Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 28, 2006, 189–198

POSITIVE SOLUTIONS FOR A CLASS OF VOLTERRA INTEGRAL EQUATIONS VIA A FIXED POINT THEOREM IN FRÉCHET SPACES

Ravi P. Agarwal — Donal O'Regan

Abstract. Motivated by the Emden differential equation we discuss in this paper the existence of positive solutions to the integral equation

$$y(t) = \int_0^t k(t,s)\, f(y(s))\, ds \quad \text{for } t \in [0,T).$$

1. Introduction

In this paper we establish the existence of positive (positive on $(0,T)$) solutions to the Volterra integral equation

$$y(t) = \int_0^t k(t,s)\, f(y(s))\, ds \quad \text{for } t \in [0,T) \tag{1.1}$$

where $0 < T \leq \infty$ is fixed. Our theory was motivated by the Emden differential equation

$$y'' - t^p\, y^q = 0, \quad p \geq 0 \text{ and } 0 < q < 1 \tag{1.2}$$

which arises in various astrophysical problems, including the study of the density of stars; of course one is interested only in positive solutions to (1.2). Differential equations including (1.2) will be discussed as a special case of (1.1) in Section 2.

2000 *Mathematics Subject Classification.* Volterra integral equation, Emden differential equation, positive solution, fixed point theorem.

Key words and phrases. and phrases 45D05, 45M20, 47H10.

We remark also when the kernel k is a convolution kernel (1.1) arises in connection with nonlinear diffusion and percolation problems (see [3] and the references therein). The results in Section 2 extend and complement the theory in [3], [5].

For notational purposes in this paper if $u \in C[0,T)$ then for every $m \in \{1,2,\dots\} = N$ we define the seminorms $\rho_m(u)$ by

$$\rho_m(u) = \sup_{t\in[0,t_m]} |u(t)|$$

where $t_m \uparrow T$. Note $C[0,T)$ is a locally convex linear topological space. The topology on $C[0,T)$, induced by the seminorms $\{\rho_m\}_{m\in\mathbb{N}}$, is the topology of uniform convergence on every compact interval of $[0,T)$.

Existence in Section 2 is based on a fixed point theorem of Agarwal and O'Regan [2] which in turn is based on Krasnoselskii's fixed point theorem in a cone. We present the result in [2] (see also [1]) for the convenience of the reader. First however we state Krasnosel'skiĭ's result.

Theorem 1.1. *Let $B = (B, \|\cdot\|)$ be a Banach space and let $C \subseteq E$ be a cone in B. Assume Ω_1 and Ω_2 are open bounded subsets of B with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let*

$$S\colon C \cap (\overline{\Omega_2} \setminus \Omega_1) \to C$$

be a continuous compact map such that either

(a) $\|Su\| \le \|u\|$ *for* $u \in C \cap \partial\Omega_1$ *and* $\|Su\| \ge \|u\|$ *for* $u \in C \cap \partial\Omega_2$, *or*
(b) $\|Su\| \ge \|u\|$ *for* $u \in C \cap \partial\Omega_1$ *and* $\|Su\| \le \|u\|$ *for* $u \in C \cap \partial\Omega_2$,

hold. Then S has a fixed point in $C \cap (\overline{\Omega_2} \setminus \Omega_1)$.

The result in [2] is based on the fact that a Fréchet space can be viewed as a projective limit of a sequence of Banach spaces $\{E_n\}_{n\in\mathbb{N}}$. We now extend Theorem 1.1 to the Fréchet space setting. Let $E = (E, \{|\cdot|_n\}_{n\in\mathbb{N}})$ be a Fréchet space with

$$|x|_1 \le |x|_2 \le |x|_3 \le \dots \quad \text{for every } x \in E.$$

Assume for each $n \in \mathbb{N}$ that $(E_n, |\cdot|_n)$ is a Banach space and suppose

$$E_1 \supseteq E_2 \supseteq \dots$$

with $|x|_n \le |x|_{n+1}$ for all $x \in E_{n+1}$. Also assume $E = \bigcap_{n=1}^\infty E_n$ where $\bigcap_1^\infty$ is the generalized intersection as described in [4, pp. 439] (i.e. E is the projective limit of $\{E_n\}_{n\in\mathbb{N}}$) with the embedding $\mu_n\colon E \to E_n$. Fix $n \in \mathbb{N}$ and let C_n will a cone in E_n and for $\rho > 0$ we let

$$U_{n,\rho} = \{x \in E_n : |x|_n < \rho\} \quad \text{and} \quad V_{n,\rho} = U_{n,\rho} \cap C_n.$$

Notice

$$\partial_{C_n} V_{n,\rho} = \partial_{E_n} U_{n,\rho} \cap C_n \quad \text{and} \quad \overline{V_{n,\rho}} = \overline{U_{n,\rho}} \cap C_n$$

(the first closure is with respect to C_n whereas the second is with respect to E_n). We are interested in establishing that F has a fixed point; here $F: E_1 \to E_1$.

Definition 1.2. Fix $k \in \mathbb{N}$. If $x, y \in E_k$ then we say $x = y$ in E_k if $|x - y|_k = 0$.

Definition 1.3. If $x, y \in E$ then we say $x = y$ in E if $x = y$ in E_k for each $k \in \mathbb{N}$.

Theorem 1.4. *For each $n \in \mathbb{N}$, let C_n be a cone in E_n and also let*

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq \dots$$

In addition suppose $F: E_1 \to E_1$. Let γ, r, R be constants with $0 < \gamma < r < R$ and assume the following conditions are satisfied:

(a) *for each $n \in \mathbb{N}$, $F: \overline{U_{n,R}} \cap C_n \to C_n$ is a continuous compact map,*
(b) *for each $n \in \mathbb{N}$, $|F\,x|_n \leq |x|_n$, for all $x \in \partial_{E_n} U_{n,r} \cap C_n$,*
(c) *for each $n \in \mathbb{N}$, $|F\,x|_n \geq |x|_n$, for all $x \in \partial_{E_n} U_{n,R} \cap C_n$, and*
(d) *for every $k \in \mathbb{N}$ and any subsequence $A \subseteq \{k, k+1, \dots\}$ if $x \in C_n$ is such that $R \geq |x|_n \geq r$ for some $n \in A$, then $|x|_k \geq \gamma$.*

Then F has a fixed point $y \in E$ (in fact $\mu_n(y) \in (\overline{U_{n,R}} \setminus U_{n,\gamma}) \cap C_n$ for every $n \in \mathbb{N}$).

Remark 1.5. Of course there is an obvious analogue of Theorem 1.2 when $U_{n,r}$ is replaced by $U_{n,R}$ in (b) and $U_{n,R}$ is replaced by $U_{n,r}$ in (c).

Proof. We know from Theorem 1.1 (part (a)) that for each $n \in \mathbb{N}$, F has a fixed point $y_n \in (\overline{U_{n,R}} \setminus U_{n,r}) \cap C_n$. Lets look at $\{y_n\}_{n \in \mathbb{N}}$. Note $y_n \in \overline{U_{1,R}} \setminus U_{1,\gamma}$ for each $n \in \mathbb{N}$. To see this notice $|y_n|_n \leq R$ and $|x|_1 \leq |x|_n$ for all $x \in E_n$ implies $|y_n|_1 \leq R$, and so $y_n \in \overline{U_{1,R}}$ for each $n \in \mathbb{N}$. On the other hand $|y_n|_n \geq r$, $y_n \in C_n$ together with (d) implies $|y_n|_1 \geq \gamma$. Thus $y_n \in (\overline{U_{1,R}} \setminus U_{1,\gamma}) \cap C_1$ and $y_n = F\,y_n$ in E_n for each $n \in \mathbb{N}$ and these together with (a) implies that there exists a subsequence N_1^* of N and a $z_1 \in (\overline{U_{1,R}} \setminus U_{1,\gamma}) \cap C_1$ with $y_n \to z_1$ in E_1 as $n \to \infty$ in N_1^*. Notice in particular that $\gamma \leq |z_1|_1 \leq R$.

Let $N_1 = N_1^* \setminus \{1\}$. Now look at $\{y_n\}_{n \in N_1}$. Again (a) guarantees that there exists a subsequence N_2^* of N_1 and a $z_2 \in (\overline{U_{2,R}} \setminus U_{2,\gamma}) \cap C_2$ with $y_n \to z_2$ in E_2 as $n \to \infty$ in N_2^* and $\gamma \leq |z_2|_2 \leq R$. Note also $|z_2 - z_1|_1 = 0$ since $N_2^* \subseteq N_1$ and $E_1 \supseteq E_2$, so $z_2 = z_1$ in E_1.

Proceed inductively to obtain subsequences of integers

$$N_1^* \supseteq N_2^* \supseteq \dots, N_k^* \subseteq \{k, k+1, \dots\}$$

and $z_k \in (\overline{U_{k,R}} \setminus U_{k,\gamma}) \cap C_k$ with $y_n \to z_k$ in E_k as $n \to \infty$ in N_k^*. Note $z_{k+1} = z_k$ in E_k for $k \in \{1, 2, \dots\}$. Also let $N_k = N_k^* \setminus \{k\}$.

Fix $k \in \mathbb{N}$. Let $y = z_k$ in E_k (i.e. $\mu_k(y) = z_k$). Notice y is well defined and $y \in E$. Now $y_n = F y_n$ in E_n for $n \in N_k$ and $y_n \to y$ in E_k as $n \to \infty$ in N_k (since $y = z_k$ in E_k) together with (a) implies $y = Fy$ in E_k. We can do this for each $k \in \mathbb{N}$ so $y = Fy$ in E. □

Remark 1.6. From the proof above notice (a) in Theorem 1.4 could be replaced by the condition:

(a') for each $n \in \mathbb{N}$, $F: (\overline{U_{n,R}} \setminus U_{n,\gamma}) \cap C_n \to C_n$ is a continuous compact map

and the result of Theorem 1.4 is again true. Also $F: E_1 \to E_1$ in the statement of Theorem 1.4 could be replaced by $F: \overline{U_{1,R}} \cap C_1 \to E_1$.

2. Volterra integral equations

We now use Theorem 1.4 to establish an existence result for (1.2). Notice T could be ∞ in Theorem 2.1.

Theorem 2.1. *Suppose the following conditions are satisfied:*

(a) *for each* $n \in \mathbb{N}$, $0 < k(t,s)$ *for all* $t \in (0, t_n]$, *a.e.* $s \in [0,t]$ *and* $k_t(s) = k(t,s) \in L^1[0,t]$ *for each* $t \in [0, t_n]$ *and* $\sup_{t \in [0,T)} \int_0^t [k_t(s)]\, ds < \infty$,

(b) *for each* $n \in \mathbb{N}$, *for any* $t, t' \in [0, t_n]$,

$$\int_0^{t^*} |k_t(s) - k_{t'}(s)|\, ds \to 0 \quad \text{as } t \to t'$$

where $t^* = \min\{t, t'\}$,

(c) *for each* $n \in \mathbb{N}$, $k(x_1, s) - k(x_2, s) \geq 0$ *for a.e.* $s \in [0, x_2]$ *where* $0 < x_2 < x_1 \leq t_n$,

(d) $f: [0,\infty) \to [0,\infty)$ *is continuous and nondecreasing with* $f(y) > 0$ *for* $y > 0$,

(e) *there exists* $a \in C[0,T)$ *such that* $a(0) = 0$, $0 < a(t) \leq 1$, $t \in (0,T)$, *and for each* $n \in \mathbb{N}$ *for any constant* $R > 0$, a *satisfies*

$$\int_0^t k(t,s) f(R\, a(s))\, ds \geq a(t) f(R) \int_0^T k(T,s)\, ds$$

for $t \in [0, t_n]$,

(f) *for each* $n \in \mathbb{N}$, *there exists* $R_1 > 0$ (*independent of* n) *with*

$$f(R_1) \int_0^{t_n} k(t_n, s)\, ds \leq R_1,$$

(g) *for each* $n \in \mathbb{N}$, *there exists* $R_2 > 0$ (*independent of* n), $R_2 \neq R_1$ *with*

$$\int_0^{t_n} k(t_n, s)\, f(R_2\, a(s))\, ds \geq R_2.$$

Then (1.1) *has at least one solution* $y \in C[0,T)$ *and either*

(A) *for each* $n \in \mathbb{N}$, $0 < \gamma \leq |y|_n \leq R_2$ *and* $y(t) \geq a(t)\,\gamma$ *for* $t \in [0, t_n]$ *if* $R_1 < R_2$ (*here* $\gamma = a(t_1)\, R_1$),

or

(B) *for each* $n \in \mathbb{N}$, $0 < \gamma \leq |y|_n \leq R_1$ *and* $y(t) \geq a(t)\gamma$ *for* $t \in [0, t_n]$ *if* $R_2 < R_1$ (*here* $\gamma = a(t_1)\, R_2$).

Remark 2.2. When $T = \infty$ notice by $\int_0^T k(T,s)\,ds$ in Theorem 2.1(e) we mean $\lim_{t\to\infty} \int_0^t k(t,s)\,ds$.

Remark 2.3. Notice if (b) in Theorem 2.1 is replaced by

(b') for each $n \in \mathbb{N}$, for any $t, t' \in [0, t_n]$,

$$\int_0^{t^*} |k_t(s) - k_{t'}(s)|\,ds + \int_{t^*}^{t^{**}} [k_{t^{**}}(s)] \to 0 \quad \text{as } t \to t'$$

where $t^* = \min\{t, t'\}$ and $t^{**} = \max\{t, t'\}$,

then automatically

$$\sup_{t\in[0,t_n]} \int_0^t [k_t(s)]\,ds < \infty$$

in Theorem 2.1(a).

Proof of Theorem 2.1. Without loss of generality assume $R_1 < R_2$. Fix $n \in \mathbb{N}$ and let $E_n = C[0, t_n]$, and

$$C_n = \{y \in C[0,t_n] : y(t) \geq a(t)|y|_n \text{ for } t \in [0,t_n]\}$$

where $|y|_n = \sup_{t\in[0,t_n]} |y(t)|$. Let

$$Fy(t) = \int_0^t k(t,s) f(y(s))\,ds$$

and $U_{n,\beta} = \{y \in C[0,t_n] : |y|_n < \beta\}$; here $\beta = R_1$ or R_2.

Now let $y \in C_n \cap \overline{U_{n,R_2}}$. Then (c) implies $Fy(t)$ is increasing in t. Also there exists $R \in [0, R_2]$ such that $|y|_n = R$ so $a(t)R \leq y(t) \leq R$ for $t \in [0, t_n]$ and as a result

$$Fy(t) \geq \int_0^t k(t,s) f(R\,a(s))\,ds, \quad t \in [0,t_n]$$

with

$$|Fy|_n = Fy(t_n) \leq \int_0^{t_n} k(t_n,s) f(R)\,ds.$$

Thus for $t \in [0, t_n]$ we have

$$Fy(t) \geq \frac{\int_0^t k(t,s) f(R\,a(s))\,ds}{\int_0^{t_n} k(t_n,s)\, f(R)\,ds} |F\,y|_n \geq \frac{\int_0^t k(t,s) f(R\,a(s))\,ds}{\int_0^T k(T,s) f(R)\,ds} |Fy|_n$$

and this together with (e) yields

$$Fy(t) \geq a(t)|Fy|_n \quad \text{for } t \in [0, t_n],$$

so $F: C_n \cap \overline{U_{n,R_2}} \to C_n$. A standard argument [5] guarantees that $F: C_n \cap \overline{U_{n,R_2}} \to C_n$ is a continuous, compact map. Next we show

$$|F\,x|_n \leq |x|_n \quad \text{for all } x \in \partial U_{n,R_1} \cap C_n \tag{2.1}$$

and

$$|F\,x|_n \geq |x|_n \quad \text{for all } x \in \partial U_{n,R_2} \cap C_n. \tag{2.2}$$

Let $x \in \partial U_{n,R_1} \cap C_n$. Then $|x|_n = R_1$ and $0 \leq a(t)\,R_1 \leq x(t) \leq R_1$ for $t \in [0, t_n]$. Also Theorem 2.1(f) guarantees that

$$|F\,x|_n = F\,x(t_n) \leq \int_0^{t_n} k(t_n, s)\, f(R_1)\, ds \leq R_1 = |x|_n,$$

so (2.1) is true.

Let $x \in \partial U_{n,R_2} \cap C_n$. Then $|x|_n = R_2$ and $0 \leq a(t)\,R_2 \leq x(t) \leq R_2$ for $t \in [0, t_n]$. Also Theorem 2.1(g) guarantees that

$$|Fx|_n = Fx(t_n) \geq \int_0^{t_n} k(t_n, s) f(a(s)R_2)\, ds \geq R_2 = |x|_n,$$

so (2.2) is true.

The result follows immediately from Theorem 1.4 once we show Theorem 1.4(d) holds (with $\gamma = a(t_1)R_1$). Fix $k \in \mathbb{N}$ and any subsequence $A \subseteq \{k, k+1, \dots\}$. Let $n \in A$ and $x \in C_n$ with $R_1 \leq |x|_n \leq R_2$. Then $R_1 \leq \sup_{t\in[0,t_n]} |x(t)| \leq R_2$ so

$$x(t) \geq a(t)|x|_n \geq a(t)R_1 \quad \text{for } t \in [0, t_n].$$

Now since $n \in A \subseteq \{k, k+1, \dots\}$ we have $n \geq k$ so (note $t_n \uparrow T$)

$$x(t) \geq a(t)R_1 \quad \text{for } t \in [0, t_k].$$

In particular $t_1 \in [0, t_k]$ so $x(t_1) \geq a(t_1)R_1$ and so

$$|x|_k = \sup_{t\in[0,t_k]} |x(t)| \geq a(t_1)R_1 = \gamma.$$

Thus Theorem 1.4(d) holds, so Theorem 1.4 guarantees that F has a fixed point $y \in C[0, T)$ with for each $n \in \mathbb{N}$,

$$\gamma \leq |y|_n \leq R \quad \text{and} \quad y(t) \geq a(t)|y|_n \geq a(t)\gamma \quad \text{for } t \in [0, t_n];$$

here $\gamma = a(t_1)R_1$. □

Example 2.4. Consider the generalized Emden equation

$$\begin{cases} y'' - h(t)y^q = 0 & \text{for } t \in [0,T), \\ y(0) = y'(0) = 0, \end{cases} \tag{2.3}$$

with $0 < q < 1$, $h\colon [0,T) \to [0,\infty)$ continuous with $h(t) \geq t^p$, $p \geq 0$ and $\int_0^T (T-s)h(s)\,ds < \infty$; here $0 < T < \infty$ is fixed. We will show (2.3) has a positive solution (positive on $(0,T)$); note $y \equiv 0$ is also a solution of (2.3).

First notice solving (2.3) is equivalent to solving the integral equation

$$y(t) = \int_0^t (t-s)h(s)[y(s)]^q\,ds \quad \text{for } t \in [0,T).$$

Let

$$k(t,s) = (t-s)\,h(s) \quad \text{and} \quad f(y) = y^q$$

in Theorem 2.1. Clearly (a)–(d) hold. Next we show (e) is satisfied with

$$a(t) = At^{(p+2)/(1-q)}$$

where

$$A = \left\{ \frac{(1-q)^2}{L\,(p+2)\,(p+q+1)} \right\}^{1/(1-q)} \quad \text{and} \quad L = \int_0^T (T-s)h(s)\,ds.$$

First we check $a(t) \leq 1$ for $t \in (0,T)$. This follows immediately if we show $A^{1-q}T^{p+2} \leq 1$, and this will be true if

$$\frac{(1-q)^2\,T^{p+2}}{(p+2)\,(p+q+1)} \leq L. \tag{2.4}$$

Now (2.4) is true since

$$\begin{aligned} L &= \int_0^t (T-s)\,h(s)\,ds \geq T\int_0^T s^p\,ds - \int_0^T s^{p+1}\,ds \\ &= \frac{1}{(p+1)\,(p+2)}T^{p+2} \geq \frac{T^{p+2}}{(p+2)}\,\frac{(1-q)^2}{(p+q+1)} \end{aligned}$$

since

$$\frac{(1-q)^2}{(p+q+1)} \leq \frac{1}{p+1}.$$

Thus $0 < a(t) \leq 1$ for $t \in (0,T)$. Now (e) follows immediately since for $n \in \mathbb{N}$, $R > 0$, and $t \in [0,t_n]$ we have

$$\begin{aligned} &\frac{\int_0^t k(t,s)\,f(R\,a(s))\,ds}{f(R)\int_0^T k(T,s)\,ds} \geq \frac{R^q\int_0^t (t-s)\,s^p\,[a(s)]^q\,ds}{R^q\int_0^T (T-s)\,h(s)\,ds} \\ &= \frac{A^q}{L}\int_0^t (t-s)\,s^{(p+2q)/(1-q)}\,ds = \frac{A^q}{L}\left[\frac{(1-q)}{p+q+1} - \frac{(1-q)}{p+2}\right]t^{(p+2)/(1-q)} \\ &= \frac{A}{A^{1-q}\,L}\,\frac{(1-q)^2}{(p+q+1)(p+2)}\,t^{(p+2)/(1-q)} = At^{(p+2)/(1-q)} = a(t). \end{aligned}$$

It remains to construct constants $R_2 > 0$, $R_1 > R_2$ so that (f) and (g) hold. Fix $n \in \mathbb{N}$ and let $R > 0$. Then

$$f(R)\int_0^{t_n} k(t_n,s)\,ds \le R^q \int_0^T (T-s)h(s)\,ds \le R$$

for R sufficiently large since $R^{1-q} \to \infty$ as $R \to \infty$. Thus there exists $R_1 > 0$ so that (f) holds. Also

$$\begin{aligned}\int_0^{t_n} k(t_n,s)f(Ra(s))\,ds &\ge R^q \int_0^{t_n} (t_n-s)[a(s)]^q\,ds \\ &\ge R^q \int_0^{t_1} (t_1-s)\,[a(s)]^q\,ds \ge R\end{aligned}$$

for R sufficiently small since $R^{1-q} \to 0$ as $R \to 0^+$. Thus there exists $R_2 > 0$ with $R_2 < R_1$ with (g) holding.

Existence of a positive (positive on $(0,T)$) solution to (2.3) follows from Theorem 2.1. In fact here one can easily show that the solution lies in $C[0,T]$.

Example 2.5. Consider the integral equation

$$y(t) = \int_0^t (t-s)^{\alpha-1}h(s)f(y(s))\,ds, \quad t \in [0,T) \tag{2.5}$$

where $h\colon [0,T) \to [0,\infty)$ is continuous and

$$\int_0^T (T-s)^{\alpha-1}\,h(s)\,ds < \infty,$$

$\alpha > 1$ and $0 < T < \infty$ is fixed. In addition assume (d) of Theorem 2.1 and the following conditions hold:

(i) $f(a\,b) = f(a)f(b)$ for $a, b \ge 0$, and
(ii) $F(1) < \infty$ where $F\colon [0,1] \to [0,\infty)$ is defined by

$$F(z) = \int_0^z \left[\frac{s}{f(s)}\right]^{1/\beta}\frac{ds}{s},$$

$z \in [0,1]$, $\beta > \alpha > 1$ and $c\int_0^T h(s)\,ds \in \operatorname{dom} F^{-1}$ where

$$c = \frac{\beta}{[K_T]^{1/\beta}(\int_0^T (T-s)^{-(\alpha-1)/(\beta-1)}h(s)\,ds)^{(\beta-1)/\beta}}$$

with $K_T = \int_0^T (T-s)^{\alpha-1}\,h(s)\,ds$.

In addition assume conditions (f) and (g) of Theorem 2.1 hold with $k(t,s) = (t-s)^{\alpha-1}h(s)$ and $a \in C[0,T)$ is given by

$$a(t) = F^{-1}\left(c\int_0^t h(s)\,ds\right) \quad \text{for } t \in [0,T)$$

where c is defined in (ii). Then (2.5) has a solution $y \in C[0,T)$.

Remark 2.6. We could define F in (ii) on $[0,\infty)$ i.e.

$$F(z)=\int_0^z\left[\frac{s}{f(s)}\right]^{1/\beta}\frac{ds}{s},\quad z>0$$

but in this case we need to assume $F^{-1}(c\int_0^t h(s)\,ds)\le 1$; here c is defined in (ii).

To see that (2.5) has a solution we will apply Theorem 2.1 with $k(t,s)=(t-s)^{\alpha-1}h(s)$. Clearly (a)–(d) are satisfied. Notice in this case (e) can be rewritten (see (i)) as

(i') there exists $a\in C[0,T)$ such that $a(0)=0$, $0<a(t)\le 1$, $t\in(0,T)$, and for each $n\in\mathbb{N}$ for any constant $R>0$, a satisfies

$$\int_0^t (t-s)^{\alpha-1}h(s)f(a(s))\,ds\ge a(t)K_T\quad\text{for } t\in[0,t_n].$$

Consider the initial value problem

$$\begin{cases} a'(t)=ca^{1-1/\beta}h(t)[f(a)]^{1/\beta} & \text{for } t\in[0,T),\\ a(0)=0, \end{cases}\tag{2.6}$$

and notice (2.6) has a solution $a\in C[0,T)$ given by

$$a(t)=F^{-1}\left(c\int_0^t h(s)\,ds\right)\quad\text{for } t\in[0,T).$$

From (ii) (see also Remark 2.6) notice $0<a(t)\le 1$ for $t\in(0,T)$. Fix $n\in\mathbb{N}$ and notice

$$a'a^{1/\beta-1}=ch[f(a)]^{1/\beta}\quad\text{for } t\in[0,t_n]$$

so

$$\beta^\beta\,a(t)=c^\beta\left(\int_0^t h(s)[f(a(s))]^{1/\beta}\,ds\right)^\beta$$

and this together with Hölder's inequality implies

$$\begin{aligned} a(t)&\le\frac{c^\beta}{\beta^\beta}\left(\int_0^t (t-s)^{\alpha-1}h(s)f(a(s))\,ds\right)\times\left(\int_0^t (t-s)^{-(\alpha-1)/(\beta-1)}h(s)\,ds\right)^{\beta-1}\\ &\le\frac{1}{K_T}\int_0^t (t-s)^{\alpha-1}h(s)f(a(s))\,ds \end{aligned}$$

from the definition of c in (ii). Thus (i') (and so Theorem 2.1(e)) is satisfied. The result now follows from Theorem 2.1.

Remark 2.7. It is also possible to construct "a" in Theorem 2.1(e) if the kernel is not of the form $(t-s)^\kappa\,h(s)$; see for example Theorem 3.1 in [5].

Example 2.8. Consider

$$y(t) = \int_0^t q(s)[y(s)]^\beta \, ds \quad \text{for } t \in [0,\infty) \tag{2.7}$$

with $q\colon [0,\infty) \to [0,\infty)$ continuous and $\int_0^\infty q(s)\,ds < \infty$ and $0 \le \beta < 1$. Now (2.3) has a positive solution (positive on $(0,T)$); note $y \equiv 0$ is also a solution of (2.7).

Let $k(t,s) = q(s)$ and $f(y) = y^\beta$. Clearly (a)–(d) of Theorem 2.1 holds and it is easy to see that (e) is satisfied with

$$a(t) = \left(\frac{(1-\beta)\int_0^t q(s)\,ds}{\int_0^\infty q(s)\,ds} \right)^{1/(1-\beta)}.$$

Finally (f) and (g) of Theorem 2.1 hold since $R^{1-\beta} \to \infty$ as $R \to \infty$ and $R^{1-\beta} \to 0$ as $R \to 0^+$. The result now follows from Theorem 2.1.

References

[1] R. P. Agarwal, M. Frigon and D. O'Regan, *A survey of recent fixed point theory in Fréchet spaces*, Nonlinear Analysis and Applications: to V. Lakshmikantham on his 80th birthday, vol. 1, Kluwer Acad. Publ., Dordrecht, 2003, pp. 75–88.

[2] R. P. Agarwal and D. O'Regan, *Cone compression and expansion fixed point theorems in Fréchet spaces with applications*, J. Differential Equations **171** (2001), 412–429.

[3] P. J. Bushell and W. Okrasinski, *Uniqueness of solutions for a class of nonlinear Volterra integral equations with convolution kernel*, Math. Proc. Cambridge Philos. Soc. **106** (1989), 547–552.

[4] L. V. Kantorovich and G. P. Akilov, *Functional Analysis in Normed Spaces*, Pergamon Press, Oxford, 1964.

[5] M. Meehan and D. O'Regan, *A note on positive solutions of Volterra integral equations using integral inequalities*, J. Inequalities Appl. **7** (2002), 285–307.

Manuscript received August 30, 2005

Ravi P. Agarwal
Department of Mathematical Science
Florida Institute of Technology
Melbourne, Florida 32901, USA

E-mail address: agarwal@fit.edu

Donal O'Regan
Department of Mathematics
National University of Ireland
Galway, IRELAND

E-mail address: donal.oregan@nuigalway.ie