Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 28, 2006, 199–233

THE SUSPENSION ISOMORPHISM FOR HOMOLOGY INDEX BRAIDS

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Abstract. Let X be a metric space, π be a local semiflow on X, $k \in \mathbb{N}$, E be a k-dimensional normed space and $\widetilde{\pi}$ be the semiflow generated by the equation $\dot{y} = Ly$, where $L\colon E \to E$ is a linear map whose all eigenvalues have positive real parts. We show in this paper that for every admissible isolated π-invariant set S there is a well-defined isomorphism of degree $-k$ from the homology categorial Conley–Morse index of $(\pi \times \widetilde{\pi}, S \times \{0\})$ to the homology categorial Conley–Morse index of (π, S) such that the family of these isomorphisms commutes with homology index sequences. In particular, given a partially ordered Morse decomposition $(M_i)_{i \in P}$ of S there is an isomorphism of degree $-k$ from the homology index braid of $(M_i \times \{0\})_{i \in P}$ to the homology index braid of $(M_i)_{i \in P}$, so C-connection matrices of $(M_i \times \{0\})_{i \in P}$ are just C-connection matrices of $(M_i)_{i \in P}$ shifted by k to the right.

1. Introduction

Let H_q, $q \in \mathbb{Z}$, be the singular homology functor with coeficients in a fixed Γ-module $\overline{G}$, where Γ is a commutative ring. Let X be a metric space, π be a local semiflow on X, $k \in \mathbb{N}$, E be a k-dimensional normed space and $\widetilde{\pi}$ be the

2000 *Mathematics Subject Classification.* Primary 37B30; Secondary 37C70, 55N10.

Key words and phrases. Conley index, homology index braid, suspension isomorphism, connection matrix.

The research of the first author was partially supported by the grants CAPES-Brazil 0710-12/04 and CNPq-Brazil 302854/2002-9. The research of the second author was partially supported by the grant DAAD-Germany D/04/40407.

semiflow generated by the equation $\dot{y} = Ly$, where $L: E \to E$ is a linear map whose all eigenvalues have positive real parts.

Consider the *local product semiflow* $\pi \times \widetilde{\pi}$ on $X \times E$ defined by $(x, y)\pi \times \widetilde{\pi} t := (x\pi t, y\widetilde{\pi} t)$ whenever $x\pi t$ is defined.

Whenever S is an isolated π-invariant set having a strongly π-admissible isolating neighbouhood, then $S \times \{0\}$, where $0 = 0_E$ is the zero of E, is an isolated $\pi \times \widetilde{\pi}$-invariant set having a strongly $\pi \times \widetilde{\pi}$-admissible isolating neighbouhood. This means that the homotopy (Conley) indices $h(\pi, S)$ and $h(\pi \times \widetilde{\pi}, S \times \{0\})$ are defined. Moreover, we have the well-known formula

$$h(\pi \times \widetilde{\pi}, S \times \{0\}) = h(\pi, S) \wedge \Sigma^k. \tag{1.1}$$

Thus applying standard results from homology theory to (1.1) we have, for $q \in \mathbb{Z}$,

$$H_q(h(\pi \times \widetilde{\pi}, S \times \{0\})) = H_{q-k}(h(\pi, S)). \tag{1.2}$$

Now formula (1.1) simply means that whenever (N_1, N_2) is an arbitrary index pair in some strongly π-admissible isolating neighbouhood of S and $(\widehat{N}_1, \widehat{N}_2)$ is an arbitrary index pair in some strongly $\pi \times \widetilde{\pi}$-admissible isolating neighbouhood of $S \times \{0\}$, then the pointed spaces $(\widehat{N}_1/\widehat{N}_2, [\widehat{N}_2])$ and $(N_1/N_2, [N_2]) \wedge (\mathbb{S}^k, s_0)$ are homotopy equivalent, where $(\mathbb{S}^k, s_0)$ is a k-dimensional sphere with a base-point. Again, formula (1.2) only means that, for $q \in \mathbb{Z}$, there is *some* Γ-module isomorphism between $H_q(\widehat{N}_1/\widehat{N}_2, \{[\widehat{N}_2]\})$ and $H_{q-k}(N_1/N_2, \{[N_2]\})$.

It is the purpose of this paper to refine formula (1.2). More specifically, we show in our main theorem that for every S there is a well-defined isomorphism $\theta_q(S)$ from the homology categorial Conley–Morse index $H'_q(S) := H_q(\mathcal{C}(\pi \times \widetilde{\pi}, S \times \{0\}))$ of $(\pi \times \widetilde{\pi}, S \times \{0\})$ to the homology categorial Conley–Morse index $H_{q-k}(S) := H_{q-k}(\mathcal{C}(\pi, S))$ of (π, S) such that the family of such isomorphisms commutes with homology index sequences. This means that, whenever (A, A^*) is an attractor-repeller pair of S relative to π, then $(A \times \{0\}, A^* \times \{0\})$ is an attractor-repeller pair of $S \times \{0\}$ relative to $\pi \times \widetilde{\pi}$ and we have the commutative diagram

$$\begin{tikzcd} \longrightarrow H'_q(A) \arrow[r] \arrow[d, "\theta_q(A)"'] & H'_q(S) \arrow[r] \arrow[d, "\theta_q(S)"'] & H'_q(A^*) \arrow[r] \arrow[d, "\theta_q(A^*)"] & H'_{q-1}(S) \longrightarrow \arrow[d, "\theta_{q-1}(A)"] \\ \longrightarrow H_{q-k}(A) \arrow[r] & H_{q-k}(S) \arrow[r] & H_{q-k}(A^*) \arrow[r] & H_{q-k-1}(A) \longrightarrow \end{tikzcd} \tag{1.3}$$

Here, the upper (resp. lower) horizontal sequence is the homology index sequence of $(\pi \times \widetilde{\pi}, S \times \{0\}, A \times \{0\}, A^* \times \{0\})$ (resp. (π, S, A, A^*)). This implies that whenever P is a finite set, $\prec$ is a strict order relation on P and $(M_i)_{i \in P}$ is a $\prec$-ordered Morse decomposition of S relative to π, then $(M_i \times \{0\})_{i \in P}$ is a $\prec$-ordered Morse decomposition of $S \times \{0\}$ relative to $\pi \times \widetilde{\pi}$ and there is a module braid isomorphism from the homology index braid of $(\pi \times \widetilde{\pi}, S \times \{0\}, (M_i \times \{0\})_{i \in P})$

to the homology index braid of $(\pi, S, (M_i)_{i\in P})$ 'shifted to the left by k'. In particular, shifting a C-connection matrix of $(\pi, S, (M_i)_{i\in P})$ by k to the right gives us a C-connection matrix of $(\pi \times \widetilde{\pi}, S \times \{0\}, (M_i \times \{0\})_{i\in P})$.

The proof of our main theorem is somewhat technical. A crucial step in the proof is an important result from homological algebra which, though already familiar to, say, Bourbaki, was only recently proved by M. Scott Osborne [12] in the explicit and general form required here. This result asserts that given a 3×3-matrix of chain maps, the diagram composed of the connecting homomorphisms of the third row, first column, first row and third column *anticommutes*!

The outline of this paper is as follows: in Section 2 we establish and collect some preliminary results required in the sequel. In particular, in Proposition 2.2 we prove existence of 'symmetric looking' FM-index pairs for product semiflows. We also extend some concepts and results from [5] about image modules of connected simple systems under covariant or contravariant functors (Definition 2.3 and Propositions 2.4 and 2.5). In Section 3 we state our main result, Theorem 3.1 and reduce its proof to the proof of a special case, Theorem 3.5. In Section 4, Theorem 4.8, we establish the existence of the suspension isomorphism claimed in Theorem 3.5 through a series of propositions and lemmas. In Section 5 we present some preliminary algebraic results. In particular, we quote Osborne's result (Proposition 5.4) and extend it to 3×3-matrices of chain maps with weakly exact rows (Proposition 5.5). We also present a technical result guaranteeing injectivity of singular chain maps (Lemma 5.7). In Section 6 we establish a technical result, Theorem 6.1, which implies the commutativity of the suspension isomorphism with respect to homology index sequences of attractor-repeller pairs. This completes the proof of Theorem 3.5 and thus also the proof of Theorem 3.1. Finally, in Section 7, Theorem 3.1 is used to establish the existence of the braid isomorphism mentioned above.

In this paper we use the notation and results from [5] without further explanation.

2. Preliminaries

In this section we will establish and collect a few preliminary results which are required for a precise statement and proof of our main result.

Proposition 2.1. *Let Y be a topological space and M_1, M_2, $\widetilde{M}_1$, $\widetilde{M}_2$ be closed in Y such that $M_2 \subset M_1 \subset \widetilde{M}_1$, $M_2 \subset \widetilde{M}_2$, $\widetilde{M}_1 \setminus \widetilde{M}_2 \subset M_1$ and $\widetilde{M}_2 \cap M_1 \subset M_2$. Then the inclusion induced map*

$$f\colon M_1/M_2 \to \widetilde{M}_1/\widetilde{M}_2$$

is a base-point preserving homeomorphism.

Proof. Proposition I.6.2 in [14] implies that f is a continuous map. Define $g: \widetilde{M}_1/\widetilde{M}_2 \to M_1/M_2$ by

$$g(z) := \begin{cases} q(x) & \text{if } z = \widetilde{q}(x) \text{ with } x \in \widetilde{M}_1 \setminus \widetilde{M}_2, \\ [M_2] & \text{otherwise,} \end{cases}$$

where $q: M_1 \to M_1/M_2$ and $\widetilde{q}: \widetilde{M}_1 \to \widetilde{M}_1/\widetilde{M}_2$ are the canonical projection maps. Since $\widetilde{M}_1 \setminus \widetilde{M}_2 \subset M_1$, it follows that g is well defined. Moreover, g is an inclusion induced map in the sense of Definition I.6.1 in [14] which is continuous by Proposition I.6.2 in [14]. Clearly, both f and g are base-point preserving.

We need to show that g is the inverse of f. Let $z \in M_1/M_2$. If $z \notin q(M_1)$ then $M_2 = \emptyset$ and $z = [M_2]$ so $(g \circ f)(z) = z$. If $z \in q(M_1)$ then let $x \in M_1 \subset \widetilde{M}_1$ be such that $z = q(x)$. Thus $f(z) = \widetilde{q}(x)$. If $x \in \widetilde{M}_1 \setminus \widetilde{M}_2$ then the definition of g implies that $g(f(z)) = q(x) = z$. Otherwise it follows that $x \in M_1 \cap \widetilde{M}_2 \subset M_2$ so, on the one hand, $g(f(z)) = [M_2]$ and, on the other hand, $z = [M_2]$, hence $g(f(z)) = z$.

Let $z \in \widetilde{M}_1/\widetilde{M}_2$. If $z = \widetilde{q}(x)$ where $x \in \widetilde{M}_1 \setminus \widetilde{M}_2$, then $g(z) = q(x)$ and $x \in M_1 \setminus M_2$. Therefore, $f(g(z)) = \widetilde{q}(x) = z$. Otherwise, $z = [\widetilde{M}_2]$ and so $f(g(z)) = z$. □

Now let X and $\widetilde{X}$ be metric spaces and let π, respectively $\widetilde{\pi}$, be a local semiflow defined on X, respectively $\widetilde{X}$. Let S, respectively $\widetilde{S}$, be a compact π-invariant, respectively $\widetilde{\pi}$-invariant.

Set $D := \{\,(t,(x,\widetilde{x})) \in [0,\infty[\, \times X \times \widetilde{X} \mid x\pi t \text{ and } \widetilde{x}\widetilde{\pi}t \text{ are defined}\,\}$. Define $(x,\widetilde{x})\pi \times \widetilde{\pi} t := (x\pi t, \widetilde{x}\widetilde{\pi}t)$ for $(t,(x,\widetilde{x})) \in D$. It follows that $\pi \times \widetilde{\pi}$ is a local semiflow on $X \times \widetilde{X}$ called the *product semiflow* of π with $\widetilde{\pi}$ (cf. Section I.10 in [14]).

We have the following result.

Proposition 2.2. *Let (N_1, N_2) (resp. $(\widetilde{N}_1, \widetilde{N}_2)$) be an* FM-*index pair for (π, S) (resp. $(\widetilde{\pi}, \widetilde{S})$) and let N be a closed subset of X such that $N_1 \subset N$. Then $(N_1 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2, N_2 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2)$ is an* FM-*index pair for $(\pi \times \widetilde{\pi}, S \times \widetilde{S})$.*

Proof. It is clear that $S \times \widetilde{S}$, $N_1 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2$ and $N_2 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2$ are closed subsets of $X \times \widetilde{X}$. Since $N_2 \subset N_1$, it follows that $N_2 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2 \subset N_1 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2$. We claim that

(2.1) $N_2 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2$ is $N_1 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2$-positively invariant relative to $\pi \times \widetilde{\pi}$.

Let $(x,\widetilde{x}) \in N_2 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2$ and let $t \in [0,\infty[$ such that $(x,\widetilde{x})\pi \times \widetilde{\pi}\,[0,t] \subset N_1 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2$. We will show that $(x,\widetilde{x})\pi \times \widetilde{\pi}\,[0,t] \subset N_2 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2$.

First assume that $(x,\widetilde{x}) \in N \times \widetilde{N}_2$. Since $(x,\widetilde{x})\pi \times \widetilde{\pi}\,[0,t] \subset N_1 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2$, $N_1 \subset N$ and $\widetilde{N}_2 \subset N_1$, it follows that $x\pi\,[0,t] \subset N$ and $\widetilde{x}\widetilde{\pi}\,[0,t] \subset \widetilde{N}_1$.

Since $\widetilde{N}_2$ is $\widetilde{N}_1$-positively invariant relative to $\widetilde{\pi}$, we have $\widetilde{x}\widetilde{\pi}\,[0,t] \subset \widetilde{N}_2$ and so $(x,\widetilde{x})\pi \times \widetilde{\pi}\,[0,t] \subset N_2 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2$.

Now assume that $(x,\widetilde{x}) \in N_2 \times \widetilde{N}_1$ and let $\tau \in [0,t]$. If $(x,\widetilde{x})\pi \times \widetilde{\pi}\,[0,\tau] \subset N_1 \times \widetilde{N}_1$, since N_2 is N_1-positively invariant relative to π, it follows that $x\pi\,[0,\tau] \subset N_2$ and so $(x,\widetilde{x})\pi \times \widetilde{\pi}\tau \in N_2 \times \widetilde{N}_1 \subset N_2 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2$. Suppose there exists a $t' \in [0,\tau]$ such that $(x,\widetilde{x})\pi \times \widetilde{\pi}t' \notin N_1 \times \widetilde{N}_1$. Hence $(x,\widetilde{x})\pi \times \widetilde{\pi}t' \in N \times \widetilde{N}_2$ and so $\widetilde{x}\widetilde{\pi}t' \in \widetilde{N}_2$. Since $\widetilde{N}_2$ is $\widetilde{N}_1$-positively invariant relative to $\widetilde{\pi}$ and $\widetilde{x}\widetilde{\pi}\,[t',\tau] \subset \widetilde{x}\widetilde{\pi}\,[0,t] \subset \widetilde{N}_1$, it follows that $\widetilde{x}\widetilde{\pi}\,[t',\tau] \subset \widetilde{N}_2$ and so $(x,\widetilde{x})\pi \times \widetilde{\pi}\tau \in N \times \widetilde{N}_2 \subset N_2 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2$. Claim (2.1) is proved.

We also claim that

(2.2) $N_2 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2$ is an exit ramp for $N_1 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2$, relative to $\pi \times \widetilde{\pi}$.

Let $(x,\widetilde{x}) \in N_1 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2$ and suppose there exists a $t' \in [0,\infty[$ such that $(x,\widetilde{x})\pi \times \widetilde{\pi}t' \notin N_1 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2$. We need to show that there exists a $t_0 \in [0,t']$ such that $(x,\widetilde{x})\pi \times \widetilde{\pi}\,[0,t_0] \subset N_1 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2$ and $(x,\widetilde{x})\pi \times \widetilde{\pi}t_0 \in N_2 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2$.

If $(x,\widetilde{x}) \in N \times \widetilde{N}_2$, setting $t_0 = 0$ we are done. Hence suppose that $(x,\widetilde{x}) \in N_1 \times \widetilde{N}_1$. There are two cases to be consider:

Case 1. Assume that $x\pi t' \notin N_1$. Since N_2 is an exit ramp for N_1, it follows that there exists a $t'' \in [0,t']$ such that $x\pi\,[0,t''] \subset N_1$ and $x\pi t'' \in N_2$. If $\widetilde{x}\widetilde{\pi}\,[0,t''] \subset \widetilde{N}_1$, it follows that $(x,\widetilde{x})\pi \times \widetilde{\pi}\,[0,t''] \subset N_1 \times \widetilde{N}_1 \subset N_1 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2$ and $(x,\widetilde{x})\pi \times \widetilde{\pi}t'' \in N_2 \times \widetilde{N}_1 \subset N_2 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2$ and we set $t_0 = t''$. Otherwise, there exists a $\widetilde{t} \in [0,t'']$ such that $\widetilde{x}\widetilde{\pi}\widetilde{t} \notin \widetilde{N}_1$. Since $\widetilde{N}_2$ is an exit ramp for $\widetilde{N}_1$, it follows that there exists a $t_0 \in [0,\widetilde{t}]$ such that $\widetilde{x}\widetilde{\pi}\,[0,t_0] \subset \widetilde{N}_1$ and $\widetilde{x}\widetilde{\pi}t_0 \in \widetilde{N}_2$. Hence, $(x,\widetilde{x})\pi \times \widetilde{\pi}\,[0,t_0] \subset N_1 \times \widetilde{N}_1 \subset N_1 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2$ and $(x,\widetilde{x})\pi \times \widetilde{\pi}t_0 \in N_1 \times \widetilde{N}_2 \subset N_2 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2$.

Case 2. Assume that $\widetilde{x}\widetilde{\pi}t' \notin \widetilde{N}_1$. Since $\widetilde{N}_2$ is an exit ramp for $\widetilde{N}_1$, it follows that there exists a $t'' \in [0,t']$ such that $\widetilde{x}\widetilde{\pi}\,[0,t''] \subset \widetilde{N}_1$ and $\widetilde{x}\widetilde{\pi}t'' \in \widetilde{N}_2$. If $x\pi\,[0,t''] \subset N_1$, it follows that $(x,\widetilde{x})\pi \times \widetilde{\pi}\,[0,t''] \subset N_1 \times \widetilde{N}_1 \subset N_1 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2$ and $(x,\widetilde{x})\pi \times \widetilde{\pi}t'' \in N_1 \times \widetilde{N}_2 \subset N_2 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2$ and we set $t_0 = t''$. Otherwise, there exists a $\widetilde{t} \in [0,t'']$ such that $x\pi\widetilde{t} \notin N_1$. Since N_2 is an exit ramp for N_1, it follows that there exists a $t_0 \in [0,\widetilde{t}]$ such that $x\pi\,[0,t_0] \subset N_1$ and $x\pi t_0 \in N_2$. Hence, $(x,\widetilde{x})\pi \times \widetilde{\pi}\,[0,t_0] \subset N_1 \times \widetilde{N}_1 \subset N_1 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2$ and $(x,\widetilde{x})\pi \times \widetilde{\pi}t_0 \in N_2 \times \widetilde{N}_1 \subset N_2 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2$. This completes the proof of claim (2.2).

To complete the proof we need to show that

(2.3) $S \times \widetilde{S} \subset \mathrm{Int}_{X \times \widetilde{X}}((N_1 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2) \setminus (N_2 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2))$ and $S \times \widetilde{S}$ is the largest $\pi \times \widetilde{\pi}$-invariant set in $\mathrm{Cl}_{X \times \widetilde{X}}((N_1 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2) \setminus (N_2 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2))$.

It clear that $(N_1 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2) \setminus (N_2 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2) = N_1 \setminus N_2 \times \widetilde{N}_1 \setminus \widetilde{N}_2$. Therefore, $\mathrm{Inv}_{\pi \times \widetilde{\pi}}(\mathrm{Cl}_{X \times \widetilde{X}}((N_1 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2) \setminus (N_2 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2))) \subset \mathrm{Inv}_{\pi \times \widetilde{\pi}}(\mathrm{Cl}_{X \times \widetilde{X}}(N_1 \setminus N_2 \times \widetilde{N}_1 \setminus \widetilde{N}_2)) = \mathrm{Inv}_{\pi \times \widetilde{\pi}}(\mathrm{Cl}_X(N_1 \setminus N_2) \times \mathrm{Cl}_{\widetilde{X}}(\widetilde{N}_1 \setminus \widetilde{N}_2)) = \mathrm{Inv}_\pi(\mathrm{Cl}_X(N_1 \setminus N_2)) \times \mathrm{Inv}_{\widetilde{\pi}}(\mathrm{Cl}_{\widetilde{X}}(\widetilde{N}_1 \setminus \widetilde{N}_2)) = S \times \widetilde{S}$. Moreover, there exist an open set V in X, respectively $\widetilde{V}$ in $\widetilde{X}$, such that $S \subset V \subset N_1 \setminus N_2$, respectively $\widetilde{S} \subset \widetilde{V} \subset \widetilde{N}_1 \setminus \widetilde{N}_2$. Hence $S \times \widetilde{S} \subset V \times \widetilde{V} \subset (N_1 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2) \setminus (N_2 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2)$ which completes the proof of claim (2.3).

Claims (2.1), (2.2) and (2.3) imply that $(N_1 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2, N_2 \times \widetilde{N}_1 \cup N \times \widetilde{N}_2)$ is an FM-index pair for $S \times \widetilde{S}$ relative to $\pi \times \widetilde{\pi}$. $\square$

Now let $\mathcal{K}$ be a category and $\mathcal{C}$ be an object of $[\mathcal{K}]$, i.e. a subcategory of $\mathcal{K}$ which is a connected simple system. Suppose Φ is a covariant (resp. contravariant) functor from $\mathcal{C}$ to the category $\mathrm{Mod}(\Gamma)$ of modules over the (commutative) ring Γ. Let $S = S_{\mathcal{C},\Phi}$ be the *disjoint* union of all $\Phi(A)$, where A is an arbitrary object of $\mathcal{C}$. Thus, formally we have

$$S = S_{\mathcal{C},\Phi} := \bigcup_{A \in \mathrm{Obj}(\mathcal{C})} \Phi(A) \times \{A\}.$$

On S define a relation $R = R_{\mathcal{C},\Phi}$ as follows:

> $(x, A)R(y, B)$ if and only if $y = \Phi(f)x$ (resp. $x = \Phi(f)y$), where f is the unique morphism in $\mathcal{C}$ from A to B.

Clearly, R is an equivalence relation on S. Let S/R be the set of equivalence classes of R and $Q = Q_{\mathcal{C},\Phi} \colon S \to S/R$ be the canonical quotient map. In the sequel we write $\widehat{\Phi}(\mathcal{C}) := S/R$.

For each $A \in \mathrm{Obj}(\mathcal{C})$, the map $Q_A = Q_{\mathcal{C},\Phi,A} \colon \Phi(A) \to \widehat{\Phi}(\mathcal{C})$ given by $Q_A(x) = Q((x, A))$ for $x \in \Phi(A)$ is easily seen to be bijective. Moreover, if $(x, A)R(y, B)$ and $(\widetilde{x}, A)R(\widetilde{y}, B)$, then $(x +_A \widetilde{x}, A)R(y +_B \widetilde{y}, B)$ and $(\lambda \cdot_A x, A)R(\lambda \cdot_B y, B)$ for every $\lambda \in \Gamma$. Here, for every $C \in \mathrm{Obj}(\mathcal{C})$, $+_C$ (resp. $\cdot_C$) is the addition (resp. scalar multiplication) in the Γ-module $\Phi(C)$. Therefore, there is a unique addition $+ = +_{\mathcal{C}}$ and scalar multiplication $\cdot = \cdot_{\mathcal{C}}$ in $\widehat{\Phi}(\mathcal{C})$ such that for every $A \in \mathrm{Obj}(\mathcal{C})$, the map Q_A is a Γ-module isomorphism. The Γ-module $\widehat{\Phi}(\mathcal{C})$ is called *the image module of* $\mathcal{C}$ *under* Φ.

Definition 2.3. Let $\mathcal{C}$ and $\mathcal{C}'$ be objects of $[\mathcal{K}]$, Φ be a functor from $\mathcal{C}$ to $\mathrm{Mod}(\Gamma)$ and Φ' be a functor from $\mathcal{C}'$ to $\mathrm{Mod}(\Gamma)$. Assume that Φ and Φ' are both covariant (resp. both contravariant) and let $A \in \mathrm{Obj}(\mathcal{C})$, $A' \in \mathrm{Obj}(\mathcal{C}')$ be arbitrary. If F is a morphism in $\mathrm{Mod}(\Gamma)$ from $\Phi(A)$ to $\Phi'(A')$, then define the map

$$\langle F \rangle = \langle F \rangle_{\mathcal{C},\Phi,\mathcal{C}',\Phi'} \colon \widehat{\Phi}(\mathcal{C}) \to \widehat{\Phi'}(\mathcal{C}')$$

by

$$\langle F \rangle := Q_{\mathcal{C}',\Phi',A'} \circ F \circ (Q_{\mathcal{C},\Phi,A})^{-1}.$$

It follows that $\langle F\rangle$ is a Γ-module homomorphism. Moreover,

Proposition 2.4. *Let $\mathcal{C}$ and $\mathcal{C}'$ be objects of $[\mathcal{K}]$. Let Φ be a functor from $\mathcal{C}$ to $\mathrm{Mod}(\Gamma)$ and Φ' be a functor from $\mathcal{C}'$ to $\mathrm{Mod}(\Gamma)$. Assume that Φ and Φ' are covariant (resp. contravariant). Suppose A, $B \in \mathrm{Obj}(\mathcal{C})$, A', $B' \in \mathrm{Obj}(\mathcal{C}')$. If the diagram*

$$\begin{tikzcd} \Phi(A) \arrow[r, "F"] \arrow[d, "\Phi(f)"'] & \Phi'(A') \arrow[d, "\Phi'(f')"] \\ \Phi(B) \arrow[r, "G"'] & \Phi'(B'), \end{tikzcd} \qquad \left(resp. \quad \begin{tikzcd} \Phi(A) \arrow[r, "F"] & \Phi'(A') \\ \Phi(B) \arrow[r, "G"'] \arrow[u, "\Phi(f)"] & \Phi'(B') \arrow[u, "\Phi'(f')"'] \end{tikzcd} \right)$$

commutes, then $\langle F\rangle = \langle G\rangle$, where f (resp. f') is the unique morphism in $\mathcal{C}$ (resp. $\mathcal{C}'$) from A to B (resp. from A' to B').

Proof. Suppose that both Φ and Φ' are covariant. Let $\eta \in \widehat{\Phi}(\mathcal{C})$ be arbitrary. Then there exist an $x \in \Phi(A)$ and a $y \in \Phi(B)$ such that $\eta = Q_{\mathcal{C},\Phi}((x,A)) = Q_{\mathcal{C},\Phi}((y,B))$. It follows that $y = \Phi(f)x$. Now

$$\begin{aligned} \langle F\rangle(\eta) &= (Q_{\mathcal{C}',\Phi',A'} \circ F \circ (Q_{\mathcal{C},\Phi,A})^{-1})(\eta) = Q_{\mathcal{C}',\Phi',A'}Fx, \\ \langle G\rangle(\eta) &= (Q_{\mathcal{C}',\Phi',B'} \circ G \circ (Q_{\mathcal{C},\Phi,B})^{-1})(\eta) \\ &= Q_{\mathcal{C}',\Phi',B'}(Gy) = Q_{\mathcal{C}',\Phi',B'}G(\Phi(f)x) = Q_{\mathcal{C}',\Phi',B'}(\Phi(f')Fx). \end{aligned}$$

Notice that

$$Q_{\mathcal{C}',\Phi',B'}(\Phi(f')Fx) = Q_{\mathcal{C}',\Phi'}((\Phi(f')Fx, B')) = Q_{\mathcal{C}',\Phi'}((Fx, A')) = Q_{\mathcal{C}',\Phi',A'}(Fx).$$

This implies that $\langle F\rangle(\eta) = \langle G\rangle(\eta)$. The proof is analogous in the contravariant case. The proposition is proved. $\square$

The following result is straightforward from Definition 2.3.

Proposition 2.5. *Let $\mathcal{C}$, $\mathcal{C}'$ and $\mathcal{C}''$ be objects of $[\mathcal{K}]$. Let Φ be a functor from $\mathcal{C}$ to $\mathrm{Mod}(\Gamma)$, Φ' be a functor from $\mathcal{C}'$ to $\mathrm{Mod}(\Gamma)$ and Φ'' be a functor from $\mathcal{C}''$ to $\mathrm{Mod}(\Gamma)$. Assume that Φ, Φ' and Φ'' are all covariant (resp. all contravariant) and let $A \in \mathrm{Obj}(\mathcal{C})$, $A' \in \mathrm{Obj}(\mathcal{C}')$, $A'' \in \mathrm{Obj}(\mathcal{C}'')$ be arbitrary. If F is a morphism in $\mathrm{Mod}(\Gamma)$ from $\Phi(A)$ to $\Phi(A')$ and F' is a morphism in $\mathrm{Mod}(\Gamma)$ from $\Phi(A')$ to $\Phi(A'')$ then*

$$\langle F' \circ F\rangle = \langle F'\rangle \circ \langle F\rangle.$$

If (F, F') is exact, i.e. $\ker F' = \operatorname{im} F$, then so is $(\langle F\rangle, \langle F'\rangle)$.

Remark 2.6. Definition 2.3 and Propositions 2.4 and 2.5 were presented in [5] in the special case of Φ covariant and $\Phi'' = \Phi' = \Phi$. However, it is the more general concept and the corresponding results presented here that were actually already used in [5].

Recall (cf. [7]) that a sequence

$$C_1 \xrightarrow{i} C_2 \xrightarrow{p} C_3$$

of chain maps is called *weakly exact* if $\ker i = 0$, $p \circ i = 0$ and the map $H_q(\rho): H_q(C_2/\operatorname{im} i) \to H_q(C_3)$ is an isomorphism for each $q \in \mathbb{Z}$. Here, the map $\rho: C_2/\operatorname{im} i \to C_3$ is the (uniquely determined) chain map with $\rho \circ Q = p$, where $Q: C_2 \to C_2/\operatorname{im} i$ is the quotient map.

Given a weakly exact sequence

$$C_1 \xrightarrow{i} C_2 \xrightarrow{p} C_3$$

and $q \in \mathbb{Z}$, define $\widehat{\partial}_q: H_q(C_3) \to H_{q-1}(C_1)$ by $\widehat{\partial}_q := \partial_{*q} \circ H_q(\rho)^{-1}$, where $\partial_{*q}: H_q(C_2/\operatorname{im} i) \to H_{q-1}(C_1)$ is the connecting homomorphism in the long exact sequence induced by the exact sequence

$$0 \longrightarrow C_1 \xrightarrow{i} C_2 \xrightarrow{Q} C_2/\operatorname{im} i \longrightarrow 0.$$

Using elementary homology theory we obtain the following result.

Proposition 2.7 (cf. [7]). *Given a weakly exact sequence*

$$C_1 \xrightarrow{i} C_2 \xrightarrow{p} C_3$$

the corresponding homology sequence

$$\longrightarrow H_q(C_1) \xrightarrow{H_q(i)} H_q(C_2) \xrightarrow{H_q(p)} H_q(C_3) \xrightarrow{\widehat{\partial}_q} H_{q-1}(C_1) \longrightarrow$$

is exact. Moreover, given a commutative diagram

$$\begin{tikzcd}
C_1 \arrow[r, "i"] \arrow[d, "f_1"'] & C_2 \arrow[r, "p"] \arrow[d, "f_2"] & C_3 \arrow[d, "f_3"] \\
\widetilde{C}_1 \arrow[r, "\widetilde{i}"'] & \widetilde{C}_2 \arrow[r, "\widetilde{p}"'] & \widetilde{C}_3
\end{tikzcd}$$

of chain maps with weakly exact rows, the induced long homology ladder

$$\begin{tikzcd}
\arrow[r] & H_q(C_1) \arrow[r, "H_q(i)"] \arrow[d, "H_q(f_1)"'] & H_q(C_2) \arrow[r, "H_q(p)"] \arrow[d, "H_q(f_2)"'] & H_q(C_3) \arrow[r, "\widehat{\partial}_q"] \arrow[d, "H_q(f_3)"] & H_{q-1}(C_1) \arrow[r] \arrow[d, "H_{q-1}(f_1)"] & {} \\
\arrow[r] & H_q(\widetilde{C}_1) \arrow[r, "H_q(\widetilde{i})"'] & H_q(\widetilde{C}_2) \arrow[r, "H_q(\widetilde{p})"'] & H_q(\widetilde{C}_3) \arrow[r, "\widetilde{\widehat{\partial}}_q"'] & H_{q-1}(\widetilde{C}_1) \arrow[r] & {}
\end{tikzcd}$$

is commutative.

In the sequel we denote by Δ the singular chain functor with coefficients in the Γ-module $\overline{G}$.

Proposition 2.8 (cf. [7] and [8]). *Let (N_1, N_2, N_3) be an* FM*-index triple for (π, S, A, A^*) with $\mathrm{Cl}_X(N_1 \setminus N_3)$ strongly π-admissible. Then the inclusion induced sequence*

$$N_2/N_3 \xrightarrow{i} N_1/N_3 \xrightarrow{p} N_1/N_2 \tag{2.4}$$

of pointed spaces induces a weakly exact sequence

$$\Delta(N_2/N_3)/\Delta(\{[N_3]\}) \xrightarrow{i} \Delta(N_1/N_3)/\Delta(\{[N_3]\}) \xrightarrow{p} \Delta(N_1/N_2)/\Delta(\{[N_2]\})$$

of chain maps.

3. Statement of the main result

We can now state the main result of this paper.

Theorem 3.1. *Let X be a metric space, π be a local semiflow on X, $k \in \mathbb{N}$, E be a k-dimensional normed space and $\widetilde{\pi}$ be the semiflow generated by the equation $\dot{y} = Ly$, where $L: E \to E$ is a linear map with all eigenvalues having positive real parts. Then there is a family of Γ-module isomorphisms*

$$\theta_q(\pi, \widetilde{\pi}, S): H_q(\pi \times \widetilde{\pi}, S \times \{0_E\}) \to H_{q-k}(\pi, S),$$

one for each $q \in \mathbb{Z}$ and each isolated π-invariant set S having a strongly π-admissible isolating neighbouhood, such that the following property is satisfied: given any isolated π-invariant set S having a strongly π-admissible isolating neighbouhood and given any attractor-repeller pair (A, A^*) of S relative to π, the following diagram commutes:

$$\begin{tikzcd}
\longrightarrow H_q(\pi', A') \arrow[r] \arrow[d, "\theta_q(\pi,\widetilde{\pi},A)"] & H_q(\pi', S') \arrow[r] \arrow[d, "\theta_q(\pi,\widetilde{\pi},S)"] & H_q(\pi', A^{*\prime}) \arrow[r] \arrow[d, "\theta_q(\pi,\widetilde{\pi},A^*)"] & H_{q-1}(\pi', S') \longrightarrow \arrow[d, "\theta_{q-1}(\pi,\widetilde{\pi},A)"] \\
\longrightarrow H_{q-k}(\pi, A) \arrow[r] & H_{q-k}(\pi, S) \arrow[r] & H_{q-k}(\pi, A^*) \arrow[r] & H_{q-k-1}(\pi, A) \longrightarrow
\end{tikzcd} \tag{3.1}$$

Here, the upper (resp. lower) horizontal sequence is the homology index sequence of $(\pi', S', A', A^{*\prime})$ (resp. (π, S, A, A^*)), where we set $\pi' := \pi \times \widetilde{\pi}$ and for $K \subset X$, $K' := K \times \{0_E\}$.

The following result will reduce the proof of Theorem 3.1 to the proof of a special case.

Theorem 3.2. *Let X and X' be metric spaces and let π (resp. π') be a local semiflow on X (resp. on X'). Let $\gamma: X \to X'$ be a homeomorphism which conjugates π with π'.*

(a) *Let S be an isolated π-invariant set and (Y, Z) be an* FM*-index pair for (π, S) such that $\mathrm{Cl}_X(Y \setminus Z)$ is strongly π-admissible. Then $\gamma(S)$ is an isolated π'-invariant set and $(\gamma(Y), \gamma(Z))$ is an* FM*-index pair for*

$(\pi', \gamma(S))$ such that $\mathrm{Cl}_{X'}(\gamma(Y) \setminus \gamma(Z))$ is strongly π'-admissible. Let $\gamma_{Y,Z}: Y/Z \to \gamma(Y)/\gamma(Z)$ be the map induced by γ and, for $q \in \mathbb{Z}$, let

$$F_q := H_q(\gamma_{Y,Z}): H_q(Y/Z, \{[Z]\}) \to H_q(\gamma(Y)/\gamma(Z), \{[\gamma(Z)]\})$$

be the induced homology map.

(b) *The map*

$$\langle F_q \rangle = \langle F_q \rangle_{\mathcal{C},\Phi,\mathcal{C}',\widehat{\Phi'}}: \widehat{\Phi}(\mathcal{C}) \to \widehat{\Phi'}(\mathcal{C}')$$

is independent of the choice of (Y, Z). Here, $\mathcal{C}$ (resp. $\mathcal{C}'$) is the categorial Conley–Morse index of (π, S) (resp. $(\pi', \gamma(S))$) as defined in [5] and Φ (resp. Φ') is the restriction of H_q to $\mathcal{C}$ (resp. $\mathcal{C}'$). Define the morphism $\kappa_q(\pi, S, \gamma): H_q(\pi, S) \to H_q(\pi', \gamma(S))$ by $\kappa_q(\pi, S, \gamma) = \langle F_q \rangle$. $\kappa_q(\pi, S, \gamma)$ is a Γ-module isomorphism.

(c) *Given an isolated π-invariant set S having a strongly π-admissible isolating neighbouhood and an attractor-repeller pair (A, A^*) of S relative to π, then $\gamma(S)$ is an isolated π'-invariant set having a strongly π'-admissible isolating neighbouhood, $(\gamma(A), \gamma(A^*))$ is an attractor-repeller pair of $\gamma(S)$ relative to π' and the diagram*

$$\begin{array}{ccccccccc} \longrightarrow & H_q(\pi, A) & \longrightarrow & H_q(\pi, S) & \longrightarrow & H_q(\pi, A^*) & \longrightarrow & H_{q-1}(\pi, A) & \longrightarrow \\ & \downarrow{\scriptstyle \kappa_q(\pi,A,\gamma)} & & \downarrow{\scriptstyle \kappa_q(\pi,S,\gamma)} & & \downarrow{\scriptstyle \kappa_q(\pi,A^*,\gamma)} & & \downarrow{\scriptstyle \kappa_{q-1}(\pi,A,\gamma)} & \\ \longrightarrow & H_q(\pi', \gamma(A)) & \longrightarrow & H_q(\pi', \gamma(S)) & \longrightarrow & H_q(\pi', \gamma(A^*)) & \longrightarrow & H_{q-1}(\pi', \gamma(A)) & \longrightarrow \end{array} \tag{3.2}$$

commutes.

Proof. Part (a) is obvious. To prove the independence of $\langle F_q \rangle$ of the choice of (Y, Z), let $(\widehat{Y}, \widehat{Z})$ be another FM-index pair for (π, S) with $\mathrm{Cl}_X(\widehat{Y} \setminus \widehat{Z})$ strongly π-admissible. By Proposition 4.6, Lemma 4.8 and Proposition 2.5 in [5] we obtain sets L_1, L_2, W and $\widehat{W}$ such that $(L_1, L_2) \subset (Y \cap \widehat{Y}, W \cap \widehat{W})$, $Z \subset W$, $\widehat{Z} \subset \widehat{W}$ and (L_1, L_2), (Y, W) and $(\widehat{Y}, \widehat{W})$ are FM-index pairs for (π, S) such that $\mathrm{Cl}_X(L_1 \setminus L_2)$, $\mathrm{Cl}_X(Y \setminus Z)$ and $\mathrm{Cl}_X(\widehat{Y} \setminus \widehat{W})$ are strongly π-admissible. We thus obtain the commutative diagram

$$\begin{array}{ccc} H_q(Y/Z, \{[Z]\}) & \xrightarrow{H_q(\gamma_{Y,Z})} & H_q(\gamma(Y)/\gamma(Z), \{[\gamma(Z)]\}) \\ \downarrow & & \downarrow \\ H_q(Y/W, \{[W]\}) & \xrightarrow{H_q(\gamma_{Y,W})} & H_q(\gamma(Y)/\gamma(W), \{[\gamma(W)]\}) \\ \uparrow & & \uparrow \\ H_q(L_1/L_2, \{[L_2]\}) & \xrightarrow{H_q(\gamma_{L_1,L_2})} & H_q(\gamma(L_1)/\gamma(L_2), \{[\gamma(L_2)]\}) \\ \downarrow & & \downarrow \\ H_q(\widehat{Y}/\widehat{W}, \{[\widehat{W}]\}) & \xrightarrow{H_q(\gamma_{\widehat{Y},\widehat{W}})} & H_q(\gamma(\widehat{Y})/\gamma(\widehat{W}), \{[\gamma(\widehat{W})]\}) \\ \uparrow & & \uparrow \\ H_q(\widehat{Y}/\widehat{Z}, \{[\widehat{Z}]\}) & \xrightarrow{H_q(\gamma_{\widehat{Y},\widehat{Z}})} & H_q(\gamma(\widehat{Y})/\gamma(\widehat{Z}), \{[\gamma(\widehat{Z})]\}) \end{array}$$

whose vertical maps are inclusion induced. Hence, by Proposition 4.5 in [5], these maps are induced by the unique morphisms in $\mathcal{C}$ (resp. in $\mathcal{C}'$) between the corresponding objects of these connected simple systems. In particular, the vertical maps are all bijective, and so we may invert the upward pointing arrows and then compose the columns to obtain the commutative diagram

$$\begin{array}{ccc} H_q(Y/Z,\{[Z]\}) & \xrightarrow{H_q(\gamma_{Y,Z})} & H_q(\gamma(Y)/\gamma(Z),\{[\gamma(Z)]\}) \\ \big\downarrow & & \big\downarrow \\ H_q(\widehat{Y}/\widehat{Z},\{[\widehat{Z}]\}) & \xrightarrow[H_q(\gamma_{\widehat{Y},\widehat{Z}})]{} & H_q(\gamma(\widehat{Y})/\gamma(\widehat{Z}),\{[\gamma(\widehat{Z})]\}) \end{array} \tag{3.3}$$

where the vertical maps are induced by the corresponding morphism in $\mathcal{C}$ (resp. in $\mathcal{C}'$). Now an application of Proposition 2.4 to diagram (3.3) completes the proof of part (b) of the theorem. To prove part (c) let (N_1,N_2,N_3) be an FM-index triple for (π,S,A,A^*) with $\mathrm{Cl}_X(N_1\setminus N_3)$ strongly π-admissible. It follows that $(N_1',N_2',N_3') := (\gamma(N_1),\gamma(N_2),\gamma(N_3))$ is an FM-index triple for $(\pi',\gamma(S),\gamma(A),\gamma(A^*))$ such that $\mathrm{Cl}_{X'}(\gamma(N_1)\setminus\gamma(N_3))$ is strongly π'-admissible. We thus have the following commutative diagram

$$\begin{array}{ccccc} \Delta(N_2/N_3)/\Delta(\{[N_3]\}) & \longrightarrow & \Delta(N_1/N_3)/\Delta(\{[N_3]\}) & \longrightarrow & \Delta(N_1/N_2)/\Delta(\{[N_2]\}) \\ \big\downarrow{\scriptstyle\Delta(\gamma_{N_2,N_3})} & & \big\downarrow{\scriptstyle\Delta(\gamma_{N_1,N_3})} & & \big\downarrow{\scriptstyle\Delta(\gamma_{N_1,N_2})} \\ \Delta(N_2'/N_3')/\Delta(\{[N_3']\}) & \longrightarrow & \Delta(N_1'/N_3')/\Delta(\{[N_3']\}) & \longrightarrow & \Delta(N_1'/N_3')/\Delta(\{[N_3']\}) \end{array} \tag{3.4}$$

with inclusion induced weakly exact rows (in view of Proposition 2.8). Applying Proposition 2.7 to diagram (3.4) we obtain the induced long commutative ladder with exact rows. An application of the $\langle\,\cdot\,,\,\cdot\,\rangle$-operation to that ladder and using part (b) we obtain diagram (3.2). This proves part (*c*). □

The following result is well-known (cf. Lemma 3 in Section 22 of [1]).

Proposition 3.3. *Let $k\in\mathbb{N}$, E be a k-dimensional normed space and $\widetilde{\pi}$ be the semiflow generated by the equation $\dot{y}=Ly$, where $L\colon E\to E$ is a linear map with all eigenvalues having positive real parts. Let π_k be the semiflow on $\mathbb{R}^k$ generated by the ordinary differential equation $\dot{u}=u$. Then there exists a homeomorphism $\alpha_k\colon E\to\mathbb{R}^k$ which conjugates $\widetilde{\pi}$ with π_k.*

Theorem 3.4. *Theorem* 3.1 *holds whenever $k\in\mathbb{N}$, $E:=\mathbb{R}^k$, and $\widetilde{\pi}:=\pi_k$, where π_k is as in Proposition* 3.3.

Proof of Theorem 3.1 using Theorem 3.4. Let $\gamma\colon X\times E\to X\times\mathbb{R}^k$ be given by $(x,u)\mapsto(x,\alpha_k(u))$, $(x,u)\in X\times E$, where α_k is as in Proposition 3.3. Then γ is a homeomorphism which conjugates $\pi\times\widetilde{\pi}$ with $\pi\times\pi_k$. If S is an isolated π-invariant set having a strongly π-admissible isolating neighbouhood then let

$$\theta_q(\pi,\pi_k,S)\colon H_q(\pi\times\pi_k,S\times\{0_{\mathbb{R}^k}\})\to H_{q-k}(\pi,S),$$

be the Γ-module isomorphism which exists by Theorem 3.4. Now use Theorem 3.2 with our choice of γ. It follows that $\gamma(S\times\{0_E\})=S\times\{0_{\mathbb{R}^k}\}$. Set

$$\theta_q(\pi,\widetilde{\pi},S):=\theta_q(\pi,\pi_k,S)\circ\kappa_q(\pi,S,\gamma).$$

The family of these Γ-isomorphisms clearly satisfies the conclusions of Theorem 3.1. $\square$

The next result is the crucial step in the proof of Theorem 3.4.

Theorem 3.5. *Theorem* 3.4 *holds for* $k=1$.

Proof of Theorem 3.4 using Theorem 3.5. The proof is by induction on $k\in\mathbb{N}$. Theorem 3.5 implies that Theorem 3.4 holds for $k=1$. Suppose that Theorem 3.4 holds for some k. Let X be a metric space and let π be a local semiflow on X. Notice that the semiflow $\pi\times\pi_{k+1}$ is conjugated to the semiflow $(\pi\times\pi_k)\times\pi_1$ by the homeomorphism $\phi\colon X\times\mathbb{R}^{k+1}\to(X\times\mathbb{R}^k)\times\mathbb{R}$ given by $\phi(x,u_1,\dots,u_{k+1})=((x,u_1,\dots,u_k),u_{k+1})$.

Let S be an isolated π-invariant set having a strongly π-admissible isolating neighbouhood. We use Theorem 3.2 with $\gamma:=\phi$ to obtain the Γ-module isomorphism $\kappa_q(\pi\times\pi_{k+1},S,\phi)$ from $H_q(\pi\times\pi_{k+1},S\times\{0_{\mathbb{R}^{k+1}}\})$ to $H_q((\pi\times\pi_k)\times\pi_1,(S\times\{0_{\mathbb{R}^k}\})\times\{0_{\mathbb{R}}\})$ as in that theorem. Using Theorem 3.5 we obtain the Γ-module isomorphism $\theta_q(\pi\times\pi_k,\pi_1,S\times\{0_{\mathbb{R}^k}\})$ from $H_q((\pi\times\pi_k)\times\pi_1,(S\times\{0_{\mathbb{R}^k}\})\times\{0_{\mathbb{R}}\})$ to $H_{q-1}(\pi\times\pi_k,S\times\{0_{\mathbb{R}^k}\})$, as in that theorem. Since the present Theorem 3.4 is valid for k, there is the Γ-module isomorphism $\theta_{q-1}(\pi,\pi_k,S)$ from $H_{q-1}(\pi\times\pi_k,S\times\{0_{\mathbb{R}^k}\})$ to $H_{(q-1)-k}(\pi,S)$, as in this theorem.

Define the Γ-module isomorphism

$$\theta_q(\pi\times\pi_{k+1},S)=\colon H_q(\pi\times\pi_{k+1},S\times\{0_{\mathbb{R}^{k+1}}\})\to H_{q-k-1}(\pi,S)$$

by

$$\theta_q(\pi\times\pi_{k+1},S):=\theta_{q-1}(\pi,\pi_k,S)\circ\theta_q(\pi\times\pi_k,\pi_1,S\times\{0_{\mathbb{R}^k}\})\circ\kappa_q(\pi\times\pi_{k+1},S,\phi).$$

The family $\theta_q(\pi\times\pi_{k+1},S)$ obviously satisfies the conclusions of Theorem 3.4 for $k+1$. $\square$

The rest of this paper is devoted to the proof of Theorem 3.5.

For the rest of this paper let X be a metric space and π be a local semiflow on X. Since Theorem 3.5 is obvious for $X=\emptyset$ we assume that $X\neq\emptyset$.

4. Construction of the suspension isomorphism

Definition 4.1. Let (N,Y,Z) be a triple of closed subsets of X with $N\neq\emptyset$ and $Z\subset Y\subset N$. Define

$$\begin{aligned} E(Y) &:= Y\times[-1,1]\cup N\times\{-1,1\},\\ E(Z) &:= Z\times[-1,1]\cup N\times\{-1,1\},\\ \Omega(Y,Z) &:= E(Y)/E(Z). \end{aligned}$$

Define further $I_0:=\{0\}$, $I_1:=[-1,0]$, $I_2:=[0,1]$ and

$$E_k(Y,Z):=Y\times I_k\cup E(Z),\quad k\in\{0,1,2\}.$$

Let $p_{Y,Z}\colon Y\times[-1,1]\cup N\times\{-1,1\}\to\Omega(Y,Z)$ be the quotient map and define

$$\begin{aligned} \Omega_1(Y,Z) &:= p_{Y,Z}(E_1(Y,Z)),\\ \Omega_2(Y,Z) &:= p_{Y,Z}(E_2(Y,Z)),\\ \Omega_0(Y,Z) &:= \Omega_1(Y,Z)\cap\Omega_2(Y,Z), \end{aligned}$$

and let z_{YZ} be the base-point of $\Omega(Y,Z)$, i.e. $\{z_{Y,Z}\}=p_{Y,Z}(E(Z))$.

Remark 4.2. It is clear that $\Omega_0(Y,Z)=p_{Y,Z}(E_0(Y,Z))$. Moreover, for $k\in\{0,1,2\}$, $\Omega_k(Y,Z)$ and $E_k(Y,Z)/E(Z)$ are identical, both as sets and as topological spaces. In fact, since $p_{Y,Z}^{-1}(p_{Y,Z}(E_k(Y,Z)))=E_k(Y,Z)$ and $E_k(Y,Z)$ is closed in $E(Y)$, it follows that the restriction of $p_{Y,Z}$ to $E_k(Y,Z)$ is a quotient map from $E_k(Y,Z)$ to $p_{Y,Z}(E_k(Y,Z))=\Omega_k(Y,Z)$.

For the rest of this section, let $N\neq\emptyset$ be closed in X, S be an isolated π-invariant set and (Y,Z) be an FM-index pair for (π,S) with $Y\subset N$ and $\mathrm{Cl}_X(Y\setminus Z)$ strongly π-admissible.

The following lemma holds.

Lemma 4.3. *Let $g_{Y,Z}\colon Y/Z\to(Y\times\{0\})/(Z\times\{0\})$ be induced by the assignment $x\mapsto(x,0)$ and $h_{Y,Z}\colon(Y\times\{0\})/(Z\times\{0\})\to\Omega_0(Y,Z)=E_0(Y,Z)/E(Z)$ be inclusion induced. The map $f_{Y,Z}\colon Y/Z\to\Omega_0(Y,Z)$ defined by $f_{Y,Z}=h_{Y,Z}\circ g_{Y,Z}$ is a base-point preserving homeomorphism. In particular,*

$$H_q(f_{Y,Z})\colon H_q(Y/Z,\{[Z]\})\to H_q(\Omega_0(Y,Z),\{z_{Y,Z}\})$$

is bijective for all $q\in\mathbb{Z}$.

Proof. The map $g_{Y,Z}\colon Y/Z\to(Y\times\{0\})/(Z\times\{0\})$ is clearly a base-point preserving homeomorphism. An application of Proposition 2.1 shows that the map $h_{Y,Z}\colon(Y\times\{0\})/(Z\times\{0\})\to E_0(Y,Z)/E(Z)$ is a base-point preserving homeomorphism. $\square$

Proposition 4.4. *Let* $\ell_{Y,Z}\colon(\Omega_1(Y,Z),\Omega_0(Y,Z)) \to (\Omega(Y,Z),\Omega_2(Y,Z))$ *be the inclusion induced map. Then the corresponding homology map*

$$H_q(\ell_{Y,Z})\colon H_q(\Omega_1(Y,Z),\Omega_0(Y,Z)) \to H_q(\Omega(Y,Z),\Omega_2(Y,Z))$$

is bijective for every $q \in \mathbb{Z}$.

To prove Proposition 4.4 we need some auxiliary results.

Recall that, for $s \in [0,\infty[$, $Z^{-s}(Y)$ is the set of all $x \in X$ such that, for some $t \in [0,s]$ and for all $\tau \in [0,t]$, $x\pi\tau$ is defined, $x\pi\tau \in Y$ and $x\pi t \in Z$.

Lemma 4.5. *There is an* $\overline{s} \in [0,\infty[$ *such that* $Z^{-\overline{s}}(Y)$ *is a (closed) neighbouhood of* Z *in* Y.

Proof. The lemma follows from the arguments given in the last part of the proof of Lemma 3.4 in [9] and the first part of the proof of Remark 3 on page 83 in [13]. □

Lemma 4.6. *The set* $\Omega_0(Y,Z)$ *is a strong deformation retract of a closed neighbouhood of itself in* $\Omega_1(Y,Z)$.

Proof. Set $p := p_{Y,Z}$ and $z := z_{Y,Z}$. Let a and b satisfy $0 < a < b < 1$. Let $U(Y)$ (resp. $U(Z)$) be an open neighbouhood of Y (resp. of Z) in N. It follows that the set $W' = U(Y)\times\,]-a,a[\,\cup U(Z)\times[-1,1]\cup N\times([-1,-b[\,\cup\,]b,1])$ is open in $N\times[-1,1]$ with $E_0(Y,Z) \subset W'$. This implies that the set $W' \cap E_1(Y,Z) = Y\times([-1,-b[\,\cup\,]-a,0]) \cup (Y\cap U(Z))\times[-1,0]\cup E(Z)$ is open in $E_1(Y,Z)$. It follows that whenever C is a closed neighbouhood of Z in Y, then $Y\times([-1,-b]\cup[-a,0])\cup C\times[-1,0]\cup E(Z)$ is a closed neighbouhood of $E_0(Y,Z)$ in $E_1(Y,Z)$. Set $C = Z^{-\overline{s}}(Y)$, and $A = Y\times([-1,-b]\cup[-a,0])\cup C\times[-1,0]\cup E(Z)$ where $\overline{s}$ is as in Lemma 4.5. There is a continuous map $\gamma\colon[-1,1]\times[0,1]\to[0,\overline{s}]$ with

$$\gamma(s,t) = \begin{cases} t\overline{s} & \text{if } (s,t)\in[-1,-a]\times[0,1], \\ (-t\overline{s}/a)s & \text{if } (s,t)\in[-a,0]\times[0,1], \\ 0 & \text{if } (s,t)\in[0,1]\times[0,1]. \end{cases}$$

Define the map $\phi\colon A\times[0,1]\to p(A)$ by $\phi((x,s),t) = p(x\pi\gamma(s,t),s)$ if $x\in Y$, $x\pi\gamma(s,t)$ is defined, $x\pi\tau\notin Z$ for all $\tau\in[0,\gamma(s,t)]$, and $\phi((x,s),t) = z$ otherwise. Since C is Y-positively invariant relative to π, it follows that ϕ is well-defined. We claim that ϕ is continuous. Suppose $((x_n,s_n),t_n)\to((x,s),t)$ in $A\times[0,1]$ as $n\to\infty$. We consider several cases.

Case 1. There is a $\tau\in[0,\gamma(s,t)]$ with $x\pi\tau$ defined and $x\pi\tau\notin \mathrm{Cl}_X(Y\setminus Z)$.

If $\gamma(s,t)=0$, then $x\notin \mathrm{Cl}_X(Y\setminus Z)$ then $x_n\notin\mathrm{Cl}_X(Y\setminus Z)$ for all $n\in\mathbb{N}$ sufficiently large. If $\gamma(s,t)>0$, then we may assume (by the continuity of π) that $\tau\in\,]0,\gamma(s,t)[$. By the continuity of γ, we see that for all $n\in\mathbb{N}$ large enough $\tau\in\,]0,\gamma(s_n,t_n)[$, $x_n\pi\tau$ is defined and $x_n\pi\tau\notin\mathrm{Cl}_X(Y\setminus Z)$. Altogether, we obtain from the definition of ϕ that $\phi((x,s),t)=z$ and $\phi((x_n,s_n),t_n)=z$ for all $n\in\mathbb{N}$ large enough. Thus $\phi((x_n,s_n),t_n)\to\phi((x,s),t)$ in this case.

Case 2. Whenever $\tau\in[0,\gamma(s,t)]$ and $x\pi\tau$ is defined, then $x\pi\tau\in\mathrm{Cl}_X(Y\setminus Z)$.

Since $x=x\pi 0$ it follows that $x\in\mathrm{Cl}_X(Y\setminus Z)$ and so, as π does not explode in $\mathrm{Cl}_X(Y\setminus Z)$, we obtain that $x\pi\gamma(s,t)$ is defined so $x_n\pi\gamma(s_n,t_n)$ is defined for all $n\in\mathbb{N}$ large enough. Moreover, $x\pi\tau\in Y$ for all $\tau\in[0,\gamma(s,t)]$ and either $x\pi\tau\notin Z$ for all $\tau\in[0,\gamma(s,t)]$ so $\phi((x,s),t)=p(x\pi\gamma(s,t),s)$ or else $x\pi\tau\in Z$ for some $\tau\in[0,\gamma(s,t)]$ so, by the Y-invariance of Z, $x\pi\gamma(s,t)\in Z$ so in this case $\phi((x,s),t)=z=p(x\pi\gamma(s,t),s)$. Thus, in both cases,

$$\phi((x,s),t)=p(x\pi\gamma(s,t),s). \tag{4.1}$$

We shall prove that for every neighbouhood V of $\phi((x,s),t)$ in $p(A)$ there is an $n_0\in\mathbb{N}$ such that $\phi((x_n,s_n),t_n)\in V$ for all $n\geq n_0$. Suppose this is not true. Then there is a neighbouhood V of $\phi((x,s),t)$ in $p(A)$ and a subsequence of $((x_n,s_n),t_n)_n$, again denoted by $((x_n,s_n),t_n)_n$, such that

$$\phi((x_n,s_n),t_n)\notin V \quad \text{for all } n\in\mathbb{N}. \tag{4.2}$$

By taking further subsequences, if necessary, we may thus assume one of the three cases:

Case 2.1. For every $n\in\mathbb{N}$: $x_n\in N\setminus Y$.

Case 2.2. For every $n\in\mathbb{N}$: $x_n\in Y$ and $x_n\pi\tau\notin Z$ all $\tau\in[0,\gamma(s_n,t_n)]$.

Case 2.3. For every $n\in\mathbb{N}$: $x_n\in Y$ and $x_n\pi\tau\in Z$ for some $\tau=\tau_n\in[0,\gamma(s_n,t_n)]$.

In Case 2.1 we have $s_n\in\{-1,1\}$ for all $n\in\mathbb{N}$ so $s\in\{-1,1\}$. It follows that $\phi((x_n,s_n),t_n)=z$ for all $n\in\mathbb{N}$. Moreover, either $\phi((x,s),t)=p(x\pi\gamma(s,t),s)$ or else $\phi((x,s),t)=z$. Since $(x\pi\gamma(s,t),s)\in N\times\{-1,1\}\subset E(Z)$ we obtain $\phi((x,s),t)=z$ so $z\in V$, so $\phi((x_n,s_n),t_n)\in V$ for all $n\in\mathbb{N}$, a contradiction to (4.2).

In Case 2.2 we have $\phi((x_n,s_n),t_n)=p(x_n\pi\gamma(s_n,t_n),s_n)$ for all $n\in\mathbb{N}$, so $\phi((x_n,s_n),t_n)\to p(x\pi\gamma(s,t),s)=\phi((x,s),t)$ in view of (4.1), a contradiction to (4.2).

Case 2.3 implies that, on the one hand, $\phi((x_n,s_n),t_n)=z$ for all $n\in\mathbb{N}$, and on the other hand, $x\pi\tau\in Z$ for some $\tau\in[0,\gamma(s,t)]$. The latter implies that $\phi((x,s),t)=z$. This again contradicts (4.2).

It follows that, indeed, ϕ is continuous. Since $\phi(E(Z)\times[0,1])=\{z\}$, ϕ induces a unique map $\Phi\colon p(A)\times[0,1]\to p(A)$ with $\Phi\circ(p\times\mathrm{Id}_{[0,1]})=\phi$. It follows from Whitehead's Lemma that Φ is continuous. Moreover, from the properties of γ it also follows that

$$\Phi(w,0) = w \quad \text{for all } w \in p(A), \tag{4.3}$$

$$\Phi(p(x,s),t) = p(x,s) \quad \text{for all } (x,s) \in E_0(Y,Z),\ t \in [0,1], \tag{4.4}$$

$$\Phi(p(x,s),1) = z \quad \text{for all } (x,s) \in C \times [-1,-a]. \tag{4.5}$$

There is a continuous function $\alpha\colon [-1,1] \times [0,1] \to [-1,1]$ such that

$$\alpha(s,t) = \begin{cases} s(1-t) - t & \text{if } (s,t) \in [-1,-b] \times [0,1], \\ s(1-t) - t + (s+b)t/(b-a) & \text{if } (s,t) \in [-b,-a] \times [0,1], \\ s(1-t) & \text{if } (s,t) \in [-a,0] \times [0,1], \\ s(1-t) + ts/b & \text{if } (s,t) \in [0,b] \times [0,1], \\ s(1-t) + t & \text{if } (s,t) \in [b,1] \times [0,1]. \end{cases}$$

The function $\psi\colon A \times [0,1] \to A$, $((x,s),t) \mapsto (x, \alpha(s,t))$, is defined, continuous and $\psi(E(Z) \times [0,1]) \subset E(Z)$. Thus ψ induces a unique map $\Psi\colon p(A) \times [0,1] \to p(A)$ with $\Psi \circ (p \times \mathrm{Id}_{[0,1]}) = p \circ \psi$. It follows from Whitehead's Lemma that Ψ is continuous. Moreover, from the properties of α it also follows that

$$\Psi(w,0) = w \quad \text{for all } w \in p(A), \tag{4.6}$$

$$\Psi(p(x,s),t) = p(x,s) \quad \text{for all } (x,s) \in E_0(Y,Z),\ t \in [0,1], \tag{4.7}$$

$$\Psi(p(x,s),1) = z \quad \text{for all } (x,s) \in Y \times [-1,-b], \tag{4.8}$$

$$\Psi(p(x,s),1) = p(x,0) \quad \text{for all } (x,s) \in Y \times [-a,0]. \tag{4.9}$$

Define the map $\Upsilon\colon p(A) \times [0,1] \to p(A)$ by

$$\Upsilon(w,t) = \begin{cases} \Phi(w,2t) & \text{if } (w,t) \in p(A) \times [0,(1/2)], \\ \Psi(\Phi(w,1), 2t-1) & \text{if } (w,t) \in p(A) \times [(1/2),1]. \end{cases}$$

It follows from (4.6) that Υ is continuous. Moreover, (4.3) implies that

$$\Upsilon(w,0) = w \quad \text{for all } w \in p(A).$$

(4.4) and (4.7) imply that

$$\Upsilon(p(x,s),t) = p(x,s) \quad \text{for all } (x,s) \in E_0(Y,Z),\ t \in [0,1].$$

(4.5), (4.8) and (4.9) together with the fact that Φ, Ψ and so Υ are base-point preserving imply that

$$\Upsilon(p(x,s),1) \in p(E_0(Y,Z)) \quad \text{for all } (x,s) \in A.$$

The lemma is proved. □

Proof of Proposition 4.4. Lemma 4.6 together with classical arguments from algebraic topology (see e.g. Theorem 1.8 in [10] and its proof) completes the proof of the proposition. □

Proposition 4.4 and standard results from algebraic topology (cf. Theorem 1.4 in [10]) imply that there exists a long exact Mayer–Vietoris sequence

$$\xrightarrow{\gamma_{q+1}} R_q \xrightarrow{\alpha_q} S_q \xrightarrow{\beta_q} T_q \xrightarrow{\gamma_q} R_{q-1} \xrightarrow{\alpha_{q-1}} S_{q-1} \xrightarrow{\beta_{q-1}} \tag{4.10}$$

where, for $q \in \mathbb{Z}$,

$$\begin{aligned} R_q &= R_q(Y,Z) = H_q(\Omega_0(Y,Z),\{z_{Y,Z}\}), \\ T_q &= T_q(Y,Z) = H_q(\Omega(Y,Z),\{z_{Y,Z}\}), \\ S_q &= S_q(Y,Z) = H_q(\Omega_1(Y,Z),\{z_{Y,Z}\}) \oplus H_q(\Omega_2(Y,Z),\{z_{Y,Z}\}), \end{aligned} \tag{4.11}$$

$$\begin{aligned} \rho_{Y,Z}&\colon (\Omega(Y,Z),\{z_{Y,Z}\}) \to (\Omega(Y,Z),\Omega_2(Y,Z)), \\ \mu_{k,Y,Z}&\colon (\Omega_0(Y,Z),\{z_{Y,Z}\}) \to (\Omega_k(Y,Z),\{z_{Y,Z}\}), \\ \nu_{k,Y,Z}&\colon (\Omega_k(Y,Z),\{z_{Y,Z}\}) \to (\Omega(Y,Z),\{z_{Y,Z}\}), \quad k \in \{1,2\}, \end{aligned} \tag{4.12}$$

are inclusions and

$$\begin{aligned} \alpha_q &:= H_q(\mu_{1,Y,Z}) \oplus H_q(\mu_{2,Y,Z}), \\ \beta_q &:= H_q(\nu_{1,Y,Z}) - H_q(\nu_{2,Y,Z}), \\ \gamma_q &:= \partial_{q*}(\Omega_1(Y,Z),\Omega_0(Y,Z),\{z_{Y,Z}\}) \circ H_q(\ell_{Y,Z})^{-1} \circ H_q(\rho_{Y,Z}). \end{aligned} \tag{4.13}$$

Here, $\partial_{q*}(\Omega_1(Y,Z),\Omega_0(Y,Z),\{z_{Y,Z}\})$ is the connecting homomorphism of the triple $(\Omega_1(Y,Z),\Omega_0(Y,Z),\{z_{Y,Z}\})$.

Lemma 4.7. *$H_q(\Omega_1(Y,Z),\{z_{Y,Z}\}) = 0$ and $H_q(\Omega_2(Y,Z),\{z_{Y,Z}\}) = 0$ for $q \in \mathbb{Z}$.*

Proof. The maps $f_1\colon E_1(Y,Z)\times[0,1] \to E_1(Y,Z)$, $((x,s),t) \mapsto (x,-|s|(1-t)-t)$, and $f_2\colon E_2(Y,Z)\times[0,1] \to E_2(Y,Z)$, $((x,s),t) \mapsto (x,|s|(1-t)+t)$, are defined, continuous and map $E(Z)\times[0,1]$ into $E(Z)$. Thus, by Whitehead's Lemma, we obtain the corresponding induced maps $F_1\colon \Omega_1(Y,Z)\times[0,1] \to \Omega_1(Y,Z)$ and $F_2\colon \Omega_2(Y,Z)\times[0,1] \to \Omega_2(Y,Z)$. These maps are continuous strong deformation retractions of $\Omega_1(Y,Z)$, resp. $\Omega_2(Y,Z)$, onto $\{z_{Y,Z}\}$. The lemma follows. □

Lemma 4.7 implies that $S_q = 0$ for all $q \in \mathbb{Z}$. The exactness of diagram (4.10) therefore shows that the map γ_q is bijective for all $q \in \mathbb{Z}$. Recalling that $(\Omega(Y,Z),\{z_{Y,Z}\}) = (E(Y)/E(Z),\{[E(Z)]\})$ and that, by Proposition 2.2, $(E(Y),E(Z))$ is an FM-index pair for $(\pi\times\pi_1, S\times\{0\})$, and using Lemma 4.3 we thus arrive at the following result:

Theorem 4.8.

(a) *For every $q \in \mathbb{Z}$, the map*

$$\xi_q = \xi_{q,Y,Z}\colon H_q(E(Y)/E(Z),\{[E(Z)]\}) \to H_{q-1}(Y/Z,\{[Z]\})$$

defined by

$$\xi_q = H_{q-1}(f_{Y,Z})^{-1} \circ \widetilde{\partial}_q(Y,Z) \circ H_q(\ell_{Y,Z})^{-1} \circ H_q(\rho_{Y,Z}),$$

is bijective, where we set

$$\widetilde{\partial}_q(Y,Z) := (-1)^q \partial_{q*}(\Omega_1(Y,Z), \Omega_0(Y,Z), \{z_{Y,Z}\}).$$

(b) *Whenever* $(\widehat{Y}, \widehat{Z})$ *is another* FM*-index pair for* (π, S) *such that* $\mathrm{Cl}_X(\widehat{Y} \setminus \widehat{Z})$ *is strongly* π*-admissible, then the diagram*

$$\begin{array}{ccc} \Phi(A) & \xrightarrow{F} & \Phi'(A') \\ {\scriptstyle \Phi(f)}\downarrow & & \downarrow{\scriptstyle \Phi'(f')} \\ \Phi(B) & \xrightarrow[G]{} & \Phi'(B') \end{array} \tag{4.14}$$

commutes. Here

$$A = (E(Y)/E(Z), \{[E(Z)]\}), \qquad B = (E(\widehat{Y})/E(\widehat{Z}), \{[E(\widehat{Z})]\}),$$
$$A' = (Y/Z, \{[Z]\}), \qquad B' = (\widehat{Y}/\widehat{Z}, \{[\widehat{Z}]\}),$$

$\mathcal{C}$ *is the categorial Morse index of* $(\pi \times \pi_1, S \times \{0\})$, $\mathcal{C}'$ *is the categorial Morse index of* (π, S), Φ *is the restriction of the functor* H_q *to* $\mathcal{C}$, Φ' *is the restriction of* H_{q-1} *to* $\mathcal{C}'$, $F = \xi_{q,Y,Z}$, $G = \xi_{q,\widehat{Y},\widehat{Z}}$ *and* f (*resp.* f') *is the unique morphism in* $\mathcal{C}$ (*resp. in* $\mathcal{C}'$) *from* A *to* B (*resp. from* A' *to* B').

(c) $\langle F \rangle_{\mathcal{C},\Phi,\mathcal{C}',\Phi'} = \langle G \rangle_{\mathcal{C},\Phi,\mathcal{C}',\Phi'}$.

Remark 4.9. The reason for including the factor $(-1)^q$ in the definition of $\xi_{q,Y,Z}$ is crucial and will become apparent in Section 6.

Proof of Theorem 4.8. We have just proved part (a). Part (c) follows from part (b) by Proposition 2.4. To prove part (b) let us first assume that $(Y,Z) \subset (\widehat{Y}, \widehat{Z})$. Given $q \in \mathbb{Z}$ consider the following diagrams in the category of Γ-modules:

$$\begin{array}{ccc} H_q(\Omega(Y,Z), \{z_{Y,Z}\}) & \xrightarrow{H_q(\rho_{Y,Z})} & H_q(\Omega(Y,Z), \Omega_2(Y,Z)) \\ \downarrow & & \downarrow \\ H_q(\Omega(\widehat{Y},\widehat{Z}), \{z_{\widehat{Y},\widehat{Z}}\}) & \xrightarrow[H_q(\rho_{\widehat{Y},\widehat{Z}})]{} & H_q(\Omega(\widehat{Y},\widehat{Z}), \Omega_2(\widehat{Y},\widehat{Z})) \end{array}$$

$$\begin{array}{ccc} H_q(\Omega(Y,Z), \Omega_2(Y,Z)) & \xrightarrow{H_q(\ell_{Y,Z})^{-1}} & H_q(\Omega_1(Y,Z), \Omega_0(Y,Z)) \\ \downarrow & & \downarrow \\ H_q(\Omega(\widehat{Y},\widehat{Z}), \Omega_2(\widehat{Y},\widehat{Z})) & \xrightarrow[H_q(\ell_{\widehat{Y},\widehat{Z}})^{-1}]{} & H_q(\Omega_1(\widehat{Y},\widehat{Z}), \Omega_0(\widehat{Y},\widehat{Z})) \end{array}$$

$$\begin{array}{ccc} H_q(\Omega_1(Y,Z),\Omega_0(Y,Z)) & \xrightarrow{\widetilde{\partial}_q(Y,Z)} & H_{q-1}(\Omega_0(Y,Z),\{z_{Y,Z}\}) \\ \downarrow & & \downarrow \\ H_q(\Omega_1(\widehat{Y},\widehat{Z}),\Omega_0(\widehat{Y},\widehat{Z})) & \xrightarrow[\widetilde{\partial}_q(\widehat{Y},\widehat{Z})]{} & H_{q-1}(\Omega_0(\widehat{Y},\widehat{Z}),\{z_{\widehat{Y},\widehat{Z}}\}) \end{array}$$

$$\begin{array}{ccc} H_{q-1}(\Omega_0(Y,Z),\{z_{Y,Z}\}) & \xrightarrow{H_{q-1}(f_{Y,Z})^{-1}} & H_{q-1}(Y/Z,\{[Z]\}) \\ \downarrow & & \downarrow \\ H_{q-1}(\Omega_0(\widehat{Y},\widehat{Z}),\{z_{\widehat{Y},\widehat{Z}}\}) & \xrightarrow[H_{q-1}(f_{\widehat{Y},\widehat{Z}})^{-1}]{} & H_{q-1}(\widehat{Y}/\widehat{Z},\{[\widehat{Z}]\}) \end{array}$$

Here, the vertical maps are inclusion induced. All these diagrams clearly commute (the third diagram commutes by the naturality of connecting homomorphisms of space triples). Composing these diagrams we thus obtain the commutative diagram

$$\begin{array}{ccc} H_q(E(Y)/E(Z),\{[E(Z)]\}) & \xrightarrow{\xi_{q,Y,Z}} & H_{q-1}(Y/Z,\{[Z]\}) \\ \downarrow & & \downarrow \\ H_q(E(\widehat{Y})/E(\widehat{Z}),\{[E(\widehat{Z})]\}) & \xrightarrow[\xi_{q,\widehat{Y},\widehat{Z}}]{} & H_{q-1}(\widehat{Y}/\widehat{Z},\{[\widehat{Z}]\}) \end{array} \tag{4.15}$$

with inclusion induced vertical maps. Diagram (4.15) is just diagram (4.14) spelled out. Now an application of Proposition 4.5 in [5] completes the proof of part (b) is in the special case $(Y,Z)\subset(\widehat{Y},\widehat{Z})$. In the general case we use Proposition 4.6, Lemma 4.8 and Proposition 2.5 in [5] to obtain sets L_1, L_2, W and $\widehat{W}$ such that $(L_1,L_2)\subset(Y\cap\widehat{Y},W\cap\widehat{W})$, $Z\subset W$, $\widehat{Z}\subset\widehat{W}$ and (L_1,L_2), (Y,W) and $(\widehat{Y},\widehat{W})$ are FM-index pairs for (π,S) such that $\mathrm{Cl}_X(L_1\setminus L_2)$, $\mathrm{Cl}_X(Y\setminus Z)$ and $\mathrm{Cl}_X(\widehat{Y}\setminus\widehat{W})$ are strongly π-admissible. By the special case just proved we thus obtain the commutative diagram

$$\begin{array}{ccc} H_q(E(Y)/E(Z),\{[E(Z)]\}) & \xrightarrow{\xi_{q,Y,Z}} & H_{q-1}(Y/Z,\{[Z]\}) \\ \downarrow & & \downarrow \\ H_q(E(Y)/E(W),\{[E(W)]\}) & \xrightarrow{\xi_{q,Y,W}} & H_{q-1}(Y/W,\{[W]\}) \\ \uparrow & & \uparrow \\ H_q(E(L_1)/E(L_2),\{[E(L_2)]\}) & \xrightarrow{\xi_{q,L_1,L_2}} & H_{q-1}(L_1/L_2,\{[L_2]\}) \\ \downarrow & & \downarrow \\ H_q(E(\widehat{Y})/E(\widehat{W}),\{[E(\widehat{W})]\}) & \xrightarrow{\xi_{q,\widehat{Y},\widehat{W}}} & H_{q-1}(\widehat{Y}/\widehat{W},\{[\widehat{W}]\}) \\ \uparrow & & \uparrow \\ H_q(E(\widehat{Y})/E(\widehat{Z}),\{[E(\widehat{Z})]\}) & \xrightarrow{\xi_{q,\widehat{Y},\widehat{Z}}} & H_{q-1}(\widehat{Y}/\widehat{Z},\{[\widehat{Z}]\}) \end{array}$$

The vertical maps in the above diagram are all inclusion induced, thus they are induced by the unique morphisms in $\mathcal{C}(\pi\times\pi_1, S\times\{0\})$ (resp. in $\mathcal{C}(\pi, S)$) between the corresponding objects of these connected simple systems. In particular, the vertical maps are all bijective and so we may invert the upward pointing arrows then compose the columns to obtain a commutative diagram of the form (4.15) where the vertical maps are induced by the corresponding morphism in $\mathcal{C}(\pi\times\pi_1, S\times\{0\})$ (resp. in $\mathcal{C}(\pi, S)$). This completes the proof of part (b) of the theorem. $\square$

In view of Theorem 4.8 we can now make the following definition.

Definition 4.10. Given an isolated π-invariant set S having a strongly π-admissible isolating neighbouhood and $q\in\mathbb{Z}$, let

$$\theta_q(\pi,\pi_1,S)\colon H_q(\pi\times\pi_1, S\times\{0_{\mathbb{R}}\})\to H_{q-1}(\pi,S)$$

be defined by $\theta_q(\pi,\pi_1,S):=\langle F\rangle_{\mathcal{C},\Phi,\mathcal{C}',\Phi'}$, where $\mathcal{C}$ is the categorial Morse index of $(\pi\times\pi_1, S\times\{0_{\mathbb{R}}\})$, $\mathcal{C}'$ is the categorial Morse index of (π,S), Φ is the restriction of the functor H_q to $\mathcal{C}$, Φ' is the restriction of H_{q-1} to $\mathcal{C}'$ and $F=\xi_{q,Y,Z}$ with $\xi_{q,Y,Z}$ defined in part (a) of Theorem 4.8. $\theta_q(\pi,\pi_1,S)$ is a well-defined Γ-module isomorphism, called the *suspension isomorphism from $H_q(\pi\times\pi_1,S\times\{0_{\mathbb{R}}\})$ to $H_{q-1}(\pi,S)$*.

Remark 4.11. The existence proof of both the suspension isomorphism $\theta_q(\pi,\pi_1,S)$ as well as general suspension isomorphism $\theta_q(\pi,\widetilde{\pi},S)$ of Theorem 3.1 does not use any particular properties of singular homology and so the result holds for an arbitrary (unreduced) homology theory with values in Γ-modules. On the other hand, the existence of long exact homology sequences for attractor-repeller pairs and the commutativity of the suspension isomorphism with such sequences, established in Section 6 below, does depend on special properties of singular homology.

5. Weakly exact sequences and anticommutativity of the connecting homomorphisms

Recall the following simple and known result.

Lemma 5.1. *In any category, consider the diagram*

$$\begin{tikzcd}
 & D \arrow[dr] \arrow[dd] & \\
B \arrow[ur] \arrow[rr] \arrow[dd] & & C \arrow[dd] \\
 & \widehat{D} \arrow[dr] & \\
\widehat{B} \arrow[ur] \arrow[rr] & & \widehat{C}
\end{tikzcd}$$

of morphisms. Suppose that the diagrams

$$
\tag{5.1}
\begin{tikzcd}
 & D \arrow[dr, "\rho"] & \\
B \arrow[ur, "Q"] \arrow[rr, "\kappa"'] & & C
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\kappa"] \arrow[d, "\gamma_B"'] & C \arrow[d, "\gamma_C"] \\
\widehat{B} \arrow[r, "\widehat{\kappa}"'] & \widehat{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \widehat{D} \arrow[dr, "\widehat{\rho}"] & \\
\widehat{B} \arrow[ur, "\widehat{Q}"] \arrow[rr, "\widehat{\kappa}"'] & & \widehat{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "Q"] \arrow[d, "\gamma_B"'] & D \arrow[d, "\gamma_D"] \\
\widehat{B} \arrow[r, "\widehat{Q}"'] & \widehat{D}
\end{tikzcd}
$$

commutes and that Q is epic. Then the diagram

$$
\begin{tikzcd}
D \arrow[r, "\rho"] \arrow[d, "\gamma_D"'] & C \arrow[d, "\gamma_C"] \\
\widehat{D} \arrow[r, "\widehat{\rho}"'] & \widehat{C}
\end{tikzcd}
$$

commutes.

Proof. We must show that $\widehat{\rho} \circ \gamma_D = \gamma_C \circ \rho$. Since Q is epic we only need to prove that $(\widehat{\rho} \circ \gamma_D) \circ Q = (\gamma_C \circ \rho) \circ Q$. The commutativity of the diagrams in (5.1) implies that

$$
\begin{aligned}
(\widehat{\rho} \circ \gamma_D) \circ Q &= \widehat{\rho} \circ (\gamma_D \circ Q) = \widehat{\rho} \circ (\widehat{Q} \circ \gamma_B) = (\widehat{\rho} \circ \widehat{Q}) \circ \gamma_B \\
&= \widehat{\kappa} \circ \gamma_B = \gamma_C \circ \kappa = \gamma_C \circ (\rho \circ Q) = (\gamma_C \circ \rho) \circ Q.
\end{aligned}
$$

The proof is complete. □

Lemma 5.2. *The sequence*

$$
0 \longrightarrow B/\operatorname{im}\alpha \xrightarrow{\zeta} C/\operatorname{im}(\beta \circ \alpha) \xrightarrow{\eta} C/\operatorname{im}\beta \longrightarrow 0
$$

induced by chain monomorphisms $\alpha: A \to B$ and $\beta: B \to C$ is exact.

Proof. Consider the following commutative diagram of chain maps

$$
\tag{5.2}
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "\mathrm{Id}"'] & B \arrow[r, "p_1"] \arrow[d, "\beta"'] & B/\operatorname{im}\alpha \arrow[r] \arrow[d, "\zeta"] & 0 \\
0 \arrow[r] & A \arrow[r, "\beta\circ\alpha"'] \arrow[d] & C \arrow[r, "p_2"'] \arrow[d, "p_3"'] & C/\operatorname{im}(\beta\circ\alpha) \arrow[r] \arrow[d, "\eta"] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & C/\operatorname{im}\beta \arrow[r, "\mathrm{Id}"'] \arrow[d] & C/\operatorname{im}\beta \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

in which p_i, $i \in \{1,2,3\}$, are quotient maps. Since the rows and the first two columns of (5.2) are exact, the third column is exact by the 3×3-lemma. $\square$

Lemma 5.3. *Consider the following commutative diagram*

$$\begin{array}{ccccc} A & \xrightarrow{\iota} & B & \xrightarrow{\kappa} & C \\ \downarrow{\scriptstyle \gamma_A} & & \downarrow{\scriptstyle \gamma_B} & & \downarrow{\scriptstyle \gamma_C} \\ \widehat{A} & \xrightarrow[\widehat{\iota}]{} & \widehat{B} & \xrightarrow[\widehat{\kappa}]{} & \widehat{C} \end{array} \tag{5.3}$$

of chain morphisms in which the first row is weakly exact and the vertical arrows are isomorphisms. Then the sequence in the second row is weakly exact.

Proof. Since ι is injective, γ_A and γ_B are isomorphims, the commutativity of diagram (5.3) implies that $\widehat{\iota}$ is injective. Moreover, $\widehat{\kappa} \circ \widehat{\iota} = (\gamma_C \circ \kappa \circ \gamma_B^{-1}) \circ (\gamma_B \circ \iota \circ \gamma_A^{-1}) = \gamma_C \circ (\kappa \circ \iota) \circ \gamma_A^{-1} = 0$ since the sequence in the first row of diagram (5.3) is weakly exact. Let $\widehat{\rho} \colon \widehat{B} / \operatorname{im} \widehat{\iota} \to \widehat{C}$ be the (uniquely determined) chain map with $\widehat{\rho} \circ \widehat{Q} = \widehat{\kappa}$, where $\widehat{Q} \colon \widehat{B} \to \widehat{B} / \operatorname{im} \widehat{\iota}$ is the quotient map. To complete the proof we need to show that

(5.4) the map $H_q(\widehat{\rho}) \colon H_q(\widehat{B} / \operatorname{im} \widehat{\iota}) \to H_q(\widehat{C})$ is an isomorphism for each $q \in \mathbb{Z}$.

Since the sequence in the first row of diagram (5.3) is weakly exact, it follows that the map $H_q(\rho) \colon H_q(B / \operatorname{im} \iota) \to H_q(C)$ is an isomorphism for each $q \in \mathbb{Z}$, where $\rho \colon B / \operatorname{im} \iota \to C$ is the (uniquely determined) chain map with $\rho \circ Q = \kappa$, where $Q \colon B \to B / \operatorname{im} \iota$ is the quotient map. We can thus write diagram (5.3) in the form

$$\begin{array}{ccccccc} A & \xrightarrow{\iota} & B & \xrightarrow{Q} & B / \operatorname{im} \iota & \xrightarrow{\rho} & C \\ \downarrow{\scriptstyle \gamma_A} & & \downarrow{\scriptstyle \gamma_B} & & & & \downarrow{\scriptstyle \gamma_C} \\ \widehat{A} & \xrightarrow[\widehat{\iota}]{} & \widehat{B} & \xrightarrow[\widehat{Q}]{} & \widehat{B} / \operatorname{im} \widehat{\iota} & \xrightarrow[\widehat{\rho}]{} & \widehat{C} \end{array}$$

This diagram can be uniquely completed to yield the diagram

$$\begin{array}{ccccccc} A & \xrightarrow{\iota} & B & \xrightarrow{Q} & B / \operatorname{im} \iota & \xrightarrow{\rho} & C \\ \downarrow{\scriptstyle \gamma_A} & & \downarrow{\scriptstyle \gamma_B} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma_C} \\ \widehat{A} & \xrightarrow[\widehat{\iota}]{} & \widehat{B} & \xrightarrow[\widehat{Q}]{} & \widehat{B} / \operatorname{im} \widehat{\iota} & \xrightarrow[\widehat{\rho}]{} & \widehat{C} \end{array}$$

in which γ is an isomorphism and the first two squares commute. Lemma 5.1 now implies that the third square commutes, too. Hence, for each $q \in \mathbb{Z}$, $H_q(\widehat{\rho}) = H_q(\gamma_C) \circ H_q(\rho) \circ H_q(\gamma)^{-1}$ and so $H_q(\widehat{\rho})$ is an isomorphism. $\square$

We require the following important result from homological algebra, which in its general and explicit form needed here is due to M. Scott Osborne:

Proposition 5.4 ([12, Proposition 9.20]). *Suppose that the diagram*

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_{1,1} \arrow[r] \arrow[d] & A_{1,2} \arrow[r] \arrow[d] & A_{1,3} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{2,1} \arrow[r] \arrow[d] & A_{2,2} \arrow[r] \arrow[d] & A_{2,3} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{3,1} \arrow[r] \arrow[d] & A_{3,2} \arrow[r] \arrow[d] & A_{3,3} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
\tag{5.5}
$$

of chain morphisms is commutative and has exact columns and rows. For every $k \in \{1,2,3\}$, let $\partial^k_q\colon H_q(A_{k,3}) \to H_{q-1}(A_{k,1})$, $q \in \mathbb{Z}$, be the connecting homomorphism of the long homology sequence associated with the k-th row of (5.5) *and $\delta^k_q\colon H_q(A_{3,k}) \to H_{q-1}(A_{1,k})$, $q \in \mathbb{Z}$, be the connecting homomorphism of the long homology sequence associated with the k-th column of* (5.5)*. Then*

$$
\delta^1_{q-1} \circ \partial^3_q = -\partial^1_{q-1} \circ \delta^3_q, \ q \in \mathbb{Z}. \tag{5.6}
$$

Proposition 5.5. *Suppose that the diagram*

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
A_{1,1} \arrow[r, "\alpha_1"] \arrow[d] & A_{1,2} \arrow[r, "\beta_1"] \arrow[d] & A_{1,3} \arrow[d] \\
A_{2,1} \arrow[r, "\alpha_2"'] \arrow[d] & A_{2,2} \arrow[r, "\beta_2"'] \arrow[d] & A_{2,3} \arrow[d] \\
A_{3,1} \arrow[r, "\alpha_3"'] \arrow[d] & A_{3,2} \arrow[r, "\beta_3"'] \arrow[d] & A_{3,3} \arrow[d] \\
0 & 0 & 0
\end{tikzcd}
\tag{5.7}
$$

of chain morphisms is commutative, has exact columns and weakly exact first two rows. If $\ker \alpha_3 = 0$*, then the third row of* (5.7) *is weakly exact. Furthermore, for every $k \in \{1,2,3\}$, let $\widehat{\partial}^k_q\colon H_q(A_{k,3}) \to H_{q-1}(A_{k,1})$, $q \in \mathbb{Z}$ be the connecting homomorphism of the long homology sequence associated with the k-th row of* (5.7) *and* $\delta_q^k \colon H_q(A_{3,k}) \to H_{q-1}(A_{1,k})$, $q \in \mathbb{Z}$, *be the connecting homomorphism of the long homology sequence associated with the* k*-th column of* (5.7). *Then*

$$\delta^1_{q-1} \circ \widehat{\partial}^3_q = -\widehat{\partial}^1_{q-1} \circ \delta^3_q, \ q \in \mathbb{Z}. \tag{5.8}$$

Proof. Since $\beta_2 \circ \alpha_2 = 0$ and the chain map from $A_{2,1}$ to $A_{3,1}$ in (5.7) is an epimorphism, it follows that $\beta_3 \circ \alpha_3 = 0$. Thus for each $k \in \{1,2,3\}$ there is a uniquely defined chain map $\rho_k \colon A'_{k,3} := A_{k,2}/\operatorname{im}\alpha_k \to A_{k,3}$ such that $\rho_k \circ p_k = \beta_k$, where $p_k \colon A_{k,2} \to A'_{k,3}$ is the canonical quotient map. Moreover, there are uniquely determined chain maps from $A'_{1,3}$ to $A'_{2,3}$ and from $A'_{2,3}$ to $A'_{3,3}$ such that the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_{1,1} \arrow[r, "\alpha_1"] \arrow[d] & A_{1,2} \arrow[r, "p_1"] \arrow[d] & A'_{1,3} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{2,1} \arrow[r, "\alpha_2"'] \arrow[d] & A_{2,2} \arrow[r, "p_2"'] \arrow[d] & A'_{2,3} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{3,1} \arrow[r, "\alpha_3"'] \arrow[d] & A_{3,2} \arrow[r, "p_3"'] \arrow[d] & A'_{3,3} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{5.9}$$

is commutative, has exact rows and exact first two columns. It follows from the 3×3-Lemma that the third column is also exact. For every $k \in \{1,2,3\}$ and $q \in \mathbb{Z}$, let $\partial_q^{k'} \colon H_q(A'_{k,3}) \to H_{q-1}(A_{k,1})$ be the connecting homomorphism of the long homology sequence associated with the k-th row of (5.9) and $\delta_q^{3'} \colon H_q(A'_{3,3}) \to H_{q-1}(A'_{1,3})$ be the connecting homomorphism of the long homology sequence associated with the 3-th column of (5.9). Lemma 5.1 together with the commutativity of (5.7) and (5.9) implies that the diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
A'_{1,3} \arrow[r, "\rho_1"] \arrow[d] & A_{1,3} \arrow[d] \\
A'_{2,3} \arrow[r, "\rho_2"'] \arrow[d] & A_{2,3} \arrow[d] \\
A'_{3,3} \arrow[r, "\rho_3"'] \arrow[d] & A_{3,3} \arrow[d] \\
0 & 0
\end{tikzcd} \tag{5.10}$$

commutes. Passing to the long commutative homology ladder

$$\begin{array}{ccccccccc} \longrightarrow & H_q(A'_{1,3}) & \longrightarrow & H_q(A'_{2,3}) & \longrightarrow & H_q(A'_{3,3}) & \xrightarrow{\delta_q^{3'}} & H_{q-1}(A'_{1,3}) & \longrightarrow \\ & \Big\downarrow{\scriptstyle H_q(\rho_1)} & & \Big\downarrow{\scriptstyle H_q(\rho_2)} & & \Big\downarrow{\scriptstyle H_q(\rho_3)} & & \Big\downarrow{\scriptstyle H_{q-1}(\rho_1)} & \\ \longrightarrow & H_q(A_{1,3}) & \longrightarrow & H_q(A_{2,3}) & \longrightarrow & H_q(A_{3,3}) & \xrightarrow[\delta_q^{3}]{} & H_{q-1}(A_{1,3}) & \longrightarrow \end{array} \tag{5.11}$$

induced by (5.10) and using the fact that both $H_q(\rho_1)$ and $H_q(\rho_2)$ are bijective for all $q \in \mathbb{Z}$, it follows from the 5-Lemma that $H_q(\rho_3)$ is bijective for all $q \in \mathbb{Z}$. This proves that the third row of (5.7) is weakly exact.

Now, applying Proposition 5.4 to diagram (5.9) we obtain that

$$\delta^1_{q-1} \circ \partial^{3'}_q = -\partial^{1'}_{q-1} \circ \delta^{3'}_q, \quad q \in \mathbb{Z}. \tag{5.12}$$

From diagram (5.11) we obtain that

$$(H_{q-1}(\rho_1))^{-1} \circ \delta^3_q = \delta^{3'}_q \circ (H_q(\rho_3))^{-1}, \quad q \in \mathbb{Z}. \tag{5.13}$$

Now $\widehat{\partial}^3_q = \partial^{3'}_q \circ (H_q(\rho_3))^{-1}$ and $\widehat{\partial}^1_{q-1} = \partial^{1'}_{q-1} \circ (H_{q-1}(\rho_1))^{-1}$, $q \in \mathbb{Z}$. This together with (5.12) and (5.13) implies (5.8). □

Note the following simple and known result.

Lemma 5.6. *Let Y and Z be topological spaces and let $f: Y \to Z$ be a continuous and injective map. Then the inclusion induced chain morphism*

$$\Delta(f): \Delta(Y) \to \Delta(Z)$$

is injective.

Lemma 5.7. *Let Y and Z be topological spaces and let $Y_1 \subset Y$ and $Z_1 \subset Z$. Let $f: Y \to Z$ be a continuous and injective map such that $Z_1 \cap f(Y) = f(Y_1)$. Then the inclusion induced chain morphism*

$$\overline{\Delta(f)}: \Delta(Y)/\Delta(Y_1) \to \Delta(Z)/\Delta(Z_1)$$

is injective.

Proof. Let $q \in \mathbb{Z}$. We must prove that, whenever $\gamma \in \Delta_q(Y)$ is such that $\Delta_q(f)(\gamma) \in \Delta_q(Z_1)$, then $\gamma \in \Delta_q(Y_1)$. This is certainly true if $\gamma = 0$, so suppose $\gamma \neq 0$. Thus $\gamma = \sum_{k \in J} r_k \sigma_k$, where J is a finite index set, and for $k \in J$, $r_k \in \overline{G} \setminus \{0\}$ and $\sigma_k: \Delta_q \to Y$ is a singular q-simplex such that $\sigma_k \neq \sigma_l$ whenever $k \neq l$. Since f is injective it follows that $f \circ \sigma_k \neq f \circ \sigma_l$ whenever $k \neq l$. Since $\Delta_q(f)(\gamma) = \sum_{k \in J} r_k(f \circ \sigma_k)$ and $\Delta_q(f)(\gamma) \in \Delta_q(Z_1)$, it thus follows that $f\sigma_k(\Delta_q) \subset Z_1$ and so $f\sigma_k(\Delta_q) \subset Z_1 \cap f(Y)$ for all $k \in J$. Therefore, for $k \in J$, $f\sigma_k(\Delta_q) \subset f(Y_1)$ and so the injectivity of f implies that $\sigma_k(\Delta_q) \subset Y_1$. The proof is complete. □

6. The suspension isomorphism and attractor-repeller pairs

In this section we will complete the proof of Theorem 3.5. For the rest of this section, let $N \neq \emptyset$ be closed in X, (A, A^*) be an attractor-repeller pair of S relative to π and (N_1, N_2, N_3) be an FM-index triple for (π, S, A, A^*) such that $N_1 \subset N$ and $\mathrm{Cl}_X(N_1 \setminus N_3)$ is strongly π-admissible. For $i,j \in \{1,2,3\}$, $i < j$, set

$$\begin{array}{llll}
\Omega^{i,j} := \Omega(N_i, N_j), & \Omega_1^{i,j} := \Omega_1(N_i, N_j), & & \\
\Omega_2^{i,j} := \Omega_2(N_i, N_j), & \Omega_0^{i,j} := \Omega_0(N_i, N_j), & & \\
z^{i,j} := z_{N_i,N_j}, & p^{i,j} := p_{N_i,N_j}, & f^{i,j} := f_{N_i,N_j}, & \\
\rho^{i,j} := \rho_{N_i,N_j}, & \ell^{i,j} := \ell_{N_i,N_j}, & \xi_q^{i,j} := \xi_{q,N_i,N_j}, &
\end{array}$$

$q \in \mathbb{Z}$. (For the notations used here cf Definition 4.1, Lemma 4.3, Proposition 4.4, formula (4.13) and Theorem 4.8.)

Theorem 3.5 will follow from Theorem 4.8, already proved, and from the following result.

Theorem 6.1.

(a) *The inclusion induced diagram*

$$\begin{tikzcd}
\Delta(\Omega^{2,3})/\Delta(\{z^{2,3}\}) \arrow[r] \arrow[d, "\Delta(\rho^{2,3})"'] & \Delta(\Omega^{1,3})/\Delta(\{z^{1,3}\}) \arrow[r] \arrow[d, "\Delta(\rho^{1,3})"'] & \Delta(\Omega^{1,2})/\Delta(\{z^{1,2}\}) \arrow[d, "\Delta(\rho^{1,2})"'] \\
\Delta(\Omega^{2,3})/\Delta(\Omega_2^{2,3}) \arrow[r] & \Delta(\Omega^{1,3})/\Delta(\Omega_2^{1,3}) \arrow[r] & \Delta(\Omega^{1,2})/\Delta(\Omega_2^{1,2}) \\
\Delta(\Omega_1^{2,3})/\Delta(\Omega_0^{2,3}) \arrow[r] \arrow[u, "\Delta(\ell^{2,3})"] & \Delta(\Omega_1^{1,3})/\Delta(\Omega_0^{1,3}) \arrow[r] \arrow[u, "\Delta(\ell^{1,3})"] & \Delta(\Omega_1^{1,2})/\Delta(\Omega_0^{1,2}) \arrow[u, "\Delta(\ell^{1,2})"]
\end{tikzcd} \tag{6.1}$$

is commutative with weakly exact rows.

(b) *The diagram*

$$\begin{tikzcd}
\Delta(N_2/N_3)/\Delta(\{[N_3]\}) \arrow[r] \arrow[d, "\Delta(f^{2,3})"'] & \Delta(N_1/N_3)/\Delta(\{[N_3]\}) \arrow[r] \arrow[d, "\Delta(f^{1,3})"'] & \Delta(N_1/N_2)/\Delta(\{[N_2]\}) \arrow[d, "\Delta(f^{1,2})"'] \\
\Delta(\Omega_0^{2,3})/\Delta(\{z^{2,3}\}) \arrow[r] & \Delta(\Omega_0^{1,3})/\Delta(\{z^{1,3}\}) \arrow[r] & \Delta(\Omega_0^{1,2})/\Delta(\{z^{1,2}\})
\end{tikzcd} \tag{6.2}$$

is commutative with weakly exact rows. Here, the horizontal maps are inclusion induced.

(c) *The following inclusion induced diagram of chain maps is commutative with exact columns and weakly exact rows:*

$$
\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\Delta(\Omega_0^{2,3})/\Delta(\{z^{2,3}\}) & \longrightarrow & \Delta(\Omega_0^{1,3})/\Delta(\{z^{1,3}\}) & \longrightarrow & \Delta(\Omega_0^{1,2})/\Delta(\{z^{1,2}\}) \\
\downarrow & & \downarrow & & \downarrow \\
\Delta(\Omega_1^{2,3})/\Delta(\{z^{2,3}\}) & \longrightarrow & \Delta(\Omega_1^{1,3})/\Delta(\{z^{1,3}\}) & \longrightarrow & \Delta(\Omega_1^{1,2})/\Delta(\{z^{1,2}\}) \\
\downarrow & & \downarrow & & \downarrow \\
\Delta(\Omega_1^{2,3})/\Delta(\Omega_0^{2,3}) & \longrightarrow & \Delta(\Omega_1^{1,3})/\Delta(\Omega_0^{1,3}) & \longrightarrow & \Delta(\Omega_1^{1,2})/\Delta(\Omega_0^{1,2}) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}
\tag{6.3}
$$

The proof of Theorem 6.1 requires several preliminary results.

Lemma 6.2. *The inclusion induced map $\iota\colon \Omega^{2,3} \to \Omega^{1,3}$ is continuous, injective and $\iota(z^{2,3}) = z^{1,3}$. Moreover, $\iota(\Omega_i^{2,3}) \subset \Omega_i^{1,3}$ and $\Omega_i^{1,3} \cap \iota(\Omega^{2,3}) \subset \iota(\Omega_i^{2,3})$, for $i \in \{1,2\}$.*

Proof. It is clear that $\iota\colon \Omega^{2,3} \to \Omega^{1,3}$ is continuous and base-point preserving.

We will prove that the map ι is injective. Consider the following commutative diagram:

$$
\begin{array}{ccc}
E(N_2) & \xrightarrow{\kappa} & E(N_1) \\
{\scriptstyle p^{2,3}}\downarrow & & \downarrow{\scriptstyle p^{1,3}} \\
\Omega^{2,3} & \xrightarrow[\iota]{} & \Omega^{1,3}
\end{array}
$$

where $\kappa\colon E(N_2) \to E(N_1)$ is the inclusion map.

Let z, $z' \in \Omega^{2,3}$ be such that $\iota(z) = \iota(z')$. Let w, $w' \in E(N_2)$ be such that $z = p^{2,3}(w)$ and $z' = p^{2,3}(w')$. Hence $\iota(p^{2,3}(w)) = \iota(p^{2,3}(w'))$ and so

$$
p^{1,3}(w) = p^{1,3}(\kappa(w)) = p^{1,3}(\kappa(w')) = p^{1,3}(w'). \tag{6.4}
$$

(6.4) implies that either $w = w'$ and so $z = z'$ or else w, $w' \in E(N_3)$ and so $z = p^{2,3}(w) = z^{2,3} = p^{2,3}(w') = z'$. Hence, ι is injective.

Let $i \in \{1,2\}$ be fixed. Let $z \in \iota(\Omega_i^{2,3})$. We want to show that $z \in \Omega_i^{1,3}$. Let $w \in E_i(N_2,N_3)$ be such that $z = \iota(p^{2,3}(w))$. Since $E_i(N_2,N_3) \subset E(N_2)$, it follows that $z = p^{1,3}(\kappa(w))$. Now, either $w \in N_2 \times I_i$ so $w \in E_i(N_1,N_3)$ and so $z \in \Omega_i^{1,3}$, or else $w \in E(N_3)$ so $w \in E_i(N_1,N_3)$ and so $z \in \Omega_i^{1,3}$.

Let $z \in \Omega_i^{1,3} \cap \iota(\Omega^{2,3})$. We will show that $z \in \iota(\Omega_i^{2,3})$. Let $w' \in E_i(N_1, N_3)$ and $w \in E(N_2)$ such that $p^{1,3}(w') = z = \iota(p^{2,3}(w))$. Hence

$$p^{1,3}(w') = p^{1,3}(w). \tag{6.5}$$

(6.5) implies that either $w = w' \notin E(N_3)$, so $w \in N_2 \times I_i$ and so $z \in \iota(\Omega_i^{2,3})$, or else $w \in E(N_3)$ and so $z = \iota(z^{2,3}) \in \iota(\Omega_i^{2,3})$. $\square$

Corollary 6.3. *$\iota(\Omega_0^{2,3}) \subset \Omega_0^{1,3}$ and $\Omega_0^{1,3} \cap \iota(\Omega^{2,3}) \subset \iota(\Omega_0^{2,3})$.*

Proof. Notice that Lemma 6.2 and the definition of the sets $\Omega_0^{i,3}$, $i \in \{1,2\}$ imply that $\iota(\Omega_0^{2,3}) = \iota(\Omega_1^{2,3} \cap \Omega_2^{2,3}) = \iota(\Omega_1^{2,3}) \cap \iota(\Omega_2^{2,3}) \subset \Omega_1^{1,3} \cap \Omega_2^{1,3} = \Omega_0^{1,3}$. Moreover, $\Omega_0^{1,3} \cap \iota(\Omega^{2,3}) = (\Omega_1^{2,3} \cap \Omega_2^{2,3}) \cap \iota(\Omega^{2,3}) = (\Omega_1^{2,3} \cap \iota(\Omega^{2,3})) \cap (\Omega_2^{2,3} \cap \iota(\Omega^{2,3})) \subset \iota(\Omega_1^{2,3}) \cap \iota(\Omega_2^{2,3}) = \iota(\Omega_0^{2,3})$. The corollary is proved. $\square$

Corollary 6.4. *The inclusion induced chain maps $\Delta(\Omega_1^{2,3}) \to \Delta(\Omega_1^{1,3})$, $\Delta(\Omega_0^{2,3}) \to \Delta(\Omega_0^{1,3})$, $\Delta(\Omega_0^{2,3}) \to \Delta(\Omega_1^{2,3})$ and $\Delta(\Omega_0^{1,3}) \to \Delta(\Omega_1^{1,3})$ are injective.*

Proof. Lemma 6.2 implies that for $i \in \{1,2\}$ the inclusion induced map $\iota|_{\Omega_i^{2,3}}\colon \Omega_i^{2,3} \to \Omega_i^{1,3}$ is well defined, continuous and injective. Moreover, Corollary 6.3 implies that the inclusion induced map $\iota|_{\Omega_0^{2,3}}\colon \Omega_0^{2,3} \to \Omega_0^{1,3}$ is well defined, continuous and injective. Moreover, $\Omega_0^{2,3} \subset \Omega_1^{2,3}$ and $\Omega_0^{1,3} \subset \Omega_1^{1,3}$ and so the inclusions $\Omega_0^{2,3} \to \Omega_1^{2,3}$ and $\Omega_0^{1,3} \to \Omega_1^{1,3}$ are well defined, continuous and injective. Now an application of Lemma 5.6 concludes the proof. $\square$

Corollary 6.5. *The inclusion induced chain maps*

$$\begin{aligned}
\Delta(\Omega^{2,3})/\Delta(\{z^{2,3}\}) &\to \Delta(\Omega^{1,3})/\Delta(\{z^{1,3}\}),\\
\Delta(\Omega^{2,3})/\Delta(\Omega_2^{2,3}) &\to \Delta(\Omega^{1,3})/\Delta(\Omega_2^{1,3}),\\
\Delta(\Omega_1^{2,3})/\Delta(\Omega_0^{2,3}) &\to \Delta(\Omega_1^{1,3})/\Delta(\Omega_0^{1,3}),\\
\Delta(\Omega_1^{2,3})/\Delta(\{z^{2,3}\}) &\to \Delta(\Omega_1^{1,3})/\Delta(\{z^{1,3}\}),\\
\Delta(\Omega_0^{2,3})/\Delta(\{z^{2,3}\}) &\to \Delta(\Omega_0^{1,3})/\Delta(\{z^{1,3}\}),\\
\Delta(N_2/N_3)/\Delta(\{[N_3]\}) &\to \Delta(N_1/N_3)/\Delta(\{[N_3]\})
\end{aligned}$$

are injective.

Proof. It clear that $\{z^{2,3}\} \subset \Omega^{2,3}$, $\{z^{1,3}\} \subset \Omega^{1,3}$, $\Omega_2^{2,3} \subset \Omega^{2,3}$, $\Omega_2^{1,3} \subset \Omega^{1,3}$, $\Omega_0^{2,3} \subset \Omega_1^{2,3}$, $\Omega_0^{1,3} \subset \Omega_1^{1,3}$, $\{z^{2,3}\} \subset \Omega_1^{2,3}$, $\{z^{1,3}\} \subset \Omega_1^{1,3}$, $\{z^{2,3}\} \subset \Omega_0^{2,3}$, $\{z^{1,3}\} \subset \Omega_0^{1,3}$, $\{[N_3]\} \subset N_2/N_3$ and $\{[N_3]\} \subset N_1/N_3$.

Moreover, since $\iota\colon \Omega^{2,3} \to \Omega^{1,3}$ is injective, it follows that $\{z^{1,3}\} \cap \iota(\Omega^{2,3}) \subset \iota(\{z^{2,3}\})$, $\{z^{1,3}\} \cap \iota(\Omega_1^{2,3}) \subset \iota(\{z^{2,3}\})$ and $\{z^{1,3}\} \cap \iota(\Omega_0^{2,3}) \subset \iota(\{z^{2,3}\})$. Lemma 6.2 implies that $\Omega_2^{1,3} \cap \iota(\Omega^{2,3}) \subset \iota(\Omega_2^{2,3})$. Similarly, $\{[N_3]\} \cap \kappa(N_2/N_3) \subset \kappa(\{[N_3]\})$, where $\kappa\colon N_2/N_3 \to N_1/N_3$ is the (continuous and injective) inclusion induced map. Since $\Omega_1^{2,3} \subset \Omega^{2,3}$, Corollary 6.3 implies that $\Omega_0^{1,3} \cap \iota(\Omega_1^{2,3}) \subset \iota(\Omega_0^{2,3})$. Now the proof follows from Lemma 6.2, Corollary 6.4 and Lemma 5.7. $\square$

Lemma 6.6. *The sequences*

$$\Delta(\Omega^{2,3})/\Delta(\{z^{2,3}\}) \longrightarrow \Delta(\Omega^{1,3})/\Delta(\{z^{1,3}\}) \longrightarrow \Delta(\Omega^{1,2})/\Delta(\{z^{1,2}\})$$

$$\Delta(N_2/N_3)/\Delta(\{[N_3]\}) \longrightarrow \Delta(N_1/N_3)/\Delta(\{[N_3]\}) \longrightarrow \Delta(N_1/N_2)/\Delta(\{[N_2]\})$$

are weakly exact.

Proof. Since (N_1, N_2, N_3) is an FM-index triple for (π, S, A, A^*) such that $\mathrm{Cl}_X(N_1 \setminus N_3)$ strongly π-admissible, Proposition 2.8 implies that the sequence

$$\Delta(N_2/N_3)/\Delta(\{[N_3]\}) \longrightarrow \Delta(N_1/N_3)/\Delta(\{[N_3]\}) \longrightarrow \Delta(N_1/N_2)/\Delta(\{[N_2]\})$$

is weakly exact.

We will prove that the first sequence is weakly exact. Recall that π_1 is the (semi)flow on $\mathbb{R}$ generated by the ordinary differential equation

$$\dot{x} = x, \qquad x \in \mathbb{R}.$$

It follows that $([-1,1], \{-1,1\})$ is an FM-index pair for $(\pi_1, \{0\})$. It is clear that $(A \times \{0\}, A^* \times \{0\})$ is an attractor-repeller pair for $S \times \{0\}$ relative to the semiflow $\pi \times \pi_1$. Notice that $E(N_3) \subset E(N_2) \subset E(N_1)$ and an application of Proposition 2.2 implies that $(E(N_1), E(N_3))$ is an FM-index pair for $(\pi \times \pi_1, S \times \{0\})$ and $(E(N_2), E(N_3))$ is an FM-index pair for $(\pi \times \pi_1, A \times \{0\})$. Hence, $(E(N_1), E(N_2), E(N_3))$ is an FM-index triple for $(\pi \times \pi_1, S \times \{0\}, A \times \{0\}, A^* \times \{0\})$. Moreover, $\mathrm{Cl}_{X \times \mathbb{R}}(E(N_1) \setminus E(N_3))$ is strongly $\pi \times \pi_1$-admissible. Now Proposition 2.8 implies that the sequence

$$\Delta(\Omega^{2,3})/\Delta(\{z^{2,3}\}) \longrightarrow \Delta(\Omega^{1,3})/\Delta(\{z^{1,3}\}) \longrightarrow \Delta(\Omega^{1,2})/\Delta(\{z^{1,2}\})$$

is weakly exact. □

Lemma 6.7. *For $j \in \{0,1,2\}$ the sequence*

$$\Delta(\Omega_j^{2,3})/\Delta(\{z^{2,3}\}) \longrightarrow \Delta(\Omega_j^{1,3})/\Delta(\{z^{1,3}\}) \longrightarrow \Delta(\Omega_j^{1,2})/\Delta(\{z^{1,2}\})$$

is weakly exact.

Proof. Define $\mathbb{R}_0 := \{0\}$, $\mathbb{R}_1 :=]-\infty, 0]$ and $\mathbb{R}_2 := [0, \infty[$. Fix $j \in \{0,1,2\}$. For $i \in \{1,2,3\}$ let $M_i = M_i^j := (N_i \times I_j) \cup (N \times (I_j \cap \{-1,1\}))$ and let $\pi_{1,j}$ be the semiflow on $\mathbb{R}_j$ generated by the ordinary differential equation

$$\dot{x} = x, \quad x \in \mathbb{R}_j.$$

It follows that $(I_j, I_j \cap \{-1,1\})$ is an FM-index pair for $(\pi_{1,j}, \{0\})$. Moreover, $(A \times \{0\}, A^* \times \{0\})$ is an attractor-repeller pair for $S \times \{0\}$ relative to the semiflow $\pi \times \pi_{1,j}$. Notice that $M_3 \subset M_2 \subset M_1$ and an application of Proposition 2.2 implies that (M_1, M_3) is an FM-index for $(\pi \times \pi_{1,j}, S \times \{0\})$ and (M_2, M_3) is an

FM-index for $(\pi \times \pi_{1,j}, A \times \{0\})$. Hence, (M_1, M_2, M_3) is an FM-index triple for $(\pi \times \pi_{1,j}, S \times \{0\}, A \times \{0\}, A^* \times \{0\})$. Moreover, $\mathrm{Cl}_{X\times\mathbb{R}_j}(M_1 \setminus M_3)$ is strongly $\pi \times \pi_{1,j}$-admissible. Now Proposition 2.8 implies that the sequence

$$\Delta(M_2/M_3)/\Delta(\{[M_3]\}) \to \Delta(M_1/M_3)/\Delta(\{[M_3]\}) \to \Delta(M_1/M_2)/\Delta(\{[M_2]\})$$

is weakly exact. Notice that, for i, $\ell \in \{1,2,3\}$ with $i < \ell$, $M_\ell \subset M_i \subset E_j(N_i, N_\ell)$, $M_\ell \subset E(N_\ell)$, $E_j(N_i, N_\ell) \setminus E(N_\ell) \subset M_i$ and $E(N_\ell) \cap M_i \subset M_\ell$. It follows from Proposition 2.1 that the inclusion induced maps

$$M_i/M_\ell \to E_j(N_i, N_\ell)/E(N_\ell), \quad i, \ell \in \{1,2,3\},\ i < \ell,$$

are base-point preserving homeomorphisms. Moreover, the following inclusion induced diagram is commutative

$$\tag{6.6}\begin{array}{ccccc} M_2/M_3 & \longrightarrow & M_1/M_3 & \longrightarrow & M_1/M_2 \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ E_j(N_2,N_3)/E(N_3) & \longrightarrow & E_j(N_1,N_3)/E(N_3) & \longrightarrow & E_j(N_1,N_2)/E(N_2) \end{array}$$

Notice that, by Remark 4.2, $\Omega_j^{i,\ell} = E_j(N_i, N_\ell)/E(N_\ell)$, i, $\ell \in \{1,2,3\}$, $i < \ell$ and $j \in \{0,1,2\}$, both as sets and as topological spaces. Hence, diagram (6.6) can be rewritten in the following way:

$$\begin{array}{ccccc} \Delta(M_2/M_3)/\Delta(\{[M_3]\}) & \longrightarrow & \Delta(M_1/M_3)/\Delta(\{[M_3]\}) & \longrightarrow & \Delta(M_1/M_2)/\Delta(\{[M_2]\}) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \Delta(\Omega_j^{2,3})/\Delta(\{z^{2,3}\}) & \longrightarrow & \Delta(\Omega_j^{1,3})/\Delta(\{z^{1,3}\}) & \longrightarrow & \Delta(\Omega_j^{1,2})/\Delta(\{z^{1,2}\}) \end{array}$$

where the vertical arrows are isomorphims. Now an application of Lemma 5.3 implies that the sequence

$$\Delta(\Omega_j^{2,3})/\Delta(\{z^{2,3}\}) \longrightarrow \Delta(\Omega_j^{1,3})/\Delta(\{z^{1,3}\}) \longrightarrow \Delta(\Omega_j^{1,2})/\Delta(\{z^{1,2}\})$$

is weakly exact. □

Lemma 6.8. *For i and $j \in \{1,2,3\}$ with $i < j$, the inclusion induced sequences*

$$0 \longrightarrow \Delta(\Omega_0^{i,j})/\Delta(\{z^{i,j}\}) \longrightarrow \Delta(\Omega_1^{i,j})/\Delta(\{z^{i,j}\}) \longrightarrow \Delta(\Omega_1^{i,j})/\Delta(\Omega_0^{i,j}) \longrightarrow 0$$

$$0 \longrightarrow \Delta(\Omega_2^{i,j})/\Delta(\{z^{i,j}\}) \longrightarrow \Delta(\Omega^{i,j})/\Delta(\{z^{i,j}\}) \longrightarrow \Delta(\Omega^{i,j})/\Delta(\Omega_2^{i,j}) \longrightarrow 0$$

are exact.

Proof. The first of the above sequences is induced by the inclusion induced chain monomorphisms $\Delta(\{z^{i,j}\}) \to \Delta(\Omega_0^{i,j})$ and $\Delta(\Omega_0^{i,j}) \to \Delta(\Omega_1^{i,j})$, while the second sequence is induced by the inclusion induced chain monomorphisms $\Delta(\{z^{i,j}\}) \to \Delta(\Omega_2^{i,j})$ and $\Delta(\Omega_2^{i,j}) \to \Delta(\Omega^{i,j})$. The lemma now follows from Lemma 5.2. □

We can finally give a

Proof of Theorem 6.1. Diagram (6.2) is commutative by the definition of the maps $f^{i,j}$ (cf. Lemma 4.3). Consider the following inclusion induced diagrams of chain maps:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
 & \Delta(\Omega_2^{2,3})/\Delta(\{z^{2,3}\}) \arrow[r] \arrow[d] & \Delta(\Omega_2^{1,3})/\Delta(\{z^{1,3}\}) \arrow[r] \arrow[d] & \Delta(\Omega_2^{1,2})/\Delta(\{z^{1,2}\}) \arrow[d] \\
(6.7) & \Delta(\Omega^{2,3})/\Delta(\{z^{2,3}\}) \arrow[r] \arrow[d] & \Delta(\Omega^{1,3})/\Delta(\{z^{1,3}\}) \arrow[r] \arrow[d] & \Delta(\Omega^{1,2})/\Delta(\{z^{1,2}\}) \arrow[d] \\
 & \Delta(\Omega^{2,3})/\Delta(\Omega_2^{2,3}) \arrow[r] \arrow[d] & \Delta(\Omega^{1,3})/\Delta(\Omega_2^{1,3}) \arrow[r] \arrow[d] & \Delta(\Omega^{1,2})/\Delta(\Omega_2^{1,2}) \arrow[d] \\
 & 0 & 0 & 0
\end{tikzcd}$$

Both (6.7) and (6.3) are clearly commutative. Lemma 6.8 implies that the columns in each of those diagrams are exact. It follows from Lemmas 6.6 and 6.7 that the first two rows in each of those diagrams are weakly exact. Now an application of Corollary 6.5 and Proposition 5.5 implies that the third row of those diagrams is weakly exact. Invoking again Lemma 6.6 we thus conclude that the rows of diagrams (6.1), (6.2) and (6.3) are weakly exact. The proof of the theorem is complete. □

We can now complete the proof of Theorem 3.5.

Proof of Theorem 3.5. For $i, j \in \{1,2,3\}$ with $i < j$ define the following sets:

$$\begin{aligned}
A_1^{i,j} &:= \Delta(\Omega^{i,j})/\Delta(\{z^{i,j}\}), & A_2^{i,j} &:= \Delta(\Omega^{i,j})/\Delta(\Omega_2^{i,j}),\\
A_3^{i,j} &:= \Delta(\Omega_1^{i,j})/\Delta(\Omega_0^{i,j}), & A_4^{i,j} &:= \Delta(\Omega_1^{i,j})/\Delta(\{z^{i,j}\}),\\
A_5^{i,j} &:= \Delta(\Omega_0^{i,j})/\Delta(\{z^{i,j}\}), & A_6^{i,j} &:= \Delta(N_i/N_j)/\Delta(\{[N_j]\}).
\end{aligned}$$

Using Proposition 4.4 and Lemma 4.3 we see that diagrams (6.1) and (6.2) of weakly exact sequences in Theorem 6.1 induce the following commutative diagrams in homology with long exact rows

$$
\begin{tikzcd}
{} \arrow[r] & H_q(A_1^{2,3}) \arrow[r] \arrow[d, "H_q(\rho^{2,3})"] & H_q(A_1^{1,3}) \arrow[r] \arrow[d, "H_q(\rho^{1,3})"] & H_q(A_1^{1,2}) \arrow[r] \arrow[d, "H_q(\rho^{1,2})"] & H_{q-1}(A_1^{2,3}) \arrow[r] \arrow[d, "H_{q-1}(\rho^{2,3})"] & {} \\
{} \arrow[r] & H_q(A_2^{2,3}) \arrow[r] \arrow[d, "H_q(\ell^{2,3})^{-1}"] & H_q(A_2^{1,3}) \arrow[r] \arrow[d, "H_q(\ell^{1,3})^{-1}"] & H_q(A_2^{1,2}) \arrow[r] \arrow[d, "H_q(\ell^{1,2})^{-1}"] & H_{q-1}(A_2^{2,3}) \arrow[r] \arrow[d, "H_{q-1}(\ell^{2,3})^{-1}"] & {} \\
{} \arrow[r] & H_q(A_3^{2,3}) \arrow[r] & H_q(A_3^{1,3}) \arrow[r] & H_q(A_3^{1,2}) \arrow[r] & H_{q-1}(A_3^{2,3}) \arrow[r] & {}
\end{tikzcd}
\tag{6.8}
$$

$$
\begin{tikzcd}
{} \arrow[r] & H_q(A_5^{2,3}) \arrow[r] \arrow[d, "H_q(f^{2,3})^{-1}"] & H_q(A_5^{1,3}) \arrow[r] \arrow[d, "H_q(f^{1,3})^{-1}"] & H_q(A_5^{1,2}) \arrow[r] \arrow[d, "H_q(f^{1,2})^{-1}"] & H_{q-1}(A_5^{2,3}) \arrow[r] \arrow[d, "H_{q-1}(f^{2,3})^{-1}"] & {} \\
{} \arrow[r] & H_q(A_6^{2,3}) \arrow[r] & H_q(A_6^{1,3}) \arrow[r] & H_q(A_6^{1,2}) \arrow[r] & H_{q-1}(A_6^{2,3}) \arrow[r] & {}
\end{tikzcd}
\tag{6.9}
$$

Note that the commutative diagram (6.3) of exact columns and weakly exact rows in Theorem 6.1 can be written as

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
A_5^{2,3} \arrow[r] \arrow[d] & A_5^{1,3} \arrow[r] \arrow[d] & A_5^{1,2} \arrow[d] \\
A_4^{2,3} \arrow[r] \arrow[d] & A_4^{1,3} \arrow[r] \arrow[d] & A_4^{1,2} \arrow[d] \\
A_3^{2,3} \arrow[r] \arrow[d] & A_3^{1,3} \arrow[r] \arrow[d] & A_3^{1,2} \arrow[d] \\
0 & 0 & 0
\end{tikzcd}
\tag{6.10}
$$

Let $\widehat{\partial}_q^3\colon H_q(A_3^{1,2}) \to H_{q-1}(A_3^{2,3})$, (resp. $\widehat{\partial}_q^1\colon H_q(A_5^{1,2}) \to H_{q-1}(A_5^{2,3})$), $q \in \mathbb{Z}$, be the connecting homomorphism of the long homology sequence associated with the third (resp. first) row of (6.10) while $\delta_q^3\colon H_q(A_3^{1,2}) \to H_{q-1}(A_5^{1,2})$, (resp. $\delta_q^2\colon H_q(A_3^{1,3}) \to H_{q-1}(A_5^{1,3})$, resp. $\delta_q^1\colon H_q(A_3^{2,3}) \to H_{q-1}(A_5^{2,3})$), $q \in \mathbb{Z}$, be the connecting homomorphism of the long homology sequence associated with the third (resp. second, resp. first) column of (6.10). Proposition 5.5 implies that

$$
\delta_{q-1}^1 \circ \widehat{\partial}_q^3 = -\widehat{\partial}_{q-1}^1 \circ \delta_q^3, \; q \in \mathbb{Z}. \tag{6.11}
$$

Thus we obtain the following diagram

$$
\begin{tikzcd}
\arrow[r] & H_q(A_3^{2,3}) \arrow[r] \arrow[d, "(-1)^q\delta_q^1"] & H_q(A_3^{1,3}) \arrow[r] \arrow[d, "(-1)^q\delta_q^2"] & H_q(A_3^{1,2}) \arrow[r, "\widehat{\partial}_q^3"] \arrow[d, "(-1)^q\delta_q^3"] & H_{q-1}(A_3^{2,3}) \arrow[r] \arrow[d, "(-1)^{q-1}\delta_{q-1}^1"] & H_{q-1}(A_3^{1,3}) \arrow[r] \arrow[d, "(-1)^{q-1}\delta_{q-1}^2"] & {} \\
\arrow[r] & H_{q-1}(A_5^{2,3}) \arrow[r] & H_{q-1}(A_5^{1,3}) \arrow[r] & H_{q-1}(A_5^{1,2}) \arrow[r, "\widehat{\partial}_{q-1}^1"'] & H_{q-2}(A_5^{2,3}) \arrow[r] & H_{q-2}(A_5^{1,3}) \arrow[r] & {}
\end{tikzcd}
\tag{6.12}
$$

The diagrams

$$
\begin{tikzcd}
H_q(A_3^{2,3}) \arrow[r] \arrow[d, "(-1)^q\delta_q^1"'] & H_q(A_3^{1,3}) \arrow[d, "(-1)^q\delta_q^2"] \\
H_{q-1}(A_5^{2,3}) \arrow[r] & H_{q-1}(A_5^{1,3})
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
H_q(A_3^{1,3}) \arrow[r] \arrow[d, "(-1)^q\delta_q^2"'] & H_q(A_3^{1,2}) \arrow[d, "(-1)^q\delta_q^3"] \\
H_{q-1}(A_5^{1,3}) \arrow[r] & H_{q-1}(A_5^{1,2})
\end{tikzcd}
$$

commute by the naturality of $(\delta_q^i)_{q\in\mathbb{Z}}$, $i \in \{1,2,3\}$ while the diagram

$$
\begin{tikzcd}
H_q(A_3^{1,2}) \arrow[r, "\widehat{\partial}_q^3"] \arrow[d, "(-1)^q\delta_q^3"'] & H_{q-1}(A_3^{2,3}) \arrow[d, "(-1)^{q-1}\delta_{q-1}^1"] \\
H_{q-1}(A_5^{1,2}) \arrow[r, "\widehat{\partial}_{q-1}^1"'] & H_{q-2}(A_5^{2,3})
\end{tikzcd}
$$

commutes in view of (6.11). It follows that diagram (6.12) is commutative.

Now, composing diagrams (6.8), (6.12) and (6.9) (from top to bottom) and using Theorem 4.8 we obtain the commutative diagram

$$
\begin{tikzcd}
\arrow[r] & H_q(A_1^{2,3}) \arrow[r] \arrow[d, "\xi_q^{2,3}"] & H_q(A_1^{1,3}) \arrow[r] \arrow[d, "\xi_q^{1,3}"] & H_q(A_1^{1,2}) \arrow[r] \arrow[d, "\xi_q^{1,2}"] & H_{q-1}(A_1^{2,3}) \arrow[r] \arrow[d, "\xi_{q-1}^{2,3}"] & {} \\
\arrow[r] & H_{q-1}(A_6^{2,3}) \arrow[r] & H_{q-1}(A_6^{1,3}) \arrow[r] & H_{q-1}(A_6^{1,2}) \arrow[r] & H_{q-2}(A_6^{2,3}) \arrow[r] & {}
\end{tikzcd}
\tag{6.13}
$$

Applying the $\langle\,\cdot\,,\,\cdot\,\rangle$-operation to diagram (6.13) and using Definition 4.10 together with Theorem 5.1 in [5] we obtain diagram (3.1). This combined with Theorem 4.8 completes the proof of Theorem 3.5. □

7. The suspension isomorphism and homology index braids

In this section let P be a finite set, $\prec$ be a strict partial order on P and $(M_i)_{i\in P}$ be a Morse decomposition of S relative to π. Using the notation of Theorem 3.1 we have that $(M_i')_{i\in P}$ is a $\prec$-ordered Morse decomposition of S' relative to π'. Given $(I,J) \in \mathcal{I}_2(\prec)$, $(M(I), M(J))$ is an attractor-repeller pair in $M(IJ)$ (where $IJ = I \cup J$) relative to π, so $(M(I)', M(J)')$ is an attractor-repeller pair in $M(IJ)'$ relative to π'. Setting, for $K \in \mathcal{I}(\prec)$ and $q \in \mathbb{Z}$, $H_q'(K) :=$

$H_q(\pi', M(K)')$, $H_q(K) := H_q(\pi, M(K))$ and $\theta_q(K) := \theta_q(\pi, \widetilde{\pi}, M(K))$ and using Theorem 3.1 we thus arrive at the commutative diagram

$$
\begin{array}{ccccccccc}
\longrightarrow & H'_q(I) & \longrightarrow & H'_q(IJ) & \longrightarrow & H'_q(J) & \longrightarrow & H'_{q-1}(I) & \longrightarrow \\
& \Big\downarrow{\scriptstyle \theta_q(I)} & & \Big\downarrow{\scriptstyle \theta_q(IJ)} & & \Big\downarrow{\scriptstyle \theta_q(J)} & & \Big\downarrow{\scriptstyle \theta_{q-1}(I)} & \\
\longrightarrow & H_{q-k}(I) & \longrightarrow & H_{q-k}(IJ) & \longrightarrow & H_{q-k}(J) & \longrightarrow & H_{q-k-1}(I) & \longrightarrow
\end{array} \tag{7.1}
$$

Here, the upper (resp. lower) horizontal sequence is the homology index sequence of $(\pi', M(IJ)', M(I)', M(J)')$ (resp. the homology index sequence of $(\pi, M(IJ), M(I), M(J))$ shifted to the left by k). We thus obtain the following result.

Theorem 7.1. *$(\theta_q(J))_{q\in\mathbb{Z}}$, $J \in \mathcal{I}(\prec)$, is an isomorphism from the homology index braid of $(\pi', S', (M'_i)_{i\in P})$ to the graded module braid obtained by shifting the homology index braid of $(\pi, S, (M_i)_{i\in P})$ to the left by k.*

Now the results of [6], [7] and [8] imply the following result.

Corollary 7.2. *Let $C\Delta(i) = (C\Delta(i)_q)_{q\in\mathbb{Z}}$, $i \in P$, be a family of graded modules. For $q \in \mathbb{Z}$ let $\widetilde{\Delta}_q: \bigoplus_{i\in P} C\Delta(i)_q \to \bigoplus_{i\in P} C\Delta(i)_{q-1}$ be a Γ-module homomorphism. Suppose that $\widetilde{\Delta} := (\widetilde{\Delta}_q)_{q\in\mathbb{Z}}$ is a C-connection matrix for the homology index braid of $(\pi, S, (M_i)_{i\in P})$. Then $\widetilde{\Delta}_{\cdot+k} := (\widetilde{\Delta}_{q+k})_{q\in\mathbb{Z}}$ is a C-connection matrix for the homology index braid of $(\pi', S', (M'_i)_{i\in P})$.*

References

[1] V. Arnold, *Ordinary Differential Equations*, Springer–Verlag, 1992.

[2] M. C. Carbinatto and K. P. Rybakowski, *Morse decompositions in the absence of uniqueness*, Topol. Methods Nonlinear Anal. **18** (2001), 205–242.

[3] ———, *Morse decompositions in the absence of uniqueness,* II, Topol. Methods Nonlinear Anal. **22** (2003), 17–53.

[4] ———, *Nested sequences of index filtrations and continuation of the connection matrix*, J. Differential Equations **207** (2004), 458–488.

[5] ———, *Homology index braids in infinite-dimensional Conley index theory*, Topol. Methods Nonlinear Anal. **26** (2005), 35–74.

[6] R. D. Franzosa, *Index filtrations and the homology index braid for partially ordered Morse decompositions*, Trans. Amer. Math. Soc. **298** (1986), 193–213.

[7] ———, *The connection matrix theory for Morse decompositions*, Trans. Amer. Math. Soc. **311** (1989), 561–592.

[8] R. D. Franzosa and K. Mischaikow, *The connection matrix theory for semiflows on* (*not necessarily locally compact*) *metric spaces*, J. Differential Equations **71** (1988), 270–287.

[9] H. L. Kurland, *Homotopy invariants of repeller-attractor pairs,* I. *The Puppe sequence of an R–A Pair*, J. Differential Equations **46** (1982), 1–31.

[10] W. Lück, *Algebraische Topologie*, Vieweg, 2005.

[11] S. Mac Lane, *Homology*, Springer–Verlag, Berlin, 1975.

[12] M. Osborne, *Basic Homological Algebra*, Springer–Verlag, Berlin, 2000.

[13] K. P. Rybakowski, *The Morse index, repeller-attractor pairs and the connection index for semiflows on noncompact spaces*, J. Differential Equations **47** (1983), 66–98.

[14] ———, *The Homotopy Index and Partial Differential Equations*, Springer–Verlag, Berlin, 1987.

[15] E.H. Spanier, *Algebraic Topology*, McGraw–Hill, New York, 1966.

Manuscript received October 24, 2005

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