POSITIVE SOLUTIONS FOR A CLASS OF VOLterra INTEGRAL EQUATIONS VIA A FIXED POINT THEOREM IN FRÉchet SPACES

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Abstract. Motivated by the Emden differential equation we discuss in this paper the existence of positive solutions to the integral equation

\[ y(t) = \int_0^t k(t, s) f(y(s)) \, ds \quad \text{for } t \in [0, T). \]

1. Introduction

In this paper we establish the existence of positive (positive on \((0, T))\) solutions to the Volterra integral equation

\[ y(t) = \int_0^t k(t, s) f(y(s)) \, ds \quad \text{for } t \in [0, T) \]

where \(0 < T \leq \infty\) is fixed. Our theory was motivated by the Emden differential equation

\[ y'' - t^p y^q = 0, \quad p \geq 0 \text{ and } 0 < q < 1 \]

which arises in various astrophysical problems, including the study of the density of stars; of course one is interested only in positive solutions to (1.2). Differential equations including (1.2) will be discussed as a special case of (1.1) in Section 2.
We remark also when the kernel \( k \) is a convolution kernel (1.1) arises in connection with nonlinear diffusion and percolation problems (see [3] and the references therein). The results in Section 2 extend and complement the theory in [3], [5].

For notational purposes in this paper if \( u \in C[0,T) \) then for every \( m \in \{1,2,\ldots\} = N \) we define the seminorms \( \rho_m(u) \) by

\[
\rho_m(u) = \sup_{t \in [0,t_m]} |u(t)|
\]

where \( t_m \uparrow T \). Note \( C[0,T) \) is a locally convex linear topological space. The topology on \( C[0,T) \), induced by the seminorms \( \{\rho_m\}_{m \in \mathbb{N}} \), is the topology of uniform convergence on every compact interval of \([0,T)\).

Existence in Section 2 is based on a fixed point theorem of Agarwal and O’Regan [2] which in turn is based on Krasnosel’skii’s fixed point theorem in a cone. We present the result in [2] (see also [1]) for the convenience of the reader. First however we state Krasnosel’skii’s result.

**Theorem 1.1.** Let \( B = (B, ||\cdot||) \) be a Banach space and let \( C \subseteq E \) be a cone in \( B \). Assume \( \Omega_1 \) and \( \Omega_2 \) are open bounded subsets of \( B \) with \( 0 \in \Omega_1, \Omega_1 \subset \Omega_2 \), and let

\[
S: C \cap (\overline{\Omega_2 \setminus \Omega_1}) \to C
\]

be a continuous compact map such that either

(a) \( ||Su|| \leq ||u|| \) for \( u \in C \cap \partial \Omega_1 \) and \( ||Su|| \geq ||u|| \) for \( u \in C \cap \partial \Omega_2 \), or

(b) \( ||Su|| \geq ||u|| \) for \( u \in C \cap \partial \Omega_1 \) and \( ||Su|| \leq ||u|| \) for \( u \in C \cap \partial \Omega_2 \),

hold. Then \( S \) has a fixed point in \( C \cap (\overline{\Omega_2 \setminus \Omega_1}) \).

The result in [2] is based on the fact that a Fréchet space can be viewed as a projective limit of a sequence of Banach spaces \( \{E_n\}_{n \in \mathbb{N}} \). We now extend Theorem 1.1 to the Fréchet space setting. Let \( E = (E, \{||\cdot||_n\}_{n \in \mathbb{N}}) \) be a Fréchet space with

\[
|x|_1 \leq |x|_2 \leq |x|_3 \leq \ldots \quad \text{for every } x \in E.
\]

Assume for each \( n \in \mathbb{N} \) that \( (E_n, ||\cdot||_n) \) is a Banach space and suppose

\[
E_1 \supseteq E_2 \supseteq \ldots
\]

with \( |x|_n \leq |x|_{n+1} \) for all \( x \in E_{n+1} \). Also assume \( E = \bigcap_{n=1}^{\infty} E_n \), where \( \bigcap_1^{\infty} \) is the generalized intersection as described in [4, pp. 439] (i.e. \( E \) is the projective limit of \( \{E_n\}_{n \in \mathbb{N}} \) with the embedding \( \mu_n: E \to E_n \). Fix \( n \in \mathbb{N} \) and let \( C_n \) will a cone in \( E_n \) and for \( \rho > 0 \) we let

\[
U_{n,\rho} = \{x \in E_n : |x|_n < \rho\} \quad \text{and} \quad V_{n,\rho} = U_{n,\rho} \cap C_n.
\]

Notice

\[
\partial C_n V_{n,\rho} = \partial E_n U_{n,\rho} \cap C_n \quad \text{and} \quad \overline{V_{n,\rho}} = \overline{U_{n,\rho}} \cap C_n
\]
(the first closure is with respect to \(C_n\) whereas the second is with respect to \(E_n\)). We are interested in establishing that \(F\) has a fixed point; here \(F: E_1 \to E_1\).

**Definition 1.2.** Fix \(k \in \mathbb{N}\). If \(x, y \in E_k\) then we say \(x = y\) in \(E_k\) if \(|x - y|_k = 0\).

**Definition 1.3.** If \(x, y \in E\) then we say \(x = y\) in \(E\) if \(x = y\) in \(E_k\) for each \(k \in \mathbb{N}\).

**Theorem 1.4.** For each \(n \in \mathbb{N}\), let \(C_n\) be a cone in \(E_n\) and also let

\[C_1 \supseteq C_2 \supseteq C_3 \supseteq \ldots\]

In addition suppose \(F: E_1 \to E_1\). Let \(\gamma, r, R\) be constants with \(0 < \gamma < r < R\) and assume the following conditions are satisfied:

(a) for each \(n \in \mathbb{N}\), \(F: \overline{U_{n,R}} \cap C_n \to C_n\) is a continuous compact map,

(b) for each \(n \in \mathbb{N}\), \(|F x|_n \leq |x|_n\), for all \(x \in \partial E_n U_{n,r} \cap C_n\),

(c) for each \(n \in \mathbb{N}\), \(|F x|_n \geq |x|_n\), for all \(x \in \partial E_n U_{n,R} \cap C_n\), and

(d) for every \(k \in \mathbb{N}\) and any subsequence \(A \subseteq \{k, k + 1, \ldots\}\) if \(x \in C_n\) is such that \(R \geq |x|_n \geq r\) for some \(n \in A\), then \(|x|_k \geq \gamma\).

Then \(F\) has a fixed point \(y \in E\) (in fact \(\mu_n(y) \in (\overline{U_{n,r}} \setminus U_{n,\gamma}) \cap C_n\) for every \(n \in \mathbb{N}\)).

**Remark 1.5.** Of course there is an obvious analogue of Theorem 1.2 when \(U_{n,r}\) is replaced by \(U_{n,R}\) in (b) and \(U_{n,R}\) is replaced by \(U_{n,r}\) in (c).

**Proof.** We know from Theorem 1.1 (part (a)) that for each \(n \in \mathbb{N}\), \(F\) has a fixed point \(y_n \in (\overline{U_{n,R}} \setminus U_{n,r}) \cap C_n\). Let \(\{y_n\}_{n \in \mathbb{N}}\). Note \(y_n \in \overline{U_{1,R}} \setminus U_{1,\gamma}\) for each \(n \in \mathbb{N}\). To see this notice \(|y_n|_n \leq R\) and \(|x|_1 \leq |x|_n\) for all \(x \in E_n\) implies \(|y_n|_1 \leq R\), and so \(y_n \in \overline{U_{1,R}}\) for each \(n \in \mathbb{N}\). On the other hand \(|y_n|_n \geq r\), \(y_n \in C_n\) together with (d) implies \(|y_n|_1 \geq \gamma\). Thus \(y_n \in (\overline{U_{1,R}} \setminus U_{1,\gamma}) \cap C_1\) and \(y_n = F y_n\) in \(E_n\) for each \(n \in \mathbb{N}\) and these together with (a) implies that there exists a subsequence \(N_1^*\) of \(N\) and a \(z_1 \in (\overline{U_{1,R}} \setminus U_{1,\gamma}) \cap C_1\) with \(y_n \to z_1\) in \(E_1\) as \(n \to \infty\) in \(N_1^*\). Notice in particular that \(\gamma \leq |z_1|_1 \leq R\).

Let \(N_1 = N_1^* \setminus \{1\}\). Now look at \(\{y_n\}_{n \in N_1}\). Again (a) guarantees that there exists a subsequence \(N_2^*\) of \(N_1\) and a \(z_2 \in (\overline{U_{2,R}} \setminus U_{2,\gamma}) \cap C_2\) with \(y_n \to z_2\) in \(E_2\) as \(n \to \infty\) in \(N_2^*\) and \(\gamma \leq |z_2|_2 \leq R\). Note also \(|z_2 - z_1|_1 = 0\) since \(N_2^* \subseteq N_1\) and \(E_1 \supseteq E_2\), so \(z_2 = z_1\) in \(E_1\).

Proceed inductively to obtain subsequences of integers

\[N_1^* \supseteq N_2^* \supseteq \ldots, N_k^* \subseteq \{k, k + 1, \ldots\}\]

and \(z_k \in (\overline{U_{n,R}} \setminus U_{n,\gamma}) \cap C_k\) with \(y_n \to z_k\) in \(E_k\) as \(n \to \infty\) in \(N_k^*\). Note \(z_{k+1} = z_k\) in \(E_k\) for \(k \in \{1, 2, \ldots\}\). Also let \(N_k = N_k^* \setminus \{k\}\).
Fix $k \in \mathbb{N}$. Let $y = z_k$ in $E_k$ (i.e. $\mu_k(y) = z_k$). Notice $y$ is well defined and $y \in E$. Now $y_n = F y_n$ in $E_n$ for $n \in N$ and $y_n \to y$ in $E_k$ as $n \to \infty$ in $N$. (since $y = z_k$ in $E_k$) together with (a) implies $y = F y$ in $E_k$. We can do this for each $k \in \mathbb{N}$ so $y = F y$ in $E$.

Remark 1.6. From the proof above notice (a) in Theorem 1.4 could be replaced by the condition:

(a') for each $n \in \mathbb{N}$, $F: (\overline{U_{n,R}} \cap U_{n,\gamma}) \cap C_n \to C_n$ is a continuous compact map

and the result of Theorem 1.4 is again true. Also $F: E_1 \to E_1$ in the statement of Theorem 1.4 could be replaced by $F: U_{1,R} \cap C_1 \to E_1$.

2. Volterra integral equations

We now use Theorem 1.4 to establish an existence result for (1.2). Notice $T$ could be $\infty$ in Theorem 2.1.

Theorem 2.1. Suppose the following conditions are satisfied:

(a) for each $n \in \mathbb{N}$, $0 < k(t, s)$ for all $t \in (0, t_n]$, a.e. $s \in [0, t]$ and $k_1(s) = k(t, s) \in L^1[0, t]$ for each $t \in [0, t_n]$ and $\sup_{t \in [0, T]} \int_0^t |k_1(s)| ds < \infty$,

(b) for each $n \in \mathbb{N}$, for any $t, t' \in [0, t_n]$,

$$\int_0^{t^*} |k_1(s) - k_1(s)| ds \to 0 \quad \text{as} \ t \to t'$$

where $t^* = \min\{t, t'\}$

(c) for each $n \in \mathbb{N}$, $k(x_1, s) - k(x_2, s) \geq 0$ for a.e. $s \in [0, x_2]$ where $0 < x_2 < x_1 \leq t_n$,

(d) $f: [0, \infty) \to [0, \infty)$ is continuous and nondecreasing with $f(y) > 0$ for $y > 0$,

(e) there exists $a \in C[0, T]$ such that $a(0) = 0$, $0 < a(t) \leq 1$, $t \in (0, T)$, and for each $n \in \mathbb{N}$ for any constant $R > 0$, $a$ satisfies

$$\int_0^t k(t, s) f(Ra(s)) ds \geq a(t) f(R) \int_0^T k(T, s) ds$$

for $t \in [0, t_n]$,

(f) for each $n \in \mathbb{N}$, there exists $R_1 > 0$ (independent of $n$) with

$$f(R_1) \int_0^{t_n} k(t_n, s) ds \leq R_1,$$

(g) for each $n \in \mathbb{N}$, there exists $R_2 > 0$ (independent of $n$), $R_2 \neq R_1$ with

$$\int_0^{t_n} k(t_n, s) f(R_2 a(s)) ds \geq R_2.$$
Then (1.1) has at least one solution $y \in C[0,T]$ and either

(A) for each $n \in \mathbb{N}$, $0 < \gamma \leq |y|_n \leq R_2$ and $y(t) \geq a(t) \gamma$ for $t \in [0,t_n]$ if $R_1 < R_2$ (here $\gamma = a(t_1) R_1$),

or

(B) for each $n \in \mathbb{N}$, $0 < \gamma \leq |y|_n \leq R_1$ and $y(t) \geq a(t) \gamma$ for $t \in [0,t_n]$ if $R_2 < R_1$ (here $\gamma = a(t_1) R_2$).

Remark 2.2. When $T = \infty$ notice by $\int_0^T k(t,s) \, ds$ in Theorem 2.1(e) we mean $\lim_{t \to \infty} \int_0^t k(t,s) \, ds$.

Remark 2.3. Notice if (b) in Theorem 2.1 is replaced by

(b') for each $n \in \mathbb{N}$, for any $t, t' \in [0,t_n]$, 

$$\int_0^{t^*} |k_t(s) - k_{t'}(s)| \, ds + \int_{t^*}^{t^{**}} [k_{t^{**}}(s)] \to 0 \quad \text{as} \quad t \to t'$$

where $t^* = \min\{t, t'\}$ and $t^{**} = \max\{t, t'\}$,

then automatically

$$\sup_{t \in [0,t_n]} \int_0^t [k_t(s)] \, ds < \infty$$

in Theorem 2.1(a).

Proof of Theorem 2.1. Without loss of generality assume $R_1 < R_2$. Fix $n \in \mathbb{N}$ and let $E_n = C[0,t_n]$, and

$$C_n = \{y \in C[0,t_n] : y(t) \geq a(t) |y|_n \text{ for } t \in [0,t_n]\}$$

where $|y|_n = \sup_{t \in [0,t_n]} |y(t)|$. Let

$$Fy(t) = \int_0^t k(t,s) f(y(s)) \, ds$$

and $U_{n,\beta} = \{y \in C[0,t_n] : |y|_n < \beta\}$; here $\beta = R_1$ or $R_2$.

Now let $y \in C_n \cap U_{n,R_2}$. Then (c) implies $Fy(t)$ is increasing in $t$. Also there exists $R \in [0,R_2]$ such that $|y|_n = R$ so $a(t) R \leq y(t) \leq R$ for $t \in [0,t_n]$ and as a result

$$Fy(t) \geq \int_0^t k(t,s) f(R a(s)) \, ds, \quad t \in [0,t_n]$$

with

$$|Fy|_n = Fy(t_n) \leq \int_0^{t_n} k(t_n,s) f(R) \, ds.$$ 

Thus for $t \in [0,t_n]$ we have

$$Fy(t) \geq \frac{\int_0^t k(t,s) f(R a(s)) \, ds}{\int_0^{t_n} k(t_n,s) f(R) \, ds} |Fy|_n \geq \frac{\int_0^t k(t,s) f(R a(s)) \, ds}{\int_0^t k(T,s) f(R) \, ds} |Fy|_n.$$
and this together with (e) yields

\[ F y(t) \geq a(t) |F y|_n \quad \text{for } t \in [0, t_n], \]

so \( F: C_n \cap \overline{U_{n,R_1}} \to C_n \). A standard argument \cite{5} guarantees that \( F: C_n \cap \overline{U_{n,R_2}} \to C_n \) is a continuous, compact map. Next we show

\[ |F x|_n \leq |x|_n \quad \text{for all } x \in \partial U_{n,R_1} \cap C_n \]

and

\[ |F x|_n \geq |x|_n \quad \text{for all } x \in \partial U_{n,R_2} \cap C_n. \]

Let \( x \in \partial U_{n,R_1} \cap C_n \). Then \( |x|_n = R_1 \) and \( 0 \leq a(t) R_1 \leq x(t) \leq R_1 \) for \( t \in [0, t_n] \). Also Theorem 2.1(f) guarantees that

\[ |F x|_n = F x(t_n) \leq \int_0^{t_n} k(t_n, s) f(R_1) \, ds \leq R_1 = |x|_n, \]

so (2.1) is true.

Let \( x \in \partial U_{n,R_2} \cap C_n \). Then \( |x|_n = R_2 \) and \( 0 \leq a(t) R_2 \leq x(t) \leq R_2 \) for \( t \in [0, t_n] \). Also Theorem 2.1(g) guarantees that

\[ |F x|_n = F x(t_n) \geq \int_0^{t_n} k(t_n, s) f(a(s) R_2) \, ds \geq R_2 = |x|_n, \]

so (2.2) is true.

The result follows immediately from Theorem 1.4 once we show Theorem 1.4(d) holds (with \( \gamma = a(t_1) R_1 \)). Fix \( k \in \mathbb{N} \) and any subsequence \( A \subseteq \{k, k + 1, \ldots \} \). Let \( n \in A \) and \( x \in C_n \) with \( R_1 \leq |x|_n \leq R_2 \). Then \( R_1 \leq \sup_{t \in [0, t_n]} |x(t)| \leq R_2 \) so

\[ x(t) \geq a(t) |x|_n \geq a(t) R_1 \quad \text{for } t \in [0, t_n]. \]

Now since \( n \in A \subseteq \{k, k + 1, \ldots \} \) we have \( n \geq k \) so (note \( t_n \uparrow T \))

\[ x(t) \geq a(t) R_1 \quad \text{for } t \in [0, t_k]. \]

In particular \( t_1 \in [0, t_k] \) so \( x(t_1) \geq a(t_1) R_1 \) and so

\[ |x|_k = \sup_{t \in [0, t_k]} |x(t)| \geq a(t_1) R_1 = \gamma. \]

Thus Theorem 1.4(d) holds, so Theorem 1.4 guarantees that \( F \) has a fixed point \( y \in C[0, T] \) with for each \( n \in \mathbb{N} \),

\[ \gamma \leq |y|_n \leq R \quad \text{and} \quad y(t) \geq a(t) |y|_n \geq a(t) \gamma \quad \text{for } t \in [0, t_n]; \]

here \( \gamma = a(t_1) R_1 \).

\[ \square \]
Example 2.4. Consider the generalized Emden equation
\begin{equation}
(2.3) \quad \begin{cases} 
    y'' - h(t)y^q = 0 & \text{for } t \in [0, T), \\
    y(0) = y'(0) = 0,
\end{cases}
\end{equation}
with $0 < q < 1$, $h : [0, T) \to [0, \infty)$ continuous with $h(t) \geq t^p$, $p \geq 0$ and $\int_0^T (T - s) h(s) \, ds < \infty$; here $0 < T < \infty$ is fixed. We will show (2.3) has a positive solution (positive on $(0, T)$); note $y \equiv 0$ is also a solution of (2.3).

First notice solving (2.3) is equivalent to solving the integral equation
\begin{equation*}
y(t) = \int_0^t (t - s) h(s)[y(s)]^q \, ds \quad \text{for } t \in [0, T).
\end{equation*}
Let
\begin{equation*}
k(t, s) = (t - s) h(s) \quad \text{and} \quad f(y) = y^q
\end{equation*}
in Theorem 2.1. Clearly (a)–(d) hold. Next we show (e) is satisfied with
\begin{equation*}
a(t) = At^{(p+2)/(1-q)}
\end{equation*}
where
\begin{equation*}
A = \left\{ \frac{(1-q)^2}{L(p+2)(p+q+1)} \right\}^{1/(1-q)} \quad \text{and} \quad L = \int_0^T (T - s) h(s) \, ds.
\end{equation*}
First we check $a(t) \leq 1$ for $t \in (0, T)$. This follows immediately if we show
\begin{equation}
(2.4) \quad \frac{(1-q)^2 T^{p+2}}{(p+2)(p+q+1)} \leq L.
\end{equation}
Now (2.4) is true since
\begin{equation*}
L = \int_0^t (T - s) h(s) \, ds \geq T \int_0^T s^p \, ds - \int_0^T s^{p+1} \, ds
= \frac{1}{(p+1)(p+2)} T^{p+2} \geq \frac{(1-q)^2}{(p+2)(p+q+1)}
\end{equation*}
since
\begin{equation*}
\frac{(1-q)^2}{(p+q+1)} \leq \frac{1}{p+1}.
\end{equation*}
Thus $0 < a(t) \leq 1$ for $t \in (0, T)$. Now (e) follows immediately since for $n \in \mathbb{N}$, $R > 0$, and $t \in [0, t_n]$ we have
\begin{align*}
&\frac{\int_0^t k(t, s) f(R a(s)) \, ds}{f(R) \int_0^t k(T, s) \, ds} \geq \frac{R^q \int_0^t (t - s) s^p \, ds}{R^q \int_0^T (T - s) h(s) \, ds} \\
&= \frac{A^q}{L} \int_0^t (t - s)^{(p+2q)/(1-q)} \, ds = \frac{A^q}{L} \left[ \frac{(1-q)}{p+q+1} - \frac{(1-q)}{p+2} \right] t^{(p+2)/(1-q)} \\
&= \frac{A}{A^{1-q} L} \frac{(1-q)^2}{(p+q+1)(p+2)} t^{(p+2)/(1-q)} = At^{(p+2)/(1-q)} = a(t).
\end{align*}
It remains to construct constants $R_2 > 0$, $R_1 > R_2$ so that (f) and (g) hold. Fix $n \in \mathbb{N}$ and let $R > 0$. Then

$$f(R) \int_0^{t_n} k(t_n, s) \, ds \leq R^q \int_0^T (T - s) h(s) \, ds \leq R$$

for $R$ sufficiently large since $R^{1-q} \to \infty$ as $R \to \infty$. Thus there exists $R_1 > 0$ so that (f) holds. Also

$$\int_0^{t_n} k(t_n, s) f(Ra(s)) \, ds \geq R^q \int_0^{t_n} (t_n - s) [a(s)]^q \, ds \geq R$$

for $R$ sufficiently small since $R_1 - q \to 0$ as $R \to 0^+$. Thus there exists $R_2 > 0$ with $R_2 < R_1$ with (g) holding.

Existence of a positive (positive on $(0, T)$) solution to (2.3) follows from Theorem 2.1. In fact here one can easily show that the solution lies in $C[0, T]$.

**Example 2.5.** Consider the integral equation

$$y(t) = \int_0^t (t - s)^{\alpha-1} h(s) f(y(s)) \, ds, \quad t \in [0, T)$$

where $h: [0, T) \to [0, \infty)$ is continuous and

$$\int_0^T (T - s)^{\alpha-1} h(s) \, ds < \infty,$$

$
\alpha > 1$ and $0 < T < \infty$ is fixed. In addition assume (d) of Theorem 2.1 and the following conditions hold:

(i) $f(ab) = f(a)f(b)$ for $a, b \geq 0$, and

(ii) $F(1) < \infty$ where $F: [0, 1] \to [0, \infty)$ is defined by

$$F(z) = \int_0^z \left[ s \int_0^s \frac{1}{s} \, ds \right]^{1/\beta} \, ds,$$

$z \in [0, 1]$, $\beta > \alpha > 1$ and $c \int_0^T h(s) \, ds \in \text{dom} F^{-1}$ where

$$c = \frac{\beta}{[K_T]^{1/\beta}(\int_0^T (T - s)^{-(\alpha-1)/(\beta-1)} h(s) \, ds)^{(\beta-1)/\beta}}$$

with $K_T = \int_0^T (T - s)^{\alpha-1} h(s) \, ds$. In addition assume conditions (f) and (g) of Theorem 2.1 hold with $k(t, s) = (t - s)^{\alpha-1} h(s)$ and $a \in C[0, T)$ is given by

$$a(t) = F^{-1} \left( c \int_0^t h(s) \, ds \right) \quad \text{for } t \in [0, T)$$

where $c$ is defined in (ii). Then (2.5) has a solution $y \in C[0, T)$.
**Remark 2.6.** We could define \( F \) in (ii) on \([0, \infty)\) i.e.

\[
F(z) = \int_0^z \left[ \frac{s}{f(s)} \right]^{1/\beta} \frac{ds}{s}, \quad z > 0
\]

but in this case we need to assume \( F^{-1}(c \int_0^t h(s) \, ds) \leq 1 \); here \( c \) is defined in (ii).

To see that (2.5) has a solution we will apply Theorem 2.1 with \( k(t, s) = (t - s)^{\alpha-1} h(s) \). Clearly (a)–(d) are satisfied. Notice in this case (e) can be rewritten (see (i)) as

(i') there exists \( a \in C[0, T) \) such that \( a(0) = 0, 0 < a(t) \leq 1, t \in (0, T) \), and for each \( n \in \mathbb{N} \) for any constant \( R > 0 \), \( a \) satisfies

\[
\int_0^t (t - s)^{\alpha-1} h(s) f(a(s)) \, ds \geq a(t) K_T \quad \text{for} \ t \in [0, t_n].
\]

Consider the initial value problem

\[
\begin{aligned}
a'(t) &= ca^{1-1/\beta} h(t) [f(a)]^{1/\beta} \quad \text{for} \ t \in [0, T), \\
a(0) &= 0,
\end{aligned}
\]

and notice (2.6) has a solution \( a \in C[0, T) \) given by

\[
a(t) = F^{-1} \left( c \int_0^t h(s) \, ds \right) \quad \text{for} \ t \in [0, T).
\]

From (ii) (see also Remark 2.6) notice \( 0 < a(t) \leq 1 \) for \( t \in (0, T) \). Fix \( n \in \mathbb{N} \) and notice

\[
a^{1/\beta - 1} a^{1/\beta - 1} = ch[f(a)]^{1/\beta} \quad \text{for} \ t \in [0, t_n]
\]

so

\[
\beta^3 a(t) = c^\beta \left( \int_0^t h(s) [f(a(s))]^{1/\beta} \, ds \right) \beta
\]

and this together with Hölder’s inequality implies

\[
a(t) \leq \frac{c^\beta}{\beta^3} \left( \int_0^t (t - s)^{\alpha-1} h(s) f(a(s)) \, ds \right) \times \left( \int_0^t (t - s)^{-(\alpha-1)/(\beta-1)} h(s) \, ds \right)^{\beta-1}
\]

\[
\leq \frac{1}{K_T} \int_0^t (t - s)^{\alpha-1} h(s) f(a(s)) \, ds
\]

from the definition of \( c \) in (ii). Thus (i') (and so Theorem 2.1(e)) is satisfied. The result now follows from Theorem 2.1.

**Remark 2.7.** It is also possible to construct “\( a \)” in Theorem 2.1(e) if the kernel is not of the form \( (t - s)^\kappa h(s) \); see for example Theorem 3.1 in [5].
Example 2.8. Consider

\[ y(t) = \int_{0}^{t} q(s)[y(s)]^\beta \, ds \quad \text{for } t \in [0, \infty) \]

with \( q: [0, \infty) \rightarrow [0, \infty) \) continuous and \( \int_{0}^{\infty} q(s) \, ds < \infty \) and \( 0 \leq \beta < 1 \). Now

(2.3) has a positive solution (positive on \((0, T)\)); note \( y \equiv 0 \) is also a solution of (2.7).

Let \( k(t, s) = q(s) \) and \( f(y) = y^\beta \). Clearly (a)–(d) of Theorem 2.1 holds and it is easy to see that (e) is satisfied with

\[ a(t) = \left( \frac{(1 - \beta) \int_{0}^{t} q(s) \, ds}{\int_{0}^{\infty} q(s) \, ds} \right)^{1/(1 - \beta)}. \]

Finally (f) and (g) of Theorem 2.1 hold since \( R^{1 - \beta} \to \infty \) as \( R \to \infty \) and \( R^{1 - \beta} \to 0 \) as \( R \to 0^+ \). The result now follows from Theorem 2.1.

References


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