Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 28, 2006, 171–188

TOPOLOGICAL DEGREE AND GENERALIZED ASYMMETRIC OSCILLATORS

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Abstract. We consider periodic perturbations of an isochronous hamiltonian system in the plane, depending on a parameter, which generalize the classical asymmetric oscillator. We compute the associated topological degree, and consider situations where large-amplitude periodic solutions can arise.

1. Introduction

We consider T-periodic differential systems in the plane which can be written in the form

$$J\dot{u} = \nabla H(u) + f(t, u, \lambda), \tag{1.1}$$

where $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is the standard symplectic matrix, and λ is a real parameter.

The interest in such type of systems comes from the study of the asymmetric oscillator

$$x'' + \mu x^+ - \nu x^- = e(t), \tag{1.2}$$

where μ and ν are positive real numbers, $e\colon \mathbb{R} \to \mathbb{R}$ is a T-periodic continuous function, $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$ (see e.g. [1], [3], [10], [12], [13]). In [5], a new method was proposed for computing the associated topological

2000 *Mathematics Subject Classification.* 34C25.
Key words and phrases. Periodic solutions, isochronoushmiltonin systems.

degree, based on phase-plane analysis of the solutions. It will be useful to briefly recall the main result of [5], focusing our attention on equation (1.2).

We look for the topological degree of $\mathcal{P}-\mathrm{Id}$ with respect to large disks, where $\mathcal{P}$ is the Poincaré map associated to (1.2) for the period T. Denote by $\phi(t)$ a nontrivial solution of the autonomous equation $x''+\mu x^+-\nu x^-=0$, which has minimal period $\tau=\pi/\sqrt{\mu}+\pi/\sqrt{\nu}$, and assume that T is a multiple of τ. It has been proved in [5] that, if the τ-periodic function

$$\Phi(\theta)=\int_0^T e(t)\phi(t+\theta)\,dt \tag{1.3}$$

only has simple zeros, and 2ζ is their number in the interval $[0,\tau[$, then

$$\deg(\mathcal{P}-\mathrm{Id},B_R)=1-\zeta,$$

where B_R denotes any disk centered at the origin with a sufficiently large radius R.

To illustrate such a situation, let

$$e(t)=\cos(nt)+\varepsilon\cos t,$$

for some integer $n\geq 1$, and let μ and ν be such that $\tau=T=2\pi$. In this case, we can choose

$$\phi(t)=\begin{cases}\dfrac{1}{\sqrt{\mu}}\sin(\sqrt{\mu}t) & \text{if } t\in\left[0,\dfrac{\pi}{\sqrt{\mu}}\right],\\ -\dfrac{1}{\sqrt{\nu}}\sin\left(\sqrt{\nu}\left(t-\dfrac{\pi}{\sqrt{\mu}}\right)\right) & \text{if } t\in\left[\dfrac{\pi}{\sqrt{\mu}},2\pi\right],\end{cases}$$

and we find

$$\Phi(\theta)=a_n cos(n\theta)+b_n\sin(n\theta)+\varepsilon\int_0^{2\pi}\cos(t)\phi(t+\theta)\,dt,$$

where, if $\mu\neq n^2$ and $\nu\neq n^2$,

$$\begin{aligned}a_n &=\int_0^{2\pi}\cos(nt)\phi(t)\,dt=\left(\cos\left(\frac{n\pi}{\sqrt{\mu}}\right)+1\right)\frac{\nu-\mu}{(\mu-n^2)(\nu-n^2)},\\ b_n &=\int_0^{2\pi}\sin(nt)\phi(t)\,dt=\sin\left(\frac{n\pi}{\sqrt{\mu}}\right)\frac{\nu-\mu}{(\mu-n^2)(\nu-n^2)}.\end{aligned}$$

Taking μ,ν in order that $\sqrt{\mu}$ and $\sqrt{\nu}$ be irrational, we have that $a_n\neq 0$ and $b_n\neq 0$ for every n, and if $|\varepsilon|$ is small (possibly $\varepsilon=0$), the function Φ has exactly $2n$ simple zeros in the interval $[0,2\pi[$, so that

$$\deg(\mathcal{P}-\mathrm{Id},B_R)=1-n,$$

for R large enough. This shows, in particular, that the degree can be any negative integer.

The above situation has been extended by Wang [14] for a Rayleigh equation and by Capietto and Wang [2] for a Liénard equation. The novelty in these results is the appearance of two functions, which permit to determine the topological degree. In the case of the asymmetric oscillator, these two functions coincide with Φ, as defined in (1.3), and Φ', its derivative. In the more general situation, the two functions differ from the ones determined for the asymmetric oscillator by two constants. The degree, again, can be any negative number.

Both the results in [2], [14] have been generalized in [7] to systems which are the same as (1.1), but with no explicit dependence on the parameter λ. There, the function f was assumed to be asymptotically positively homogeneous of some degree $\beta \in [0,1[$, the precise meaning of which will be recalled in Section 3, and some different types of applications were proposed.

A further step made in [7] was to consider functions f which can be asymptotically positively homogeneous of degree 1, as well, provided that f itself be multiplied by a small parameter. Even in this situation, the degree can be computed by essentially the same method. As a simple example, f could be a linear perturbation of a nonlinear equation, which at first seemed rather surprising.

It is one aim of this paper to show how the above results can be viewed as particular cases of a general theorem for the parameter-dependent system (1.1). Besides this, we propose a simpler approach to compute the topological degree, which should clarify the situations considered previously. Moreover, we are able to give examples where the degree can be an arbitrary positive number, as well as any negative number.

In the same setting, in Section 4 we propose a theorem on the existence of large-amplitude periodic solutions, which completes the theory developed in [6] and shows its connection with the above results on the computation of the degree. This type of theorem has been used in [6] to prove the occurrence of bifurcations from infinity of periodic solutions for second order scalar differential equations modelling asymmetric oscillators with varying parameters.

Besides the study of periodic solutions, the related problem of the boundedness of the solutions has attracted much attention. We just mention that, starting with the asymmetric oscillator, the results in [1], [13] have been recently extended to systems like (1.1) in [8], [9].

2. The Poincaré map

We consider system (1.1), with the following assumptions which generalize the situation for the asymmetric oscillator, as will be shown in Section 3.

The C^1-function $H:\mathbb{R}^2 \to \mathbb{R}$, with locally Lipschitz continuous gradient, is positively homogeneous of degree 2 and positive: we have

$$H(\sigma u) = \sigma^2 H(u) > 0,$$

for every $u \in \mathbb{R}^2 \setminus \{0\}$ and $\sigma > 0$. Under these assumptions, the origin 0 is an isochronous center for the autonomous system

$$J\dot{u} = \nabla H(u). \tag{2.1}$$

Let us fix a reference solution $\varphi: \mathbb{R} \to \mathbb{R}^2$, such that

$$J\dot{\varphi}(t) = \nabla H(\varphi(t)) \quad \text{and} \quad H(\varphi(t)) = 1/2,$$

so that, by Euler's identity,

$$\langle J\dot{\varphi}(t) | \varphi(t)\rangle = \langle \nabla H(\varphi(t)) | \varphi(t)\rangle = 2H(\varphi(t)) = 1, \tag{2.2}$$

for every $t \in \mathbb{R}$. The minimal period of φ will be denoted by τ. Any nontrivial solution of (2.1) is of the type $\rho\varphi(t+\theta)$, for some $\rho > 0$ and $\theta \in [0, \tau[$.

The function $f: \mathbb{R} \times \mathbb{R}^2 \times [1, \infty[\to \mathbb{R}^2$ is assumed to be continuous in its first two variables, T-periodic in its first variable, and locally Lipschitz continuous in its second variable (here and in the following, we denote by $\mathbb{R}_+$ the set of positive real numbers).

A finite number of directions being given,

$$\alpha_1 < \alpha_2 < \ldots < \alpha_l < \alpha_{l+1} = \alpha_1 + 2\pi,$$

we define the set $\Sigma = \{\rho e^{i\alpha_k} : \rho \geq 0,\ k = 1, \ldots, l\}$, which is made of l rays starting from the origin. If $l = 0$, there are none of these directions, and we set $\Sigma = \{0\}$. Let $a: \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that

$$\lim_{\lambda \to \infty} \frac{a(\lambda)}{\lambda} = 0, \tag{2.3}$$

and assume that there exists a continuous function $F: \mathbb{R} \times (\mathbb{R}^2 \setminus \Sigma) \to \mathbb{R}^2$ such that

$$F(t, u) = \lim_{\lambda \to \infty} \frac{f(t, \lambda u, \lambda)}{a(\lambda)}, \tag{2.4}$$

the above limit being uniform with respect to (t, u) when u varies in compact subsets of $\mathbb{R}^2 \setminus \Sigma$. Moreover, for some $\gamma > 0$,

$$\|f(t, \lambda u, \lambda)\| \leq \gamma a(\lambda)(\|u\| + 1), \tag{2.5}$$

for every $t \in \mathbb{R}$, $\lambda \geq 1$ and $u \in \mathbb{R}^2$. It then follows, in particular, that F transforms bounded subsets of $\mathbb{R} \times (\mathbb{R}^2 \setminus \Sigma)$ into bounded sets in $\mathbb{R}^2$.

In this paper, we assume throughout that

T is an integer multiple of τ.

In this situation, it is said that system (1.1) is "at resonance".

By the change of variable $v = \lambda^{-1} u$, system (1.1) becomes

$$J\dot{v} = \nabla H(v) + \frac{1}{\lambda} f(t, \lambda v, \lambda). \tag{2.6}$$

We denote by $\widetilde{\mathcal{P}}_\lambda: \mathbb{R}^2 \to \mathbb{R}^2$ the Poincaré map associated to (2.6) for the period T. It is well defined, since the right hand side of (2.6) is locally Lipschitz continuous and, by (2.5) and the fact that ∇H is positively homogeneous of degree 1, has at most linear growth in v, for every fixed $\lambda \geq 1$.

If $v(t)$ is a solution of (2.6) with starting point $v(0) \neq 0$, we can write

$$v(t) = \rho(t)\varphi(t + \theta(t)),$$

with $\rho(0) > 0$. As long as $\rho(t) > 0$, the functions $\theta(t)$ and $\rho(t)$ are of class C^1 and, since ∇H is positively homogeneous of degree 1,

$$\rho'(t)J\varphi(t + \theta(t)) + \rho(t)\theta'(t)J\dot\varphi(t + \theta(t)) = \frac{1}{\lambda} f(t, \lambda\rho(t)\varphi(t + \theta(t)), \lambda).$$

Since $\varphi(t+\theta(t))$ and $\dot\varphi(t+\theta(t))$ are linearly independent, for every t, as long as $\rho(t) > 0$, by (2.2) the system (2.6) is equivalent to

$$\begin{cases} \theta' = \dfrac{1}{\lambda\rho}\langle f(t, \lambda\rho\varphi(t+\theta), \lambda) | \varphi(t+\theta)\rangle, \\[2mm] \rho' = -\dfrac{1}{\lambda}\langle f(t, \lambda\rho\varphi(t+\theta), \lambda) | \dot\varphi(t+\theta)\rangle. \end{cases} \tag{2.7}$$

Denote by $(\theta(t;\theta_0,\rho_0;\lambda), \rho(t;\theta_0,\rho_0;\lambda))$ the solution of (2.7) with starting point

$$\theta(0;\theta_0,\rho_0;\lambda) = \theta_0 \in [0,\tau[, \quad \rho(0;\theta_0,\rho_0;\lambda) = \rho_0 > 0.$$

Writing briefly $\rho(t)$ for $\rho(t;\theta_0,\rho_0;\lambda)$ and $\theta(t)$ for $\theta(t;\theta_0,\rho_0;\lambda)$, by (2.5) we have, for some constants c_1, c_2 depending only on φ,

$$\begin{aligned} |\rho(t) - \rho_0| &= \left| \frac{1}{\lambda} \int_0^t \langle f(s, \lambda\rho(s)\varphi(s+\theta(s)), \lambda) | \dot\varphi(s+\theta(s))\rangle \, ds \right| \\ &\leq \int_0^t \gamma \frac{a(\lambda)}{\lambda} (\|\rho(s)\varphi(s+\theta(s))\| + 1)\|\dot\varphi(s+\theta(s))\| \, ds \\ &\leq c_1\gamma \frac{a(\lambda)}{\lambda} \int_0^t |\rho(s) - \rho_0| \, ds + c_2\gamma\frac{a(\lambda)}{\lambda}(\rho_0 + 1)t, \end{aligned}$$

so that, by Gronwall inequality,

$$|\rho(t) - \rho_0| \leq c_2\gamma\frac{a(\lambda)}{\lambda}(\rho_0+1)t \ \exp\left(c_1\gamma\frac{a(\lambda)}{\lambda}t\right).$$

Hence, by (2.3),

$$\lim_{\lambda\to\infty} \rho(t;\theta_0,\rho_0;\lambda) = \rho_0, \tag{2.8}$$

uniformly with respect to $t \in [0,T]$, $\theta_0 \in [0,\tau[$ and ρ_0 in a compact subset of $\mathbb{R}_+$. In particular, for λ large enough, we have $\rho(t) > 0$ for every $t \in [0,T]$. Concerning

$\theta(t)$, for λ sufficiently large we have, for some constants c_3, c_4 depending only on φ,

$$
\begin{aligned}
|\theta(t)-\theta_0| &= \left|\frac{1}{\lambda}\int_0^t \frac{1}{\rho(s)}\langle f(s,\lambda\rho(s)\varphi(s+\theta(s)),\lambda)|\varphi(s+\theta(s))\rangle\,ds\right| \\
&\le \int_0^t \gamma\frac{a(\lambda)}{\lambda}\left(\|\varphi(s+\theta(s))\|+\frac{1}{|\rho(s)|}\right)\|\varphi(s+\theta(s))\|\,ds \\
&\le \gamma\frac{a(\lambda)}{\lambda}\left(c_3 t+\frac{c_4}{\rho_0}\right),
\end{aligned}
$$

so that, again by (2.3),

$$
\lim_{\lambda\to\infty}\theta(t;\theta_0,\rho_0;\lambda)=\theta_0, \tag{2.9}
$$

uniformly with respect to $t\in[0,T]$, $\theta_0\in[0,\tau[$ and ρ_0 in a compact subset of $\mathbb{R}_+$.

For $\theta_0\in[0,\tau[$ and $\rho_0>0$, writing $\widetilde{\mathcal{P}}_\lambda(\rho_0\varphi(\theta_0))=\rho_1\varphi(\theta_1)$, for λ large enough we have

$$
\begin{cases}
\theta_1=\theta_0+\displaystyle\int_0^T \frac{1}{\lambda\rho(t)}\langle f(t,\lambda\rho(t)\varphi(t+\theta(t)),\lambda)|\varphi(t+\theta(t))\rangle\,dt, \\
\rho_1=\rho_0-\displaystyle\int_0^T \frac{1}{\lambda}\langle f(t,\lambda\rho(t),\lambda\varphi(t+\theta(t)))|\dot\varphi(t+\theta(t))\rangle\,dt,
\end{cases}
$$

where, as before, $(\theta(t),\rho(t))$ denotes the solution of (2.7) with starting point (θ_0,ρ_0). Define the two functions

$$
\begin{aligned}
\mathcal{F}_1(\theta,\rho) &= -\frac{1}{\rho}\int_0^T \langle F(t,\rho\varphi(t+\theta))|\varphi(t+\theta)\rangle\,dt, \\
\mathcal{F}_2(\theta,\rho) &= \int_0^T \langle F(t,\rho\varphi(t+\theta))|\dot\varphi(t+\theta)\rangle\,dt.
\end{aligned} \tag{2.10}
$$

Because of the properties of φ, the set $\{t\in[0,T]:\varphi(t+\theta)\in\Sigma\}$ is finite, so that, since F maps bounded sets into bounded sets, $\mathcal{F}_1$ and $\mathcal{F}_2$ are well defined on $\mathbb{R}\times\mathbb{R}_+$ and they are continuous.

Lemma 2.1. *We have*

$$
\begin{cases}
\theta_1=\theta_0-\dfrac{a(\lambda)}{\lambda}[\mathcal{F}_1(\theta_0,\rho_0)+R_1(\theta_0,\rho_0;\lambda)], \\
\rho_1=\rho_0-\dfrac{a(\lambda)}{\lambda}[\mathcal{F}_2(\theta_0,\rho_0)+R_2(\theta_0,\rho_0;\lambda)],
\end{cases}
$$

where R_1 and R_2 are such that

$$
\lim_{\lambda\to\infty}R_1(\theta_0,\rho_0;\lambda)=\lim_{\lambda\to\infty}R_2(\theta_0,\rho_0;\lambda)=0,
$$

uniformly for $\theta_0\in[0,\tau[$ and ρ_0 in a compact subset of $\mathbb{R}_+$.

Proof. We have to show that

$$
\lim_{\lambda\to\infty}\frac{1}{a(\lambda)}\int_0^T \frac{1}{\rho(t)}\langle f(t,\lambda\rho(t)\varphi(t+\theta(t)),\lambda)|\varphi(t+\theta(t))\rangle\,dt=-\mathcal{F}_1(\theta_0,\rho_0),
$$

and

$$\lim_{\lambda\to\infty}\frac{1}{a(\lambda)}\int_0^T\langle f(t,\lambda\rho(t)\varphi(t+\theta(t)),\lambda)|\dot\varphi(t+\theta(t))\rangle\,dt=\mathcal{F}_2(\theta_0,\rho_0),$$

uniformly with respect to $\theta_0\in[0,\tau[$ and $\rho_0\in[a,b]$, with $0<a<b$. We prove the first of the two, the second one being similar.

Fix $\varepsilon>0$. Corresponding to each direction α_k defining the set Σ, we consider a small cone determined by $[\alpha_k-\eta,\alpha_k+\eta]$, for some $\eta>0$. Let Σ_η be the union of these cones, and define

$$A_\eta(\theta_0)=\{t\in[0,T]:\varphi(t+\theta_0)\in\Sigma_\eta\}.$$

Writing the above integral and the one defining $\mathcal{F}_1(\theta_0,\rho_0)$ as

$$\int_0^T\ldots=\int_{A_\eta(\theta_0)}\ldots+\int_{[0,T]\setminus A_\eta(\theta_0)}\ldots,$$

we have that, taking η small enough, since F transforms bounded sets into bounded sets,

$$\bigg|\int_{A_\eta(\theta_0)}\langle F(t,\varphi(t+\theta_0))|\varphi(t+\theta_0)\rangle\,dt\bigg|\le\frac{\varepsilon}{4},$$

and, for λ large enough, by (2.5) and (2.8),

$$\begin{aligned}\bigg|\int_{A_\eta(\theta_0)}&\frac{1}{a(\lambda)\rho(t)}\langle f(t,\lambda\rho(t)\varphi(t+\theta(t)),\lambda)|\varphi(t+\theta(t))\rangle\,dt\bigg|\\&\le\int_{A_\eta(\theta_0)}\gamma\bigg(\|\varphi(t+\theta(t))\|+\frac{2}{\rho_0}\bigg)\|\varphi(t+\theta(t))\|\,dt\le\frac{\varepsilon}{4}.\end{aligned}$$

On the other hand, for $t\in[0,T]\setminus A_\eta(\theta_0)$, by (2.4), (2.8) and (2.9),

$$\begin{aligned}\lim_{\lambda\to\infty}\frac{1}{\rho(t)}&\bigg\langle\frac{f(t,\lambda\rho(t)\varphi(t+\theta(t)),\lambda)}{a(\lambda)}\bigg|\varphi(t+\theta(t))\bigg\rangle\\&=\frac{1}{\rho_0}\langle F(t,\rho_0\varphi(t+\theta_0))|\varphi(t+\theta_0)\rangle,\end{aligned}$$

uniformly in $t\in[0,T]\setminus A_\eta(\theta_0)$, $\theta_0\in[0,\tau[$, and $\rho_0\in[a,b]$, so that, for λ large enough,

$$\begin{aligned}\bigg|\int_{[0,T]\setminus A_\eta(\theta_0)}&\bigg[\frac{1}{a(\lambda)\rho(t)}\langle f(t,\lambda\rho(t)\varphi(t+\theta(t)),\lambda)|\varphi(t+\theta(t))\rangle\\&-\frac{1}{\rho_0}\langle F(t,\rho_0\varphi(t+\theta_0))|\varphi(t+\theta_0)\rangle\bigg]\,dt\bigg|\le\frac{\varepsilon}{2}.\end{aligned}$$

So, taking η small enough and λ large enough, for every $\theta_0\in[0,\tau[$ and $\rho_0\in[a,b]$, we have

$$\bigg|\frac{1}{a(\lambda)}\int_0^T\frac{1}{\rho(t)}\langle f(t,\lambda\rho(t)\varphi(t+\theta(t)),\lambda)|\varphi(t+\theta(t))\rangle\,dt+\mathcal{F}_1(\theta_0,\rho_0)\bigg|\le\varepsilon,$$

and the lemma is thus proved. □

3. The computation of the degree

We want to compute the degree of $\widetilde{\mathcal{P}}_\lambda - \mathrm{Id}$ on the set

$$\Omega = \{\rho\varphi(\theta) : \theta \in [0, \tau[,\ \rho \in [0, 1[\}.$$

To this aim, it will be sufficient to consider the solutions of (2.7) starting from the boundary $\partial\Omega$, i.e. with $\rho_0 = 1$. The generalized polar coordinates (θ, ρ) used to define the set Ω permit to identify it with the unit ball, and its boundary $\partial\Omega$ with S^1. It is also convenient to identify $\mathbb{R}/\tau\mathbb{Z}$ with S^1, and to define the τ-periodic function

$$\Gamma: S^1 \to \mathbb{R}^2, \quad \Gamma(\theta) = (\Gamma_1(\theta), \Gamma_2(\theta)),$$

where $\Gamma_1(\theta) = \mathcal{F}_1(\theta, 1)$ and $\Gamma_2(\theta) = \mathcal{F}_2(\theta, 1)$, i.e.

$$\begin{aligned} \Gamma_1(\theta) &= -\int_0^T \langle F(t, \varphi(t+\theta)) | \varphi(t+\theta) \rangle \, dt, \\ \Gamma_2(\theta) &= \int_0^T \langle F(t, \varphi(t+\theta)) | \dot{\varphi}(t+\theta) \rangle \, dt. \end{aligned}$$

Assuming that $\Gamma(\theta) \neq 0$, for every $\theta \in S^1$, we denote by $\mathrm{rot}(\Gamma, S^1)$ its rotation number (sometimes called the Kronecker degree): it is the number of rotations around the origin, in clockwise direction, performed by $\Gamma(\theta)$ as θ varies from 0 to τ. We will now see how it is related to the topological degree associated to our periodic problem.

Proposition 3.1. *Assume that $\Gamma(\theta) \neq 0$, for every $\theta \in S^1$. Then, for λ sufficiently large,*

$$\deg(\widetilde{\mathcal{P}}_\lambda - \mathrm{Id}, \Omega) = 1 + \mathrm{rot}(\Gamma, S^1).$$

Proof. Define

$$\mathcal{V}_\lambda(\theta_0) = \rho_1\varphi(\theta_1) - \varphi(\theta_0),$$

so that

$$\deg(\widetilde{\mathcal{P}}_\lambda - \mathrm{Id}, \Omega) = \mathrm{rot}(\mathcal{V}_\lambda, S^1).$$

By Lemma 1, with $\rho_0 = 1$,

$$\begin{aligned} \mathcal{V}_\lambda(\theta_0) &= \Big(1 - \frac{a(\lambda)}{\lambda}\Gamma_2(\theta_0)\Big)\varphi\Big(\theta_0 - \frac{a(\lambda)}{\lambda}\Gamma_1(\theta_0)\Big) - \varphi(\theta_0) + \widetilde{R}_1(\theta_0; \lambda) \\ &= \Big(1 - \frac{a(\lambda)}{\lambda}\Gamma_2(\theta_0)\Big)\Big(\varphi(\theta_0) - \frac{a(\lambda)}{\lambda}\Gamma_1(\theta_0)\dot{\varphi}(\theta_0)\Big) - \varphi(\theta_0) + \widetilde{R}_2(\theta_0; \lambda) \\ &= -\frac{a(\lambda)}{\lambda}[\Gamma_1(\theta_0)\dot{\varphi}(\theta_0) + \Gamma_2(\theta_0)\varphi(\theta_0)] + \widetilde{R}_3(\theta_0; \lambda), \end{aligned}$$

where $\widetilde{R}_1, \widetilde{R}_2, \widetilde{R}_3$ are such that

$$\lim_{\lambda\to\infty} \frac{\lambda}{a(\lambda)}\widetilde{R}_1(\theta_0; \lambda) = \lim_{\lambda\to\infty} \frac{\lambda}{a(\lambda)}\widetilde{R}_2(\theta_0; \lambda) = \lim_{\lambda\to\infty} \frac{\lambda}{a(\lambda)}\widetilde{R}_3(\theta_0; \lambda) = 0,$$

uniformly for $\theta_0 \in [0, \tau[$. Let us introduce the auxiliary function

$$\mathcal{W}(\theta_0) = \Gamma_1(\theta_0)\dot{\varphi}(\theta_0) + \Gamma_2(\theta_0)\varphi(\theta_0).$$

For each θ_0, the couple $\{\dot{\varphi}(\theta_0), \varphi(\theta_0)\}$ is a basis for $\mathbb{R}^2$ and $(\Gamma_1(\theta_0), \Gamma_2(\theta_0))$ are the coordinates of $\mathcal{W}(\theta_0)$ with respect to this basis. Recalling the definition of $\Gamma(\theta_0)$, we conclude that $\mathcal{W}(\theta_0)$ rotates exactly $\mathrm{rot}(\Gamma, S^1)$ times around the origin, in clockwise direction, with respect to this basis, as θ_0 varies from 0 to τ. Since the basis itself rotates once in clockwise direction, we finally have

$$\mathrm{rot}(\mathcal{W}, S^1) = 1 + \mathrm{rot}(\Gamma, S^1).$$

As seen above, for any $\varepsilon > 0$ there is a $\lambda_\varepsilon > 0$ such that, if $\lambda \geq \lambda_\varepsilon$, then

$$\left\| \frac{\lambda}{a(\lambda)} \mathcal{V}_\lambda(\theta_0) + \mathcal{W}(\theta_0) \right\| \leq \varepsilon.$$

If ε is small enough, Rouché's theorem applies and, for $\lambda \geq \lambda_\varepsilon$, we have

$$\mathrm{rot}(\mathcal{V}_\lambda, S^1) = \mathrm{rot}(-\mathcal{W}, S^1) = \mathrm{rot}(\mathcal{W}, S^1) = 1 + \mathrm{rot}(\Gamma, S^1)\,.$$

The proof is thus completed. □

We are now ready to compute the topological degree associated to the periodic problem for system (1.1). Let $\mathcal{P}_\lambda \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the associated Poincaré map for the period T. Since

$$\mathcal{P}_\lambda(u) = \lambda \widetilde{\mathcal{P}}_\lambda(\lambda^{-1}u),$$

it is well defined, for $\lambda \geq 1$. Consequently, defining $\Omega_\lambda = \lambda\Omega$, i.e.

$$\Omega_\lambda = \{\rho\varphi(\theta) : \theta \in [0, \tau, \ \rho \in [0, \lambda[\},$$

we can conclude as follows.

Theorem 3.2. *Assume*

$$\Gamma(\theta) \neq 0, \quad \textit{for all } \theta \in S^1. \tag{3.1}$$

Then, for λ sufficiently large,

$$\deg(\mathcal{P}_\lambda - \mathrm{Id}, \Omega_\lambda) = 1 + \mathrm{rot}(\Gamma, S^1)\,.$$

We now have a series of corollaries, in some of which we need the following notation: for a vector $v \in \mathbb{R}^2 \setminus \{0\}$, we write

$$\mathbb{R}_+ v = \{tv : t > 0\}.$$

The first one deals with a situation where there is a "missing direction".

Corollary 3.3. *Assume* (3.1) *and that there is a vector* $v \in \mathbb{R}^2 \setminus \{0\}$ *such that*

$$\Gamma(\theta) \notin \mathbb{R}_+ v, \quad \text{for all } \theta \in S^1.$$

Then, for λ *sufficiently large, system* (1.1) *has a* T*-periodic solution.*

Proof. In this case, $\mathrm{rot}(\Gamma, S^1) = 0$, so that $\deg(\mathcal{P}_\lambda - \mathrm{Id}, \Omega_\lambda) = 1$, for λ sufficiently large. □

Taking $v = (0,1)$ or $v = (0,-1)$, we immediately get the following.

Corollary 3.4. *Assume* (3.1) *and that* Γ_1 *has constant sign, or that* Γ_2 *has constant sign on the zeros of* Γ_1. *Then, for* λ *sufficiently large, system* (1.1) *has a* T*-periodic solution.*

The above result generalizes a classical situation, first introduced by Lazer and Leach, which is better known as Landesman–Lazer type of situation (see [4] and the references therein).

The next corollary considers a situation in which $\Gamma(\theta)$ rotates in clockwise direction.

Corollary 3.5. *Assume* (3.1) *and that there is a vector* $v \in \mathbb{R}^2 \setminus \{0\}$ *such that, if* $\Gamma(\theta) \in \mathbb{R}_+ v$ *for some* $\theta \in [0, \tau[$, *then* $\frac{d}{d\theta}\langle \Gamma(\theta) | Jv \rangle$ *exists and is negative. Then, for* λ *sufficiently large, system* (1.1) *has a* T*-periodic solution.*

Proof. In this case, $\mathrm{rot}(\Gamma, S^1) \geq 0$, so that $\deg(\mathcal{P}_\lambda - \mathrm{Id}, \Omega_\lambda) \geq 1$, for λ sufficiently large. □

It is now useful to define the sets

$$\begin{aligned} \mathcal{A}_+ &= \{\theta \in S^1 : \Gamma_1(\theta) = 0,\ \Gamma_2(\theta) > 0\}, \\ \mathcal{A}_- &= \{\theta \in S^1 : \Gamma_1(\theta) = 0,\ \Gamma_2(\theta) < 0\}. \end{aligned}$$

Taking $v = (0,1)$ or $v = (0,-1)$ in Corollary 3.5, we have the following.

Corollary 3.6. *Assume* (3.1), *and that* Γ_1 *is differentiable and one of the following two situations holds:*

(a) $\Gamma_1'(\theta) > 0$ *for every* $\theta \in \mathcal{A}_+$*;*
(b) $\Gamma_1'(\theta) < 0$ *for every* $\theta \in \mathcal{A}_-$*.*

Then, for λ *sufficiently large, system* (1.1) *has a* T*-periodic solution.*

As a particular case, we have the following.

Corollary 3.7. *Assume* (3.1), *and that* Γ_1 *is differentiable and there are two constants* $c_1, c_2 \in \mathbb{R}$, *with* $c_1 > 0$, *for which* $\Gamma_2 = c_1 \Gamma_1' + c_2$. *Then, for* λ *sufficiently large, system* (1.1) *has a* T*-periodic solution.*

Proof. If $c_2 \leq 0$, we are in the situation (a) of Corollary 3.6, while if $c_2 > 0$ we have (b). Then, Corollary 3.6 directly applies. □

In the next corollary, $\Gamma(\theta)$ rotates in counter-clockwise direction at least twice.

Corollary 3.8. *Assume* (3.1) *and that there is a vector* $v \in \mathbb{R}^2 \setminus \{0\}$ *such that, if* $\Gamma(\theta) \in \mathbb{R}_+ v$ *for some* $\theta \in [0, \tau[$, *then* $\frac{d}{d\theta}\langle \Gamma(\theta) | Jv \rangle$ *exists and is positive. If the set*

$$\mathcal{A}_v = \{\theta \in S^1 : \Gamma(\theta) \in \mathbb{R}_+ v\}$$

has at least two elements, then, for λ *sufficiently large, system* (1.1) *has a* T*-periodic solution.*

Proof. Here, $\mathrm{rot}(\Gamma, S^1) \leq -2$, so that $\deg(\mathcal{P}_\lambda - \mathrm{Id}, \Omega_\lambda) \leq -1$, for λ sufficiently large. □

Taking $v = (0,1)$ or $v = (0,-1)$ in Corollary 3.8, we readily get the following.

Corollary 3.9. *Assume* (3.1)*, and that* Γ_1 *is differentiable and one of the following two situations holds:*

(a) $\Gamma_1'(\theta) < 0$ *for every* θ *in the set* $\mathcal{A}_+$*, which has at least two elements;*
(b) $\Gamma_1'(\theta) > 0$ *for every* θ *in the set* $\mathcal{A}_-$*, which has at least two elements.*

Then, for λ *sufficiently large, system* (1.1) *has a* T*-periodic solution.*

As a particular case which has often appeared in the applications, we have the following.

Corollary 3.10. *Assume* (3.1)*, and that* Γ_1 *is differentiable and there are two constants* $c_1, c_2 \in \mathbb{R}$*, with* $c_1 < 0$*, for which* $\Gamma_2 = c_1\Gamma_1' + c_2$*. If* Γ_2 *changes sign more than twice on the zeros of* Γ_1 *in* $[0, \tau[$*, then, for* λ *sufficiently large, system* (1.1) *has a* T*-periodic solution.*

Proof. If $c_2 \leq 0$, we are in the situation (a) of Corollary 3.9, while if $c_2 > 0$ we have (b). Then, Corollary 3.9 directly applies. □

4. Two particular cases

We first assume that the function f in (1.1) does not depend explicitly on λ, and is asymptotically positively homogeneous of some degree $\beta \in [0,1[$, in its second variable.

We consider the system

$$J\dot{u} = \nabla H(u) + g(t,u), \tag{4.1}$$

and assume that there are some constants $\alpha > 0$ and $\beta \in [0,1[$ for which

$$\|g(t,u)\| \leq \alpha(\|u\|^\beta + 1), \tag{4.2}$$

for every $t \in \mathbb{R}$ and $u \in \mathbb{R}^2$, and that there is a continuous function $G: \mathbb{R} \times (\mathbb{R}^2 \setminus \Sigma) \to \mathbb{R}^2$ such that

$$G(t,u) = \lim_{\lambda \to \infty} \frac{g(t, \lambda u)}{\lambda^\beta}, \tag{4.3}$$

uniformly with respect to (t,u) as u varies in compact subsets of $\mathbb{R}^2 \setminus \Sigma$ (the set Σ being defined as in Section 2). In this setting, choosing $a(\lambda) = \lambda^\beta$, we see that (2.3)–(2.5) hold, with $F = G$.

The Poincaré map associated to (4.1), not depending on λ, will now be denoted by $\mathcal{P}$. Assuming (3.1), the excision property of the degree permits, in this situation, to replace $\deg(\mathcal{P} - \mathrm{Id}, \Omega_\lambda)$ in Theorem 3.2 with $\deg(\mathcal{P} - \mathrm{Id}, B_R)$, for any sufficiently large disk $B_R = \{x \in \mathbb{R}^2 : \|x\| < R\}$, for which we have

$$\deg(\mathcal{P} - \mathrm{Id}, B_R) = 1 + \mathrm{rot}(\Gamma, S^1).$$

If Γ_1 is not identically zero, it can be shown by classical results from degree theory (see e.g. [11]) that

$$\mathrm{rot}(\Gamma, S^1) = \deg(\Gamma_1,]a, a+\tau[\cap \{\Gamma_2 > 0\}),$$

where a is chosen so that $\Gamma_1(a) \neq 0$. Hence, the excision property lead to

$$\begin{aligned} \deg(\mathcal{P} - \mathrm{Id}, B_R) &= 1 + \deg(\Gamma_1,]a, a+\tau[\cap \{\Gamma_2 > 0\}) \\ &= 1 - \deg(\Gamma_1,]a, a+\tau[\cap \{\Gamma_2 < 0\}), \end{aligned}$$

for R sufficiently large. This result has been proved in [7, Theorem 2].

We have already presented various examples of applications of this result in [7] to scalar second order equations of the kind

$$x'' + \mu x^+ - \nu x^- + h(x, x') = e(t), \tag{4.4}$$

generalizing equation (1.2) for the asymmetric oscillator. Here, we take

$$H(x,y) = \frac{1}{2}[\mu(x^+)^2 + \nu(x^-)^2 + y^2],$$

with $\mu > 0$ and $\nu > 0$. Our results in [7] generalize those by Wang [14] and Capietto and Wang [2]. However, for equation (4.4), it can be seen that $\Gamma_2 = -\Gamma_1' + c$, for some constant $c \in \mathbb{R}$, so that the degree can be at most 1. In particular, for the asymmetric oscillator, we have that $\Gamma_1 = \Phi$, as defined in (1.3), and $\Gamma_2 = -\Phi'$, so that we recover the situation described in the Introduction.

A different situation appears for the equation

$$x'' + n^2 x = \cos(nt)\left(1 - \sigma \frac{x^2}{1 + (nx)^2 + (x')^2}\right) + \varepsilon \cos t, \tag{4.5}$$

where n is a positive integer. Here, $\beta = 0$, $\Sigma = \{0\}$, and $n\tau = T = 2\pi$. Choosing $\varphi(t) = (\sin(nt), n\cos(nt))$, for $n \geq 2$ we have

$$\Gamma_1(\theta) = \pi\left(1 - \frac{3\sigma}{4n^2}\right)\sin(n\theta), \quad \Gamma_2(\theta) = n\pi\left(\frac{\sigma}{4n^2} - 1\right)\cos(n\theta).$$

Then, $\Gamma_2(\theta) = c_{n,\sigma}\Gamma_1'(\theta)$, with

$$c_{n,\sigma} = \frac{\sigma - 4n^2}{4n^2 - 3\sigma},$$

so that $\mathrm{rot}(\Gamma, S^1)$ will be $\pm n$, depending on the sign of $c_{n,\sigma}$. Hence,

$$\deg(\mathcal{P} - \mathrm{Id}, B_R) = \begin{cases} 1 + n & \text{if } \sigma \in \,]4n^2/3, 4n^2[, \\ 1 - n & \text{if } \sigma \notin [4n^2/3, 4n^2], \end{cases}$$

for R large enough. The same is also true for $n = 1$, provided that ε is sufficiently small.

We remark that an equation similar to (4.5) was proposed in [8] as an example of apparently chaotic dynamics.

A second situation considered in [7] deals with a system like

$$J\dot{u} = \nabla H(u) + \varepsilon g(t, u), \tag{4.6}$$

with a small parameter $\varepsilon > 0$, where the function g is asymptotically positively homogeneous of degree 1. We assume here that (4.2) and (4.3) hold, with $\beta = 1$. The Poincaré map associated to (4.6), taking into account the small parameter ε appearing in the equation, will now be denoted by $\widehat{\mathcal{P}}_\varepsilon$.

We can interpret such a situation by considering a function $\varepsilon(\lambda)$ such that

$$\lim_{\lambda \to +\infty} \varepsilon(\lambda) = 0, \tag{4.7}$$

and defining

$$f(t, u, \lambda) = \varepsilon(\lambda)g(t, u).$$

We are now in the situation of system (1.1), with $a(\lambda) = \lambda\varepsilon(\lambda)$. Theorem 1 above tells us that, for λ large enough,

$$\deg(\widehat{\mathcal{P}}_{\varepsilon(\lambda)} - \mathrm{Id}, \Omega_\lambda) = \deg(\mathcal{P}_\lambda - \mathrm{Id}, \Omega_\lambda) = 1 + \mathrm{rot}(\Gamma, S^1).$$

Since the function $\varepsilon(\lambda)$ satisfying (4.7) is arbitrary, we can conclude that, for any ε sufficiently small and λ large enough,

$$\deg(\widehat{\mathcal{P}}_\varepsilon - \mathrm{Id}, \Omega_\lambda) = 1 + \mathrm{rot}(\Gamma, S^1).$$

An equivalent version of this result has been given in [7, Theorem 3]. Again, we can replace $\deg(\widehat{\mathcal{P}}_\varepsilon - \mathrm{Id}, \Omega_\lambda)$ by $\deg(\widehat{\mathcal{P}}_\varepsilon - \mathrm{Id}, B_R)$, with R large enough.

As a simple example, consider the equation

$$x'' + x = \varepsilon\cos(nt)(|x| + \sigma|x'|).$$

If n is even, we find that $\Gamma_1(\theta) = \Gamma_2(\theta) = 0$, for every $\theta \in \mathbb{R}$, in which case our theory does not apply. If n is odd, we find

$$\begin{aligned}\Gamma_1(\theta) &= -\frac{4(2 + n\sigma(-1)^{(n-1)/2})}{n(n^2-4)}\sin(n\theta),\\ \Gamma_2(\theta) &= \frac{4(n + 2\sigma(-1)^{(n+1)/2})}{n(n^2-4)}\cos(n\theta),\end{aligned}$$

so that

$$\deg(\widehat{\mathcal{P}}_\varepsilon - \mathrm{Id}, B_R) = \begin{cases} 1+n & \text{if } \sigma \in \left]\dfrac{2}{n}(-1)^{(n+1)/2}, \dfrac{n}{2}(-1)^{(n+1)/2}\right[, \\ 1-n & \text{if } \sigma \notin \left[\dfrac{2}{n}(-1)^{(n+1)/2}, \dfrac{n}{2}(-1)^{(n+1)/2}\right], \end{cases}$$

for ε small enough, and R large enough. (Here we adopt the convention $[a,b] = [b,a]$ and $]a,b[=]b,a[$.)

5. Large-amplitude periodic solutions

In this section we consider situations where large-amplitude T-periodic solutions for system (1.1) appear. Similar results were proposed in [6] for scalar second order equations depending on a parameter.

Considering again (θ,ρ) as generalized polar coordinates, we can identify the set $\{(\theta,\rho) : \rho > 0\}$ with $\mathbb{R}^2 \setminus \{0\}$ and define the function

$$\mathcal{F}\colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2, \quad \mathcal{F}(\theta,\rho) = (\mathcal{F}_1(\theta,\rho), \mathcal{F}_2(\theta,\rho)),$$

with $\mathcal{F}_1$ and $\mathcal{F}_2$ as in (2.10).

Theorem 5.1. *Assume that there is an open bounded set U, whose closure is contained in $\mathbb{R}^2 \setminus \{0\}$, for which $\deg(\mathcal{F}, U) \neq 0$. Then, there is a $\overline{\lambda} \geq 1$ such that, for every $\lambda \geq \overline{\lambda}$, system (1.1) has a T-periodic solution of the form*

$$u_\lambda(t) = \lambda\rho_\lambda(t)\varphi(t + \theta_\lambda(t)), \tag{5.1}$$

with $\lambda^{-1}u_\lambda(t) \in U$, for every $t \in \mathbb{R}$.

Proof. Let $r > 0$ be such that $\overline{U} \cap B_r = \emptyset$, and denote by $(\theta(t),\rho(t))$ the solution of (2.7) with starting point $\theta(0) = \theta_0 \in [0,\tau[$, $\rho(0) = \rho_0 \geq r$. By Lemma 2.1, the function $\psi_\lambda\colon \mathbb{R}^2 \setminus B_r \to \mathbb{R}^2$, defined for λ sufficiently large by

$$\begin{aligned}\psi_\lambda(\theta_0,\rho_0) = \bigg(&-\int_0^T \frac{1}{\rho(t)}\langle f(t, \lambda\rho(t)\varphi(t+\theta(t)), \lambda) | \varphi(t+\theta(t))\rangle\, dt,\\ &\int_0^T \langle f(t, \lambda\rho(t)\varphi(t+\theta(t)), \lambda) | \dot\varphi(t+\theta(t))\rangle\, dt\bigg),\end{aligned}$$

has the following property: for every $\varepsilon > 0$, there is a λ_ε such that, if $\lambda \geq \lambda_\varepsilon$, then
$$\left\| \frac{1}{a(\lambda)} \psi_\lambda(\theta_0, \rho_0) - \mathcal{F}(\theta_0, \rho_0) \right\| \leq \varepsilon,$$
for every (θ_0, ρ_0) in the closure of U. Then, taking ε sufficiently small, we have that
$$\deg(\psi_\lambda, U) = \deg(\mathcal{F}, U) \neq 0,$$
so that there is a $(\theta_0, \rho_0) \in U$ such that $\psi_\lambda(\theta_0, \rho_0) = 0$. Consequently, (θ_0, ρ_0) is the starting point for a T-periodic solution $(\theta_\lambda(t), \rho_\lambda(t))$ of (2.7). Defining $u_\lambda(t)$ as in (5.1), we have $\lambda^{-1} u_\lambda(0) = \rho_0 \varphi(\theta_0) \in U$ and, by (2.8) and (2.9), if λ is sufficiently large, then $\lambda^{-1} u_\lambda(t) \in U$, for every $t \in [0, T]$. □

Corollary 5.2. *Assume that the function $\mathcal{F}$ is differentiable and that there is a point (θ^*, ρ^*), with $\rho^* > 0$, for which $\mathcal{F}(\theta^*, \rho^*) = 0$ and the jacobian matrix $\mathcal{F}'(\theta^*, \rho^*)$ is invertible. Then, there is a $\overline{\lambda} \geq 1$ such that, for every $\lambda \geq \overline{\lambda}$, system* (1.1) *has a T-periodic solution of the form* (5.1), *with*
$$\lim_{\lambda \to \infty} \theta_\lambda(t) = \theta^*, \qquad \lim_{\lambda \to \infty} \rho_\lambda(t) = \rho^*.$$

Proof. Since $\det \mathcal{F}'(\theta^*, \rho^*) \neq 0$, there is an open bounded neighborhood U of (θ^*, ρ^*) on which $\mathcal{F}$ is a diffeomorphism, so that $|\deg(\mathcal{F}, U)| = 1$. Theorem 5.1 then applies to give a solution of the form (5.1), with $(\theta_\lambda(t), \rho_\lambda(t)) \in U$, for every $t \in \mathbb{R}$, if λ is sufficiently large. Since the neighborhood U can be taken as small as desired, the proof is easily completed. □

An alternative proof of the above corollary can be provided by the use of the implicit function theorem.

As an example, consider the system
$$J\dot{u} = \nabla H(u) + \varepsilon \mathbb{A} u + g(t, u),$$
where $\mathbb{A}$ is a 2×2 matrix and ε is a small parameter. We assume (4.2) and (4.3), for some $\beta \in [0, 1[$, and setting $\lambda = \varepsilon^{1/(\beta - 1)}$, we define
$$f(t, u, \lambda) = \lambda^{\beta - 1} \mathbb{A} u + g(t, u).$$
Then, taking $a(\lambda) = \lambda^\beta$, we have
$$F(t, u) = \mathbb{A} u + G(t, u).$$
Define
$$\begin{aligned} \Lambda_1(\theta) &= -\int_0^T \langle G(t, \varphi(t + \theta)) | \varphi(t + \theta) \rangle \, dt, \\ \Lambda_2(\theta) &= \int_0^T \langle G(t, \varphi(t + \theta)) | \dot{\varphi}(t + \theta) \rangle \, dt, \end{aligned}$$

and $\Lambda(\theta) = (\Lambda_1(\theta), \Lambda_2(\theta))$. Then,

$$\mathcal{F}_1(\theta, \rho) = -\kappa_1 + \frac{1}{\rho}\Lambda_1(\theta), \quad \mathcal{F}_2(\theta, \rho) = \kappa_2 \rho + \Lambda_2(\theta),$$

where

$$\kappa_1 = \int_0^T \langle \mathbb{A}\varphi(t) | \varphi(t)\rangle \, dt, \quad \kappa_2 = \int_0^T \langle \mathbb{A}\varphi(t) | \dot{\varphi}(t)\rangle \, dt.$$

Assume that there are $\theta^* \in [0, \tau[$ and $\rho^* > 0$, for which Λ_1 and Λ_2 are differentiable at θ^*,

$$\Lambda_1(\theta^*) = \kappa_1 \rho^*, \quad \Lambda_2(\theta^*) = -\kappa_2 \rho^*,$$

and

$$\Lambda_1(\theta^*)\Lambda_2'(\theta^*) - \Lambda_2(\theta^*)\Lambda_1'(\theta^*) \neq 0,$$

i.e. $\Lambda(\theta^*)$ and $\Lambda'(\theta^*)$ are linearly independent. Then,

$$\mathcal{F}(\theta^*, \rho^*) = 0 \quad \text{and} \quad \det \mathcal{F}'(\theta^*, \rho^*) \neq 0,$$

so that Corollary 5.2 applies.

The above situation is illustrated by the following two examples, where for simplicity we assume $\tau = T = 2\pi$. First, consider the asymmetric oscillator with a positive damping

$$x'' + \varepsilon x' + \mu x^+ - \nu x^- = e(t), \tag{5.2}$$

which was treated in [5], [6]. Here we have $\mathbb{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, while $\beta = 0$ and the set Σ is determined choosing $\alpha_1 = 0$ and $\alpha_2 = \pi$. Then,

$$\Lambda_1(\theta) = \int_0^T e(t)\phi(t + \theta) \, dt,$$

i.e. $\Lambda_1 = \Phi$, as defined in (1.3), and $\Lambda_2 = -\Phi'$. Moreover, $\kappa_1 = \int_0^{2\pi} (\phi\phi') = 0$, and $\kappa_2 = \int_0^{2\pi} |\phi'|^2 \, dt = \pi$. If θ^* is such that

$$\Phi(\theta^*) = 0 \quad \text{and} \quad \Phi'(\theta^*) > 0,$$

then (5.2) has large amplitude 2π-periodic solutions of the form

$$(x_\varepsilon(t), x_\varepsilon'(t)) = \frac{1}{\varepsilon}\rho_\varepsilon(t)(\phi(t + \theta_\varepsilon(t)), \phi'(t + \theta_\varepsilon(t))), \tag{5.3}$$

the functions $\theta_\varepsilon(t), \rho_\varepsilon(t)$ being such that

$$\lim_{\varepsilon \to 0^+} \theta_\varepsilon(t) = \theta^*, \quad \lim_{\varepsilon \to 0^+} \rho_\varepsilon(t) = \frac{\Phi'(\theta^*)}{\pi}.$$

Since $\varepsilon > 0$, it can be seen that these solutions are asymptotically stable (see [5]).

As a second example consider the asymmetric oscillator

$$x'' + (\mu + \varepsilon)x^+ - (\nu + \varepsilon)x^- = e(t), \tag{5.4}$$

which was treated in [6]. Here we have $\mathbb{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, while $\beta = 0$ and the set Σ is determined choosing $\alpha_1 = \pi/2$ and $\alpha_2 = 3\pi/2$. Then, $\Lambda_1 = \Phi$ and $\Lambda_2 = -\Phi'$. Moreover,

$$\kappa_1 = \int_0^{2\pi} |\phi|^2 = \frac{\pi(\mu^{-3/2} + \nu^{-3/2})}{2} \quad \text{and} \quad \kappa_2 = \int_0^{2\pi} (\phi\phi') = 0.$$

If there is a θ^* such that

$$\Phi(\theta^*) > 0, \quad \Phi'(\theta^*) = 0, \quad \Phi''(\theta^*) \neq 0,$$

then (5.4) has large amplitude 2π-periodic solutions of the form (5.3), the functions $\theta_\varepsilon(t), \rho_\varepsilon(t)$ being such that

$$\lim_{\varepsilon \to 0^+} \theta_\varepsilon(t) = \theta^*, \quad \lim_{\varepsilon \to 0^+} \rho_\varepsilon(t) = \frac{2\Phi(\theta^*)}{\pi(\mu^{-3/2} + \nu^{-3/2})}.$$

Acknowledgements. The author warmly thanks Christian Fabry for his precious suggestions and friendly discussions.

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Manuscript received March 8, 2006

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