1 - \frac{\Phi(u_j)}{\rho_j^p} = \int_\Omega \frac{pF(x,u_j)}{\rho_j^p} \leq \int_\Omega \lambda_{l+1}|\widetilde{u}_j|^p + \frac{V}{\rho_j^p}$$

gives a contradiction. □

PROOF OF THEOREM 1.3. Take a sequence $(\varepsilon_j) \subset (0,\varepsilon]$ decreasing to 0 and let

$$\Phi_j(u) = \Phi(u) - \varepsilon_j \int_\Omega |u|^p.$$

Then

$$(\lambda_l + \varepsilon_j)|t|^p - V(x) \leq pF(x,t) + \varepsilon_j|t|^p \leq \lambda_{l+1}|t|^p + V(x)$$

by (1.10), so there is a sequence $(u_j) \subset W_0^{1,p}(\Omega)$ such that

$$\Phi_j(u_j) \text{ is bounded}, \quad \Phi_j'(u_j) \to 0 \tag{3.1}$$

as in the proof of Theorem 1.2.

We claim that (u_j) is bounded and hence a subsequence converges to a critical point of Φ. If $\rho_j = \|u_j\| \to \infty$, a subsequence of $\widetilde{u}_j = u_j/\rho_j$ converges to some $\widetilde{u}$ weakly in $W_0^{1,p}(\Omega)$, strongly in $L^p(\Omega)$, and a.e. in Ω. Then

$$\int_\Omega \frac{H(x,u_j)}{\rho_j} = \frac{\Phi_j'(u_j)\, u_j/p - \Phi_j(u_j)}{\rho_j} \to 0$$

by (3.1) and

$$\underline{\lim}\int_\Omega \frac{H(x,u_j)}{\rho_j} \geq \int_\Omega \underline{\lim}\,\frac{H(x,u_j)}{|u_j|}\,|\widetilde{u}_j| \geq \int_\Omega \underline{H}(x)|\widetilde{u}| \geq 0$$

by (1.11). Since $\underline{H} > 0$ a.e. it follows that $\widetilde{u} = 0$ a.e. Now passing to the limit in

$$1 - \frac{\Phi_j(u_j)}{\rho_j^p} = \int_\Omega \frac{pF(x,u_j)}{\rho_j^p} + \varepsilon_j|\widetilde{u}_j|^p \leq \int_\Omega \lambda_{l+1}|\widetilde{u}_j|^p + \frac{V}{\rho_j^p}$$

gives a contradiction. □

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