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SYMMETRIC HOMOCLINIC SOLUTIONS TO THE PERIODIC ORBITS IN THE MICHELSON SYSTEM

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Abstract. The Michelson system [6] $x'''+x'+0.5x^2 = c^2$ for the parameter value $c = 1$ is investigated. It was proven in [8] that the system possesses two odd periodic solutions. We shall show that there exist infinitely many homoclinic and heteroclinic connections between them. Moreover, we shall show that the family of homoclinic solutions contains a countable set of odd homoclinic solutions.

1. Introduction

Consider the third order ODE

$$x''' + x' + \frac{1}{2}x^2 = c^2. \tag{1.1}$$

The above equation has attracted attention of the scientists and it has been studied in several papers [3], [4], [6]–[10] because of its relevance to the Kuramoto–Sivashinsky PDE

$$u_t + u_{xxxx} + u_{xx} + uu_x = 0. \tag{1.2}$$

In particular (1.1) arises as the equation for a steady state or a traveling wave solutions of (1.2).

The numerical simulations show that for the parameter values $0 < c < 1.2$ the equation (1.1) possesses extremely complicated and chaotic dynamics. For

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$c \gg 0$ it was shown [5] that there exists only one bounded non-stationary solution — a heteroclinic orbit connecting equilibrium points $(x, x', x'') = (\pm c\sqrt{2}, 0, 0)$.

When the parameter decreases the cocoon bifurcations of the two-dimensional unstable and stable manifolds of the equilibrium points appear [4] and lead to the very complicated dynamics. In particular, for $c = 1$ it was proven in [9], [10] the existence of symbolic dynamics on two symbols and the existence of infinitely many heteroclinic connections between equilibrium points $(\pm c\sqrt{2}, 0, 0)$.

In [8] it was proven that for the parameter value $c = 1$ the system possesses two odd periodic solutions. Let us denote these solutions by S_1 and S_2. These periodic orbits are presented in Figure 1. In this paper we would like to prove the following results.

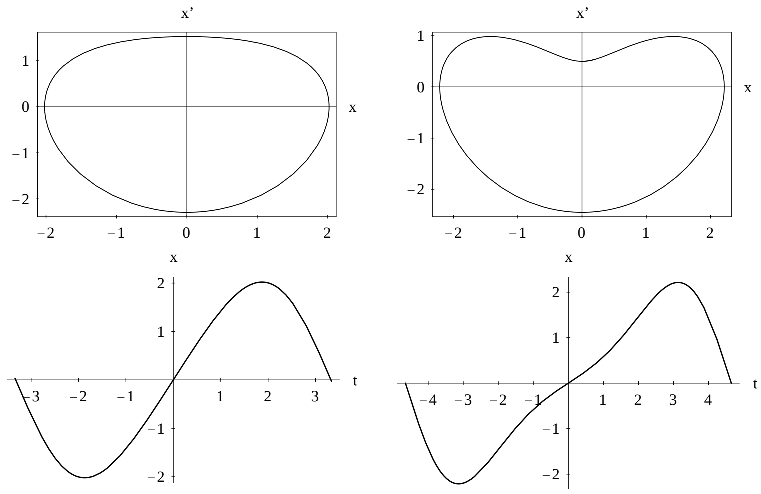

Figure 1. The two odd periodic solutions S_1 (left panel) and S_2 (right panel) established in [8].

Theorem 1.1. *The equation* (1.1) *with the parameter value* $c = 1$ *possesses infinitely many heteroclinic solutions connecting periodic orbits* S_1, S_2 *in both directions.*

Theorem 1.2. *The equation* (1.1) *with the parameter value* $c = 1$ *possesses infinitely many homoclinic solutions both to the* S_1 *and* S_2 *periodic orbits. Moreover, both families of such homoclinic solutions contain a countable set of odd homoclinic solutions.*

Let us briefly explain how the set of homoclinic and heteroclinic orbits looks like. One observes that the periodic orbits presented in Figures 1 and 6 are 'close' to each other in some neighbourhood of $x = x'' = 0$ and $x' = -2.35$. We will show that there are solutions of (1.1) defined for all $t \in \mathbb{R}$ which stay close to the periodic orbits S_1 and S_2. These solutions can be coded by the sequences $(i_j)_{j\in\mathbb{Z}} \in \{1,2\}^{\mathbb{Z}}$ in the following way: if $i_j = 1$ then the orbit makes loop close to S_1 and if $i_j = 2$ then the solution makes loop close to S_2. The heteroclinic or homoclinic orbits correspond to the sequences which satisfy $i_m = i_{m+1}$ for all $m \geq M > 0$ and $i_n = i_{n-1}$ for all $n \leq N < 0$. Hence, heteroclinic and homoclinic solutions can make some finite number of loops close to S_1 and S_2 as it is coded by $(i_N, i_{N+1}, \ldots, i_{M-1}, i_M)$ and finally they converge to the periodic orbits S_{i_N} and S_{i_M} in negative and positive time, respectively.

The paper is organized as follows. In Sections 2 and 3 we recall the main topological tools proved in [1], [2], [12]. In Section 4 we present a more general statement of Theorem 1.1 and Theorem 1.2 and we give the proofs.

2. Topological tools: h-sets and covering relations

In this section we present main topological tools used in this paper. The crucial notion is that of *covering relation* [2].

2.1. h-sets. Notation. For a given norm in $\mathbb{R}^n$ by $B_n(c,r)$ we will denote an open ball of radius r centered at $c \in \mathbb{R}^n$. When the dimension n is obvious from the context we will drop the subscript n. By B_n we will denote the unit ball $B_n(0,1)$. We set $\mathbb{R}^0 = \{0\}$, $B_0(0,r) = \{0\}$, $\partial B_0(0,r) = \emptyset$.

For a given set Z, by $\operatorname{int} Z$, $\overline{Z}$, ∂Z we denote the interior, the closure and the boundary of Z, respectively. For a map $h\colon [0,1] \times Z \to \mathbb{R}^n$ we set $h_t = h(t,\cdot)$. By Id we denote the identity map. For a map f, by $\operatorname{dom}(f)$ we will denote the domain of f. For $N \subset \Omega$, N-open and $c \in \mathbb{R}^n$ by $\deg(f,N,c)$ we denote the local Brouwer degree. For the properties of this notion we refer the reader to [4] (see also Appendix in [2]).

Definition 2.1 ([2, Definition 1]). A h-set N is an object consisting of the following data:

(a) $|N|$ — a compact subset of $\mathbb{R}^n$, we define $\dim(N) = n$,
(b) $u(N), s(N) \in \{0,1,2,\ldots\}$, such that $u(N) + s(N) = \dim(N) = n$,
(c) a homeomorphism $c_N\colon \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}$, such that

$$c_N(|N|) = \overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1).$$

We set

$$\begin{aligned}
N_c &= \overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1),\\
N_c^- &= \partial\overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1),\\
N_c^+ &= \overline{B_{u(N)}}(0,1) \times \partial\overline{B_{s(N)}}(0,1),\\
N^- &= c_N^{-1}(N_c^-), \quad N^+ = c_N^{-1}(N_c^+).
\end{aligned}$$

Hence a h-set N is a product of two closed balls in some coordinate system. The numbers, $u(N)$ and $s(N)$, stand for the dimensions of nominally unstable and nominally stable directions, respectively. The subscript c refers to the new coordinates given by homeomorphism c_N. Observe that if $u(N)=0$, then $N^-=\emptyset$ and if $s(N)=0$, then $N^+=\emptyset$. The geometry of Definition 2.1 is presented in Figure 2.

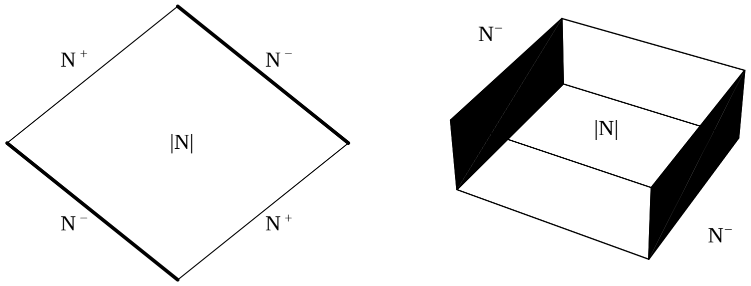

Figure 2. (left) an h-set with $u(N)=s(N)=1$, (right) an h-set with $u(N)=1$ and $s(N)=2$.

Definition 2.2 ([2, Definition 3]). Let N be a h-set.

(a) We define a h-set N^T as follows
- $|N^T|=|N|$,
- $u(N^T)=s(N)$, $s(N^T)=u(N)$.

(b) We define a homeomorphism $c_{N^T}\colon \mathbb{R}^n\to\mathbb{R}^n=\mathbb{R}^{u(N^T)}\times\mathbb{R}^{s(N^T)}$, by

$$c_{N^T}(x)=j(c_N(x)),$$

where $j\colon\mathbb{R}^{u(N)}\times\mathbb{R}^{s(N)}\to\mathbb{R}^{s(N)}\times\mathbb{R}^{u(N)}$ is given by $j(p,q)=(q,p)$.

Observe that $N^{T,+}=N^-$ and $N^{T,-}=N^+$. This operation is useful in the context of inverse maps.

2.2. Covering relations. Here we present a definition and some properties of covering relations — the main topological tool used in this paper.

Definition 2.3 ([2, Definition 6]). Assume N,M are h-sets in $\mathbb{R}^n$, such that $u(N)=u(M)=u$ and $s(N)=s(M)=s$. Let $f\colon|N|\to\mathbb{R}^n$ be a continuous map. Let $f_c=c_M\circ f\circ c_N^{-1}\colon N_c\to\mathbb{R}^u\times\mathbb{R}^s$. We say that

$$N\overset{f}{\Longrightarrow}M\quad (N\ f\text{-covers } M)$$

if and only if the following conditions are satisfied:

(a) there exists a continuous homotopy $h\colon [0,1] \times N_c \to \mathbb{R}^u \times \mathbb{R}^s$, such that the following conditions hold true

$$\begin{aligned} h_0 &= f_c, \\ h([0,1], N_c^-) \cap M_c &= \emptyset, \\ h([0,1], N_c) \cap M_c^+ &= \emptyset. \end{aligned}$$

(b) There exists a linear map $A\colon \mathbb{R}^u \to \mathbb{R}^u$, such that

$$\begin{gathered} h_1(p,q) = (A(p), 0), \quad \text{for } p \in \overline{B_u}(0,1) \text{ and } q \in \overline{B_s}(0,1), \\ A(\partial B_u(0,1)) \subset \mathbb{R}^u \setminus \overline{B_u}(0,1). \end{gathered}$$

Intuitively, $N \overset{f}{\Longrightarrow} M$ if f stretches N in the 'nominally unstable' direction, so that its projection onto 'unstable' direction in M covers in topologically nontrivial manner projection of M. In the 'nominally stable' direction N is contracted by f. As a result N is mapped across M in the unstable direction, without touching M^+. The geometry of this concept is presented in Figure 3.

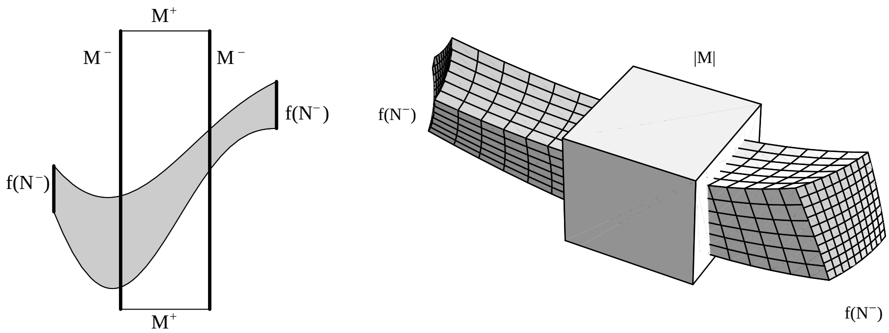

Figure 3. The geometry of covering relations (left) $u(N) = u(M) = 1$, $s(N) = s(M) = 1$ and (right) $u(N) = u(M) = 1$, $s(N) = s(M) = 2$.

Definition 2.4 ([2, Definition 7]). Assume N, M are h-sets in $\mathbb{R}^n$, such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $g\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$. Assume that $g^{-1}\colon |M| \to \mathbb{R}^n$ is well defined and continuous. We say that

$$N \overset{g}{\Longleftarrow} M \quad (N \ g\text{-backcovers } M)$$

if and only if $M^T \overset{g^{-1}}{\Longrightarrow} N^T$.

DEFINITION 2.5. Assume N, M are h-sets and f is a continuous map. We say that

$$N \overset{f}{\Longleftrightarrow} M \quad (N \text{ generally } f\text{-covers } M)$$

if one of the two following conditions is satisfied:

$$\begin{aligned} N &\subset \operatorname{dom}(f) && \text{and} \quad N \overset{f}{\Longrightarrow} M, \\ |M| &\subset \operatorname{dom}(f^{-1}) && \text{and} \quad N \overset{f}{\Longleftarrow} M. \end{aligned}$$

We would like to stress that the relation $N \overset{P}{\Longleftrightarrow} M$ is not symmetric.

DEFINITION 2.6. Let N be an h-set. Let $b\colon \overline{B_{u(N)}} \to |N|$ be continuous and let $b_c = c_N \circ b$. We say that b is a horizontal disk in N if there exists a homotopy $h\colon [0,1] \times \overline{B_{u(N)}} \to N_c$, such that

$$\begin{aligned} h_0 &= b_c, \\ h_1(x) &= (x,0), && \text{for all } x \in \overline{B_{u(N)}}, \\ h(t,x) &\in N_c^-, && \text{for all } t \in [0,1] \text{ and } x \in \partial B_{u(N)}. \end{aligned}$$

DEFINITION 2.7. Let N be an h-set. Let $b\colon \overline{B_{s(N)}} \to |N|$ be continuous and let $b_c = c_N \circ b$. We say that b is a vertical disk in N if there exists a homotopy $h\colon [0,1] \times \overline{B_{s(N)}} \to N_c$, such that

$$\begin{aligned} h_0 &= b_c, \\ h_1(x) &= (0,x), && \text{for all } x \in \overline{B_{s(N)}}, \\ h(t,x) &\in N_c^+, && \text{for all } t \in [0,1] \text{ and } x \in \partial B_{s(N)}. \end{aligned}$$

The geometry of these definitions is presented in Figure 4. In this case the horizontal disc is a curve ($u(N) = 1$) which can be deformed into the line connecting center points of the both components of N^-. Moreover, the end points of this curve belong to N^- throughout this deformation. Similarly, the vertical disc is a curve ($s(N) = 1$) which can be deformed into the line connecting center points of the top and bottom walls. Moreover, the end points of this curve belong to N^+ throughout this deformation.

Notice, that if $u(N) = s(N)$ then we can find a disc which is both horizontal and vertical in N — see Figure 5. Obviously, the required homotopies in the Definitions 2.6 and 2.7 are different.

The following theorem is one of the basic results in the covering relations method.

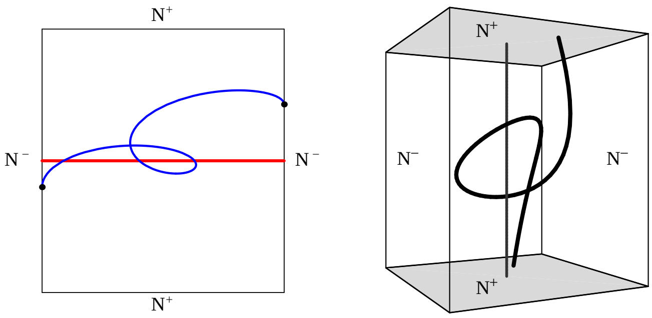

Figure 4. A horizontal disc in a h-set N with $u(N) = 1$ and $s(N) = 1$ (left). A vertical disc in a h-set N with $u(N) = 2$ and $s(N) = 1$ (right).

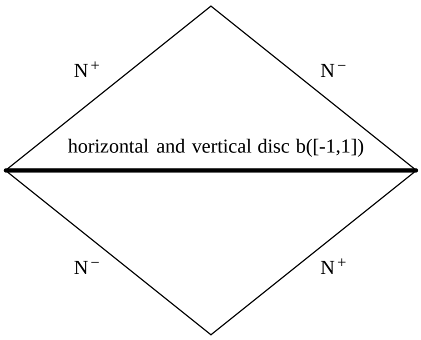

Figure 5. A curve b forms both horizontal and vertical discs in N.

Theorem 2.8 ([12, Theorem 3]). *Let $k \geq 1$. Assume N_i, $i = 0, \ldots, k$, are h-sets and for each $i = 1, \ldots, k$ we have either*

$$N_{i-1} \overset{f_i}{\Longrightarrow} N_i$$

or $|N_i| \subset \operatorname{dom}(f_i^{-1})$ and

$$N_{i-1} \overset{f_i}{\Longleftarrow} N_i.$$

Assume that b_h is a horizontal disk in N_0 and b_v is a vertical disk in N_k. Then there exists a point $x \in \operatorname{int}|N_0|$, such that:

$$\begin{aligned} x &= b_h(t), && \text{for some } t \in B_{u(N_0)}(0,1),\\ f_i \circ f_{i-1} \circ \cdots \circ f_1(x) &\in \operatorname{int}|N_i|, && \text{for } i = 1, \ldots, k,\\ f_k \circ f_{k-1} \circ \cdots \circ f_1(x) &= b_v(z), && \text{for some } z \in B_{s(N_k)}(0,1). \end{aligned}$$

A direct consequence of Theorem 2.8 is the following

Corollary 2.9. *Assume N_i, $i = 0, 1, \dots,$ are h-sets and for each $i = 1, 2, \dots$ we have either*

$$N_{i-1} \stackrel{f_i}{\Longrightarrow} N_i$$

or $|N_i| \subset \operatorname{dom}(f_i^{-1})$ and

$$N_{i-1} \stackrel{f_i}{\Longleftarrow} N_i.$$

Assume that b_h is a horizontal disk in N_0. Then there exists a point $x \in \operatorname{int}|N_0|$, such that:

$$x = b_h(t), \quad \text{for some } t \in B_{u(N_0)}(0, 1),$$
$$f_i \circ f_{i-1} \circ \dots \circ f_1(x) \in \operatorname{int}|N_i|, \quad i = 1, 2, \dots$$

3. Hyperbolicity

The goal of this section is to describe the tools which allow for a map, in the presence of hyperbolic fixed points, to prove an existence of homo- and heteroclinic trajectories.

In this section we recall the results from [1] with some additions.

3.1. General theorems. Let $P\colon \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-map. For any set X we define an interval matrix $[DP(X)] \subset \mathbb{R}^{n\times n}$ to be an interval enclosure of $DP(X)$ given by

$$M \in [DP(X)] \quad \text{if and only if} \quad \inf_{x\in X} DP(x)_{ij} \le M_{ij} \le \sup_{x\in X} DP(x)_{ij}$$

for $i, j = 1, \dots, n$.

Lemma 3.1 ([11, Lemma 4.1]). *Let N be a convex set. Assume $x_0, x_1 \in N$. Then*

$$P(x_1) - P(x_0) \in [DP(N)] \cdot (x_1 - x_0).$$

Moreover, there exists a matrix $M \in [DP(N)]$ such that

$$P(x_1) - P(x_0) = M \cdot (x_1 - x_0)$$

Consider a two-dimensional function $f(x) = (f_1(x), f_2(x))^T$, where $x = (x_1, x_2)^T$. We assume that $f(0) = 0$, i.e. 0 is a fixed point of f. For a convex set U, such that $0 \in U$ we define intervals $\boldsymbol{\lambda}_1(U)$, $\boldsymbol{\varepsilon}_1(U)$, $\boldsymbol{\varepsilon}_2(U)$ and $\boldsymbol{\lambda}_2(U)$ by

$$Df(U) = \begin{pmatrix} \boldsymbol{\lambda}_1(U) & \boldsymbol{\varepsilon}_1(U) \\ \boldsymbol{\varepsilon}_2(U) & \boldsymbol{\lambda}_2(U) \end{pmatrix}.$$

Since $f(0) = 0$ then from Lemma 3.1 it follows that

$$f_1(x) \in \boldsymbol{\lambda}_1(U)x_1 + \boldsymbol{\varepsilon}_1(U)x_2, \quad f_2(x) \in \boldsymbol{\varepsilon}_2(U)x_1 + \boldsymbol{\lambda}_2(U)x_2.$$

Let

$$\begin{aligned} \varepsilon_1'(U) &= \sup\{|\varepsilon| : \varepsilon \in \boldsymbol{\varepsilon}_1(U)\}, & \varepsilon_2'(U) &= \sup\{|\varepsilon| : \varepsilon \in \boldsymbol{\varepsilon}_2(U)\}, \\ \lambda_1'(U) &= \inf\{|\lambda_1| : \lambda_1 \in \boldsymbol{\lambda}_1(U)\}, & \lambda_2'(U) &= \sup\{|\lambda_2| : \lambda_2 \in \boldsymbol{\lambda}_2(U)\}. \end{aligned}$$

Let us define the rectangle N_{α_1,α_2} by

$$N_{\alpha_1,\alpha_2} = [-\alpha_1, \alpha_1] \times [-\alpha_2, \alpha_2] \quad \text{for } \alpha_1, \alpha_2 > 0.$$

Definition 3.2 ([1, Definition 1]). Let x_* be a fixed point for the map f. We say that f is *hyperbolic* on $N \ni x_*$, if there exists a local coordinate system on N, such that in this coordinate system

$$\begin{aligned} x_* &= 0, \\ \varepsilon_1'(N)\varepsilon_2'(N) &< (1 - \lambda_2'(N))(\lambda_1'(N) - 1), \\ N &= N_{\alpha_1,\alpha_2}, \end{aligned}$$

where $\alpha_1 > 0$, $\alpha_2 > 0$ are such that the following conditions are satisfied

$$\frac{\varepsilon_1'(N)}{\lambda_1'(N) - 1} < \frac{\alpha_1}{\alpha_2} < \frac{1 - \lambda_2'(N)}{\epsilon_2'(N)}.$$

It is easy to see that for the map f to be hyperbolic on N it is necessary that $\lambda_1' > 1$, $\lambda_2' < 1$ and the linearization of f at x_* is hyperbolic with one stable and unstable direction.

Theorem 3.3 ([1, Theorem 3]). *Assume that f is hyperbolic on N. Then*

(a) *if $f^k(x) \in N$ for $k \geq 0$, then $\lim_{k\to\infty} f^k(x) = x_*$,*
(b) *if $y_k \in N$ and $f(y_{k-1}) = y_k$ for $k \leq 0$, then $\lim_{k\to-\infty} y_k = x_*$.*

The next theorem shows how we can combine covering relations and hyperbolicity in order to prove the existence of asymptotic orbits with prescribed itinerary.

Theorem 3.4 ([1, Theorem 4]). *Assume that g is hyperbolic on N_m and f hyperbolic on N_0. Let $x_g \in N_m$ be a fixed point for g and $x_f \in N_0$ be a fixed point for f. If*

$$N_0 \overset{f}{\Longleftrightarrow} N_0 \overset{f_0}{\Longleftrightarrow} N_1 \overset{f_1}{\Longleftrightarrow} N_2 \overset{f_2}{\Longleftrightarrow} \dots \overset{f_{m-1}}{\Longleftrightarrow} N_m \overset{g}{\Longleftrightarrow} N_m \tag{3.1}$$

then there exists a sequence $(x_k)_{k=-\infty}^0$, $f(x_k) = x_{k+1}$ for $k < 0$ such that

$$\begin{aligned} x_k &\in N_0 & &\textit{for } k \leq 0, \\ f_{i-1} \circ f_{i-2} \circ \dots \circ f_0(x_0) &\in N_i & &\textit{for } i = 1, \dots, m, \\ g^n \circ f_{m-1} \circ \dots \circ f_0(x_0) &\in N_m & &\textit{for } n > 0, \end{aligned}$$

$$\lim_{k\to-\infty} x_k = x_f,$$
$$\lim_{k\to\infty} g^k \circ f_{m-1} \circ \cdots \circ f_0(x_0) = x_g.$$

The above theorem can be used without any modifications for proving the existence of trajectories converging to periodic orbits. In this case we consider higher iterates of maps f and g in (3.1).

4. The application to the Michelson system

In this section we show how the presented methods may be applied in order to prove Theorems 1.1 and 1.2. We rewrite the equation (1.1) as a first order system in $\mathbb{R}^3$ with a parameter value fixed to $c = 1$, i.e.

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = 1 - y - \dfrac{1}{2}x^2 \end{cases} \tag{4.1}$$

Let us observe that the system (4.1) possesses the following reversing symmetry

$$R(x, y, z, t) = (-x, y, -z, -t) \tag{4.2}$$

which means that if $t \to (x(t), y(t), z(t))$ is a solution of (4.1) then $t \to (-x(-t), y(-t), -z(-t))$ is a solution too. The existence of symmetry (4.2) implies that each solution of (1.1) with initial condition $y = y'' = 0$, $y' \in \mathbb{R}$ is an odd function.

4.1. Representation of h-sets. In this section we deal with h-sets possessing exactly one unstable direction and one stable direction which are parallelograms. Therefore we use the following representation. A h-set N in $\mathbb{R}^2$ may be defined by specifying (x, u, s), where $x, u, s \in \mathbb{R}^2$, are such that u, s are linearly independent. We then set

$$\begin{aligned} |N| &:= \{v \in \mathbb{R}^2 \mid \text{there exists } t_1, t_2 \in [-1,1] \text{ such that } v = x + t_1 s + t_2 u\} \\ &= x + [-1,1]\cdot u + [-1,1]\cdot s \end{aligned}$$

and take u as the nominally unstable direction and s as the nominally stable direction. The homeomorphism c_N is taken as the affine map $c_N(v) = M^{-1}(v - x)$, where $M = [u^T, s^T]$ is a square matrix. In this representation $N_c = \overline{B_1} \times \overline{B_1} = [-1,1]^2$ is a product of unit balls in the maximum norm.

In such a situation we will write $N = \mathfrak{h}(x, u, s)$.

4.2. Definition of the sets. Consider the Poincaré section $\Theta = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$. Since the third coordinate on Θ is constant and equal to zero we will use x, y coordinates only to describe the points on Θ and we will identify Θ with $\mathbb{R}^2$. The vector field is tangent to the section on parabola $\{(x, 1 - x^2/2, 0) \in \mathbb{R}^3 \mid x, \in \mathbb{R}\}$, hence the Poincaré return map is not defined on the whole section.

However, as it has been proven in [9], [10] the Poincaré map is well defined and continuous on some subset of Θ. Here we recall the main definitions.

We define five h-sets $N_i = \mathfrak{h}(x_i, u_i, s_i)$, $i = 1, \ldots, 5$, where

$$\begin{aligned}
x_1 &= (0.00, 1.55), & u_1 &= (0.14, 0.06), & s_1 &= (-0.14, 0.06),\\
x_2 &= (0.00, 0.51), & u_2 &= (0.09, 0.13), & s_2 &= (-0.09, 0.13),\\
x_3 &= (1.41, 0.97), & u_3 &= (0.06, 0.05), & s_3 &= (-0.06, 0.05),\\
x_4 &= (0.00, -2.35), & u_4 &= (0.06, 0.10), & s_4 &= (-0.06, 0.10),\\
x_5 &= (-1.41, 0.97), & u_5 &= (-0.06, 0.05), & s_5 &= (0.06, 0.05).
\end{aligned}$$

These sets are chosen as neighbourhoods of the intersections of periodic orbits found by Troy [8] with the Poincaré section Θ — see Figure 6.

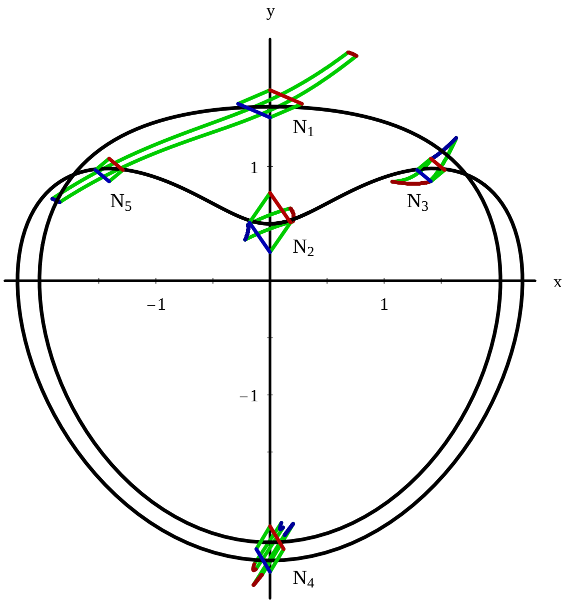

Figure 6. The numerical evidence of the existence of covering relations established in Lemma 4.1. Red and blue colors (when in color) correspond to N_i^- and their images.

Let $N = \bigcup_{i=1}^{5} |N_i|$. The following lemma has been proven in [9].

Lemma 4.1 ([9, Lemma 5.1]). *Let $P\colon \Theta \multimap \Theta$ denote the Poincaré return map for the Michelson system. Then $N \subset \operatorname{dom}(P)$ and*

$$\begin{aligned}
N_2 \overset{P}{\Longrightarrow} N_3 \overset{P}{\Longrightarrow} N_4 \overset{P}{\Longrightarrow} N_5 \overset{P}{\Longrightarrow} N_2,\\
N_1 \overset{P}{\Longrightarrow} N_4 \overset{P}{\Longrightarrow} N_1.
\end{aligned} \tag{4.3}$$

Note that in (4.3) there are two different loops of covering relations corresponding to the two symmetric periodic solutions S_1 and S_2. The main observation is that these loops contain the common h-set N_4 which allows to construct essentially different sequences of covering relations of an arbitrary length.

Now we define the hyperbolic sets around the odd periodic solutions. Define

$$\begin{aligned} q_1 &= 1.5259617305037, & q_2 &= 0.5000256485352, & \eta &= 4 \cdot 10^{-13}, \\ I_1 &= [q_1 - \eta, q_1 + \eta], & I_2 &= [q_2 - \eta, q_2 + \eta]. \end{aligned}$$

Lemma 4.2 ([10, Lemmas 5.12–5.14]). *There are two points $S_1^*, S_2^* \in \mathrm{Fix}(R)$ and there are two pairs of h-sets H_1, G_1 and H_2, G_2 centered at S_1^*, S_2^* respectively, such that the following conditions hold true:*

(a) *$S_1^* \in \{0\} \times I_1$ is a unique fixed point of P^2 in $|H_1|$,*
(b) *$S_2^* \in \{0\} \times I_2$ is a unique fixed point of P^4 in $|H_2|$,*
(c) *$|H_1| \subset |G_1| \subset |N_1|$ and $|H_2| \subset |G_2| \subset |N_2|$,*
(d) *P^2 is well defined and continuous on $|G_1|$,*
(e) *P^4 is well defined and continuous on $|G_2|$,*
(f) *P^2 is hyperbolic on H_1 and $H_1 \overset{P^2}{\Longrightarrow} H_1$,*
(g) *P^4 is hyperbolic on H_2 and $H_2 \overset{P^4}{\Longrightarrow} H_2$,*
(h) *$H_1 \overset{P^2}{\Longrightarrow} G_1 \overset{P^2}{\Longrightarrow} N_1$ and $N_1 \overset{P^2}{\Longleftarrow} G_1 \overset{P^2}{\Longleftarrow} H_1$,*
(i) *$H_2 \overset{P^4}{\Longrightarrow} G_2 \overset{P^4}{\Longrightarrow} N_2$ and $N_2 \overset{P^4}{\Longleftarrow} G_2 \overset{P^4}{\Longleftarrow} H_2$.*

The proof of the existence of backcovering relations

$$N_1 \overset{P^2}{\Longleftarrow} G_1 \overset{P^2}{\Longleftarrow} H_1, \qquad N_2 \overset{P^4}{\Longleftarrow} G_2 \overset{P^4}{\Longleftarrow} H_2 \tag{4.4}$$

is not explicitly presented in [10]. The assertion is a consequence of the symmetry of these sets in the sense of the following definitions.

Definition 4.3. Let Q be a h-set in $\mathbb{R}^n$. Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a homeomorphism. We define a h-set $L * Q$ as follows:

(a) $|L * Q| = L(|Q|)$,
(b) $u(L * Q) = u(Q)$ and $s(L * Q) = s(Q)$,
(c) $c_{L*Q} = c_Q \circ L^{-1}$.

We define a h-set $L^T * Q$ by $L^T * Q = (L * Q)^T$.

Informally speaking, $L * Q$ is just a natural symmetric image of Q and $L^T * Q$ is a symmetric image of Q, but we additionally switch the 'expanding' and 'contracting' directions.

Definition 4.4. A h-set Q is called R-symmetric if $R^T * Q = Q$.

We have the following

Lemma 4.5. *Let* $R\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ *be a reversing symmetry for a map* f *and* $Q_1 \overset{f}{\Longleftrightarrow} Q_2$*, then* $R^T * Q_2 \overset{f}{\Longleftrightarrow} R^T * Q_1$.

Proof. From the definition of covering relations and reversing symmetry it follows immediately that

$$\begin{aligned} &\text{if} \quad Q_1 \overset{f}{\Longrightarrow} Q_2 \quad \text{then} \quad R^T * Q_2 \overset{f}{\Longleftarrow} R^T * Q_1, \\ &\text{if} \quad Q_1 \overset{f}{\Longleftarrow} Q_2 \quad \text{then} \quad R^T * Q_2 \overset{f}{\Longrightarrow} R^T * Q_1. \end{aligned} \qquad \square$$

In fact, the sets H_1, H_2, G_1, G_2, N_1, N_2 are R-symmetric, hence from Lemma 4.5 we get that if $Q_1 \overset{P}{\Longrightarrow} Q_2$ holds for some $Q_1, Q_2 \in \{H_1, H_2, G_1, G_2, N_1, N_2\}$ then $Q_2 \overset{P}{\Longleftarrow} Q_1$. Therefore (4.4) holds true.

Let S_1 and S_2 denote the periodic orbits associated with the fixed points S_1^* S_2^* of the suitable Poincaré map, i.e.

$$S_j = \{S_j^*(t) \mid t \in \mathbb{R}\} \quad \text{for } j = 1, 2,$$

where $t \to S_j^*(t)$ is a solution of (4.1) with an initial condition $S_j^*(0) = S_j^*$.

Definition 4.6. We say that the sequence $(i_0, \ldots, i_n) \in \{1, \ldots, 5\}^{n+1}$ is admissible with respect to P if

$$N_{i_j} \overset{P}{\Longrightarrow} N_{i_{j+1}} \quad \text{for } j = 0, \ldots, n-1.$$

Now we are in the position to present a more general statement of Theorems 1.1 and 1.2.

Theorem 4.7. *Assume* $(i_0, \ldots, i_n) \in \{1, \ldots, 5\}^{n+1}$ *is an admissible sequence with respect to* P *such that* $i_0, i_n \in \{1, 2\}$. *Then there exists a solution* $u\colon \mathbb{R} \to \mathbb{R}^3$ *of* (4.1) *satisfying the following conditions:*

(a) *the solution* $u(t)$ *is defined for* $t \in \mathbb{R}$;
(b) *there are real numbers* $0 = t_0 < \ldots < t_n$ *such that* $u(t_j) \in |N_{i_j}|$ *for* $j = 0, \ldots, n$;
(c) *the* ω*-limit set* $\omega(u) = S_{i_n}$;
(d) *the* α*-limit set* $\alpha(u) = S_{i_0}$.

Hence, if $i_0 = i_n$ *then* u *is a homoclinic solution to* $S_{i_0} = S_{i_n}$ *and if* $i_0 \neq i_n$ *then* u *is a heteroclinic connection between* S_{i_0} *and* S_{i_n}.

Proof. From the assumptions and Lemma 4.2 we get that

$$H_{i_0} \overset{P^k}{\Longrightarrow} H_{i_0} \overset{P^k}{\Longrightarrow} G_{i_0} \overset{P^k}{\Longrightarrow} N_{i_0} \overset{P}{\Longrightarrow} \cdots \overset{P}{\Longrightarrow} N_{i_n} \overset{P^l}{\Longleftarrow} G_{i_n} \overset{P^l}{\Longleftarrow} H_{i_n} \overset{P^l}{\Longleftarrow} H_{i_n},$$

where $k = 2$ if $i_0 = 1$ and $k = 4$ if $i_0 = 2$. Similarly $l = 2$ if $i_n = 1$ and $l = 4$ if $i_n = 2$. From Lemma 4.2 we know that P^k is hyperbolic on H_{i_0} and P^l is hyperbolic on H_{i_n}. Now, Theorem 3.4 implies that there exists $u_0 \in |N_{i_0}|$ such that

$$P^j(u_0) \in |N_{i_j}|, \quad \text{for } j = 1, \ldots, n, \tag{4.5}$$

$$\lim_{j\to-\infty} P^{kj}(u_0) = S^*_{i_0}, \quad \lim_{j\to\infty} P^{n+lj}(u_0) = S^*_{i_n}. \tag{4.6}$$

This implies that the solution u of (4.1) satisfying $u(0) = u_0$ is defined for $t \in \mathbb{R}$. From the definition of the Poincaré map and (4.5) it follows that there are real numbers $0 = t_0 < \ldots < t_n$ such that $u(t_j) = P^j(u_0) \in |N_{i_j}|$ for $j = 1, \ldots, n$.

Finally, from (4.6) and the continuity of the local dynamical system induced by (4.1) we get that $\omega(u) = S_{i_n}$ and $\alpha(u) = S_{i_0}$. □

Denote by $\mathrm{Fix}(R)$ the set of fixed points for the symmetry, i.e.

$$\mathrm{Fix}(R) = \{(0, y, 0) \in \mathbb{R}^3 \mid y \in \mathbb{R}\}.$$

Now we prove the existence of odd homoclinic solutions.

Theorem 4.8. *Assume $(i_0, \ldots, i_n) \in \{1, \ldots, 5\}^{n+1}$ is an admissible sequence with respect to P such that $i_0 \in \{1, 2, 4\}$ and $i_n \in \{1, 2\}$. Then there exists a solution $u: \mathbb{R} \to \mathbb{R}^3$ of (4.1) satisfying the following conditions:*

(a) *$u(t)$ is defined for all $t \in \mathbb{R}$,*
(b) *$u(0) \in \mathrm{Fix}(R)$,*
(c) *there are real numbers $0 = t_0 < \ldots < t_n$ such that $u(t_j) \in |N_{i_j}|$ and $u(-t_j) \in R(|N_{i_j}|)$ for $j = 0, \ldots, n$,*
(d) *$\omega(u) = \alpha(u) = S_{i_n}$.*

Proof. Define the horizontal disc in N_{i_0}

$$b_h: B_1 \ni y \to x_{i_0} + y \cdot s_{i_0} + y \cdot u_{i_0} \in |N_{i_0}|,$$

where the required homotopy from the Definition 2.7 of vertical disc is given by

$$h_h: [0, 1] \times B_1 \ni (t, y) \to ((1 - t)y, y) \in (N_{i_0})_c.$$

Observe that the unstable and stable vectors used to define h-sets N_1, N_2 and N_4 are symmetric, i.e. $R(s_i) = u_i$ for $i = 1, 2, 4$. Moreover, $x_i \in \mathrm{Fix}(R)$ for $i = 1, 2, 4$. Therefore $b_h(B_1) \subset \mathrm{Fix}(R)$.

From Lemma 4.2 we get

$$N_{i_0} \overset{P}{\Longrightarrow} N_{i_1} \overset{P}{\Longrightarrow} \cdots \overset{P}{\Longrightarrow} N_{i_n} \overset{P^k}{\Longleftarrow} G_{i_n} \overset{P^k}{\Longleftarrow} H_{i_n} \overset{P^k}{\Longleftarrow} H_{i_n}, \tag{4.7}$$

where $k = 2$ if $i_n = 1$ and $k = 4$ if $i_n = 2$.

Now, Corollary 2.9 applied to the sequence (4.7) and the horizontal disc b_h in N_{i_0} implies that there exists $t \in B_1$ such that $b_h(t) \in |N_{i_0}| \cap \mathrm{Fix}(R)$ and

$$\begin{aligned} P^j(b_h(t)) &\in |N_{i_j}|, \quad \text{for } j = 1, \ldots, n, \\ P^{n+kj}(b_h(t)) &\in |N_{i_n}|, \quad \text{for } j = 1, 2, \ldots . \end{aligned}$$

From Lemma 4.2 we know that P^k is hyperbolic on N_{i_n}, therefore Theorem 3.3 implies that

$$\lim_{j\to\infty} P^{n+jk}(b_h(t)) = S^*_{i_n}. \tag{4.8}$$

Let u be the solution of (4.1) with initial condition $u(0) = b_h(t)$. Clearly $u(0) \in b_h(B_1) \subset \mathrm{Fix}(R)$. From the definition of the Poincaré map there are real numbers $0 = t_0 < \ldots, < t_n$ such that $u(t_j) = P^j(b_h(t)) \in |N_{i_j}|$. From (4.8) we get that $\omega(u) = S_{i_n}$.

Since $u(0) \in \mathrm{Fix}(R)$ the reversing symmetry property of (4.1) implies that $u(-t_j) = R(u(t_j)) \in R(|N_{i_j}|)$ for $j = 1, \ldots, n$ and $\alpha(u) = R(\omega(u)) = S_{i_n}$. □

Now we present the proof of the main theorems.

Proof of Theorems 1.1 and 1.2. All the assertions follow from Lemma 4.1 Theorems 4.7 and 4.8. The existence of infinitely many heteroclinic and homoclinic solutions is a consequence of Theorem 4.7 because we can find infinitely many sequences satisfying the assumptions of Theorem 4.7 which give geometrically different solutions. Take for example:

- $(1, \underbrace{4, 5, 2, 3, \ldots, 4, 5, 2, 3}_{n \text{ times}}, 4, 1) \in \{1, 2, 3, 4, 5\}^{4n+3}$, $n > 0$ for homoclinic solutions to S_1,
- $(2, 3, \underbrace{4, 1 \ldots, 4, 1}_{n \text{ times}}, 4, 5, 2) \in \{1, 2, 3, 4, 5\}^{2n+5}$, $n > 0$ for homoclinic solutions to S_2,
- $(1, \underbrace{4, 5, 2, 3, 4, 1, \ldots, 4, 5, 2, 3, 4, 1}_{n \text{ times}}, 4, 5, 2) \in \{1, 2, 3, 4, 5\}^{6n+4}$, $n > 0$ for heteroclinic orbit connecting S_1 with S_2,
- $(2, 3, \underbrace{4, 5, 2, 3, 4, 1, \ldots, 4, 5, 2, 3, 4, 1}_{n \text{ times}}) \in \{1, 2, 3, 4, 5\}^{6n+2}$, $n > 0$ for heteroclinic orbit connecting S_2 with S_1.

Similarly we can find infinitely many sequences satisfying assumptions of Theorem 4.8. Take for example:

- $(\underbrace{4, 5, 2, 3, \ldots, 4, 5, 2, 3}_{n \text{ times}}, 4, 1) \in \{1, 2, 3, 4, 5\}^{4n+2}$, $n > 0$ for odd homoclinic solutions to S_1,

- $(\underbrace{4,1,\ldots,4,1}_{n \text{ times}},4,5,2) \in \{1,2,3,4,5\}^{2n+3}$, $n > 0$ for odd homoclinic solutions to S_2. $\square$

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